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SOME RESULTS ON THE UNIFORM POLYNOMIAL BEHAVIOR IN AVERAGE OF COCYCLES OVER MAPS

Tian Yue[1], Kaituo Liu[2], Jie Zhang[3]

The objective of this paper is to establish necessary and sufficient conditions for the uniform polynomial stability in average of cocycles over maps. To this end, the results encompass a Hai-type characterization, a logarithmic characterization, and two Datko-type characterizations. Concurrently, analogous characterizations for the uniform polynomial instability in average are also provided.

Keywords: cocycle, uniform polynomial stability in average, uniform polynomial instability in average, logarithmic criterion, Datko type characterization

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1. Introduction

The qualitative theory exploring the exponential asymptotic behavior of dynamical systems in Banach spaces, including aspects such as stability, instability, and dichotomy, has seen considerable advancements over recent decades. This progress has been documented in numerous studies (see [1, 2, 3, 4, 5, 9, 10, 14, 15, 17] and the references therein), encompassing both deterministic and stochastic settings. Within the domain of dynamical systems and ergodic theory, cocycles are fundamental as they encapsulate the linear evolution of vectors along the trajectories of a dynamical system. Research on cocycles over dynamical systems has been instrumental in understanding the stability properties of linear perturbations evolving alongside nonlinear dynamical behaviors. For example, Stoica [14] obtained a Perron type theorem for uniform exponential dichotomy in mean square of stochastic cocycles in Hilbert spaces. Furthermore, Dragičević [10] established several continuous and discrete versions of the Datko type condition for exponential stability in average of cocycles. Recently, Yue [16] was motivated by the work of Dragičević [10] to describe the exponential instability in average of cocycles, using the Datko-Pazy theorem and Lyapunov functions.

Conversely, the classical concept of exponential (in)stability may appear overly restrictive. As such, it is crucial to explore more general behaviors, for example, by considering polynomial growth rates (see [6, 11, 12, 18]). In [12], Hai proposed several discrete and continuous versions of Datko type theorems for uniform polynomial (in)stability in the mean of stochastic skew-evolutionary semiflows, using Banach spaces of functions or sequences techniques. Furthermore, Boruga studied a logarithmic condition and a Datko type condition for polynomial stability in average of cocycles over semiflows in [8]. It should be noted that the author in [8] focused solely on the continuous time case. A pertinent question arises as to whether the logarithmic criterion and Datko type theorem can be extended to the discrete time scenario. This paper will address this inquiry in the affirmative.

[1]Associate Professor, Department of Mathematics, Hubei University of Automotive Technology, Shiyan 442002, China, e-mail: : yuet@huat.edu.cn

[2]Associate Professor, Department of Mathematics, Hubei University of Automotive Technology, Shiyan 442002, China, e-mail: : liukaituo@huat.edu.cn

[3]Lecturer, Department of Mathematics, Hubei University of Automotive Technology, Shiyan 442002, China, e-mail: : 905793623@qq.com

Drawing inspiration from [8, 10, 16], this paper delves into the study of uniform polynomial (in)stability in average behavior of cocycles over maps. The primary objective is to provide necessary and sufficient conditions for both uniform polynomial stability and instability in average behavior of these cocycles. Specifically, we establish four distinct necessary and sufficient conditions for uniform polynomial stability in average: a Hai type characterization, a logarithmic characterization, and two Datko type characterizations. Concurrently, we present analogous results for uniform polynomial instability in average. Notably, compared to continuous time approaches, our methodology relies on discrete time techniques, offering enhanced convenience for hypothesis verification and computational processes.

2. **Preliminaries**

In this section, we give some notations and definitions that will be used in the sequel. We denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{N}_{\geq\delta} = \{x \in \mathbb{N} : x \geq \delta\}$ and by $\Delta = \left\{(m,n) \in \mathbb{N}^2 : m \geq n \geq 1\right\}$. For a real number t, $[t]$ represents the largest integer less than or equal to t. Let $\Omega = (\Omega, \mathcal{B}, \mu)$ be a probability space, X a Banach space, $\mathcal{L}(X)$ the set of all invertible bounded linear operators from X to itself.

Definition 2.1. (see [7, 10]) *Let $f : \Omega \to \Omega$ be an invertible measurable map. A measurable map $\mathcal{A} : \mathbb{N} \times \Omega \to \mathcal{L}(X)$ is said to be a cocycle over f if the following conditions hold:*

(i) $\mathcal{A}(0,\omega) = \mathrm{Id}$ for all $\omega \in \Omega$;

(ii) $\mathcal{A}(n+m,\omega) = \mathcal{A}(n, f^m(\omega))\mathcal{A}(m,\omega)$ for all $n, m \in \mathbb{N}$ and $\omega \in \Omega$.

In what follows, we denote by

$$\mathcal{A}_\omega(m,n) = \mathcal{A}(m,\omega)\mathcal{A}(n,\omega)^{-1}.$$

Moreover, we denote by $\mathcal{F}$ the Banach space of all Bochner measurable functions $z : \Omega \to X$ such that

$$\|z\|_1 := \int_\Omega \|z(\omega)\| \, d\mu(\omega) < \infty,$$

identified if they are equal $\mu-$a.e.

Given a cocycle $\mathcal{A}$ over a map f, we shall always suppose that $\mathcal{A}$ is uniformly polynomially bounded in average, that is, there exist $K \geq 1$ and $\alpha > 0$ such that

$$\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\| \, d\mu(\omega) \leq K \left(\frac{m}{n}\right)^\alpha \int_\Omega \|z(\omega)\| \, d\mu(\omega) \tag{1}$$

for all $(m,n,z) \in \Delta \times \mathcal{F}$.

Definition 2.2. *Assume a cocycle $\mathcal{A}$ over a map f.*

(i) $\mathcal{A}$ is uniformly polynomially stable in average if there exist $N \geq 1$ and $v > 0$ such that

$$\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\| \, d\mu(\omega) \leq N \left(\frac{m}{n}\right)^{-v} \int_\Omega \|z(\omega)\| \, d\mu(\omega), \ \forall (m,n,z) \in \Delta \times \mathcal{F}. \tag{2}$$

(ii) $\mathcal{A}$ is uniformly polynomially unstable in average if there exist $N \geq 1$ and $v > 0$ such that

$$N \int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\| \, d\mu(\omega) \geq \left(\frac{m}{n}\right)^{v} \int_\Omega \|z(\omega)\| \, d\mu(\omega), \ \forall (m,n,z) \in \Delta \times \mathcal{F}. \tag{3}$$

(iii) $\mathcal{A}$ has uniform polynomial decay in average if there exist $M \geq 1$ and $\beta > 0$ such that

$$M \int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\| \, d\mu(\omega) \geq \left(\frac{m}{n}\right)^{-\beta} \int_\Omega \|z(\omega)\| \, d\mu(\omega), \ \forall (m,n,z) \in \Delta \times \mathcal{F}. \tag{4}$$

Proposition 2.1. (see [13]) *The following two relations hold for all* $\gamma > 0$, $m \in \mathbb{N}_{\geq 1}$ *and* $(n,k) \in \mathbb{N}^2$ *with* $n \geq k$.

(i) $\sum_{i=m}^{\infty} \frac{1}{i^{\gamma+1}} \leq \frac{2^{\gamma+1}}{\gamma m^{\gamma}}$;

(ii) $\sum_{i=k}^{n} i^{\gamma-1} \leq \frac{2n^{\gamma}}{\gamma}$.

3. Uniform polynomial stability in average

The study in this section is devoted to establishing some discrete characterizations for the uniform polynomial stability in average of cocycles. Firstly, we will present a Hai [11] type characterization for the concept of uniform polynomial stability in average.

Theorem 3.1. *A cocycle* $\mathcal{A}$ *is uniformly polynomially stable in average if and only if there exist* $c \in (0,1)$ *and* $\lambda \in \mathbb{N}_{\geq 2}$ *such that*

$$\int_{\Omega} \|\mathcal{A}_{\omega}(\lambda n, n) z(\omega)\| \, d\mu(\omega) \leq c \int_{\Omega} \|z(\omega)\| \, d\mu(\omega), \tag{5}$$

for all $(n,z) \in \mathbb{N}_{\geq 1} \times \mathcal{F}$.

Proof. Necessity. It is a simple verification that (5) holds for $\lambda = 1 + [N^{\frac{1}{v}}]$ and $c = \frac{N}{\lambda^{v}}$, where N, v are given by Definition 2.2(i).

Sufficiency. Let $(m,n,z) \in \Delta \times \mathcal{F}$ and $p = max\left\{j \in \mathbb{N} : m \geq n\lambda^{j}\right\}$. Then we have $\lambda^{p} \leq \frac{m}{n} < \lambda^{p+1}$. By the relations (1) and (5), it follows that

$$\begin{aligned}
\int_{\Omega} \|\mathcal{A}_{\omega}(m,n) z(\omega)\| \, d\mu(\omega) &= \int_{\Omega} \|\mathcal{A}_{\omega}(m, n\lambda^{p}) \mathcal{A}_{\omega}(n\lambda^{p}, n) z(\omega)\| \, d\mu(\omega) \\
&\leq K \left(\frac{m}{n\lambda^{p}}\right)^{\alpha} \int_{\Omega} \|\mathcal{A}_{\omega}(n\lambda^{p}, n) z(\omega)\| \, d\mu(\omega) \\
&\leq K\lambda^{\alpha} \int_{\Omega} \|\mathcal{A}_{\omega}(n\lambda^{p}, n) z(\omega)\| \, d\mu(\omega) \\
&\leq K\lambda^{\alpha} c \int_{\Omega} \left\|\mathcal{A}_{\omega}(n\lambda^{p-1}, n) z(\omega)\right\| \, d\mu(\omega) \leq \dots \\
&\leq \frac{K\lambda^{\alpha}}{c} c^{p+1} \int_{\Omega} \|z(\omega)\| \, d\mu(\omega) \\
&= \frac{K\lambda^{\alpha}}{c} \left(\lambda^{p+1}\right)^{-v} \int_{\Omega} \|z(\omega)\| \, d\mu(\omega) \\
&\leq N \left(\frac{m}{n}\right)^{-v} \int_{\Omega} \|z(\omega)\| \, d\mu(\omega),
\end{aligned}$$

where $N = \frac{K\lambda^{\alpha}}{c}$ and $v = -\frac{\ln c}{\ln \lambda}$. Hence, $\mathcal{A}$ is uniformly polynomially stable in average. □

Next, we give below a logarithmic criterion for the uniform polynomial stability in average concept by making use of Theorem 3.1.

Theorem 3.2. *A cocycle* $\mathcal{A}$ *is uniformly polynomially stable in average if and only if there exists a constant* $L > 1$ *such that*

$$\left(\ln \frac{m}{n}\right) \int_{\Omega} \|\mathcal{A}_{\omega}(m,n) z(\omega)\| \, d\mu(\omega) \leq L \int_{\Omega} \|z(\omega)\| \, d\mu(\omega), \tag{6}$$

for all $(m,n,z) \in \Delta \times \mathcal{F}$.

Proof. Necessity. If $\mathcal{A}$ is uniformly polynomially stable in average, then by Definition 2.2(i), there are $N \geq 1$ and $v > 0$ such that (2) holds. By (2), we have

$$\begin{aligned}\left(\ln\frac{m}{n}\right)\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega) &\leq N\left(\ln\frac{m}{n}\right)\left(\frac{m}{n}\right)^{-v}\int_\Omega \|z(\omega)\|\,d\mu(\omega)\\ &= \frac{N}{v}\frac{\ln\left(\frac{m}{n}\right)^v}{\left(\frac{m}{n}\right)^v}\int_\Omega \|z(\omega)\|\,d\mu(\omega)\\ &\leq L\int_\Omega \|z(\omega)\|\,d\mu(\omega),\end{aligned}$$

where $L = 1 + \frac{N}{ve}$.

Sufficiency. Let c is an arbitrary constant belongs to (0,1) and $\lambda = \left[e^{\frac{L}{c}}\right] + 1$. From (6) we have that

$$\int_\Omega \|\mathcal{A}_\omega(\lambda n,n)z(\omega)\|\,d\mu(\omega) \leq \frac{L}{\ln\lambda}\int_\Omega \|z(\omega)\|\,d\mu(\omega) \leq c\int_\Omega \|z(\omega)\|\,d\mu(\omega)$$

for all $(n,z) \in \mathbb{N}_{\geq 1} \times \mathcal{F}$. Now by Theorem 3.1, we conclude that $\mathcal{A}$ is uniformly polynomially stable in average. □

Finally, we present two Datko [9] type characterizations for the uniform polynomial stability in average.

Theorem 3.3. *A cocycle $\mathcal{A}$ is uniformly polynomially stable in average if and only if there exist $D > 1$ and $d > 0$ such that*

$$\sum_{i=n}^{\infty}\frac{1}{i}\left(\frac{i}{n}\right)^d\int_\Omega \|\mathcal{A}_\omega(i,n)z(\omega)\|\,d\mu(\omega) \leq D\int_\Omega \|z(\omega)\|\,d\mu(\omega), \tag{7}$$

for all $(n,z) \in \mathbb{N}_{\geq 1} \times \mathcal{F}$.

Proof. Necessity. Assume that $\mathcal{A}$ is uniformly polynomially stable in average, then by Definition 2.2(i), there are $N \geq 1$ and $v > 0$ such that the relation (2) holds. Let $d \in (0,v)$. From (2) and Proposition 2.1(i) we have

$$\begin{aligned}\sum_{i=n}^{\infty}\frac{1}{i}\left(\frac{i}{n}\right)^d\int_\Omega \|\mathcal{A}_\omega(i,n)z(\omega)\|\,d\mu(\omega) &\leq N\sum_{i=n}^{\infty}\frac{1}{i}\left(\frac{i}{n}\right)^d\left(\frac{i}{n}\right)^{-v}\int_\Omega \|z(\omega)\|\,d\mu(\omega)\\ &= Nn^{v-d}\sum_{i=n}^{\infty}\frac{1}{i^{v-d+1}}\int_\Omega \|z(\omega)\|\,d\mu(\omega)\\ &\leq Nn^{v-d}\frac{2^{v-d+1}}{(v-d)n^{v-d}}\int_\Omega \|z(\omega)\|\,d\mu(\omega)\\ &= \frac{2^{v-d+1}N}{v-d}\int_\Omega \|z(\omega)\|\,d\mu(\omega),\end{aligned}$$

for all $(n,z) \in \mathbb{N}_{\geq 1} \times \mathcal{F}$. Hence, (7) holds with $D = \frac{2^{v-d+1}N}{v-d} + 1$.

Sufficiency. If $(m,n)\in\Delta$ with $m\geq 2n$, then from (1) and (7) we have that

$$\begin{aligned}
&\left(\frac{m}{n}\right)^d\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega)\\
&\quad=\frac{1}{m-[m/2]+1}\sum_{i=[m/2]}^{m}\left(\frac{m}{n}\right)^d\int_\Omega \|\mathcal{A}_\omega(m,i)\mathcal{A}_\omega(i,n)z(\omega)\|\,d\mu(\omega)\\
&\quad\leq \frac{2K}{m}\sum_{i=[m/2]}^{m}\left(\frac{m}{n}\right)^d\left(\frac{m}{i}\right)^\alpha\int_\Omega \|\mathcal{A}_\omega(i,n)z(\omega)\|\,d\mu(\omega)\\
&\quad\leq 2K\sum_{i=[m/2]}^{m}\frac{1}{i}\left(\frac{i}{n}\right)^d\left(\frac{m}{[m/2]}\right)^{d+\alpha}\int_\Omega \|\mathcal{A}_\omega(i,n)z(\omega)\|\,d\mu(\omega)\\
&\quad\leq 2^{2(d+\alpha)+1}K\sum_{i=n}^{\infty}\frac{1}{i}\left(\frac{i}{n}\right)^d\int_\Omega \|\mathcal{A}_\omega(i,n)z(\omega)\|\,d\mu(\omega)\\
&\quad\leq 2^{2(d+\alpha)+1}KD\int_\Omega \|z(\omega)\|\,d\mu(\omega).
\end{aligned}$$

If $(m,n)\in\Delta$ with $n\leq m<2n$, then it follows from (1) that

$$\begin{aligned}
\left(\frac{m}{n}\right)^d\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega) &\leq K\left(\frac{m}{n}\right)^{d+\alpha}\int_\Omega \|z(\omega)\|\,d\mu(\omega)\\
&\leq 2^{d+\alpha}K\int_\Omega \|z(\omega)\|\,d\mu(\omega).
\end{aligned}$$

From the above two cases, we conclude that (2) holds with $N=2^{2(d+\alpha)+1}KD$ and $v=d$. □

Theorem 3.4. *A cocycle $\mathcal{A}$ is uniformly polynomially stable in average if and only if there exist $D>1$ and $d>0$ such that*

$$\sum_{i=n}^{m}\frac{1}{i}\left(\frac{m}{i}\right)^d\frac{1}{\int_\Omega \|\mathcal{A}_\omega(i,n)z(\omega)\|\,d\mu(\omega)}\leq\frac{D}{\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega)},\tag{8}$$

for all $(m,n,z)\in\Delta\times(\mathcal{F}\setminus\{0\})$.

Proof. Necessity. Assume that $\mathcal{A}$ is uniformly polynomially stable in average, then by Definition 2.2(i), there are $N\geq 1$ and $v>0$ such that the relation (2) holds. Let $d\in(0,v)$. From (2) and Proposition 2.1(ii) we have

$$\begin{aligned}
\sum_{i=n}^{m}\frac{1}{i}\left(\frac{m}{i}\right)^d\frac{1}{\int_\Omega \|\mathcal{A}_\omega(i,n)z(\omega)\|\,d\mu(\omega)} &\leq \frac{Nm^{d-v}}{\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega)}\sum_{i=n}^{m}i^{v-d-1}\\
&\leq \frac{Nm^{d-v}}{\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega)}\cdot\frac{2m^{v-d}}{v-d}\\
&=\frac{2N}{v-d}\frac{1}{\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega)},
\end{aligned}$$

for all $(m,n,z)\in\Delta\times(\mathcal{F}\setminus\{0\})$. Hence, (8) holds with $D=\frac{2N}{v-d}+1$.

Sufficiency. Let $(m,n,z)\in\Delta\times(\mathcal{F}\setminus\{0\})$.

If $m \geq 2n$, then from (1) and (8) we have that

$$\begin{aligned}\left(\frac{m}{n}\right)^d \frac{1}{\int_\Omega \|z(\omega)\|\, d\mu(\omega)} &= \frac{1}{n+1}\sum_{i=n}^{2n}\left(\frac{m}{n}\right)^d \frac{1}{\int_\Omega \|z(\omega)\|\, d\mu(\omega)}\\ &\leq \frac{1}{n}\sum_{i=n}^{2n} K\left(\frac{m}{n}\right)^d\left(\frac{i}{n}\right)^\alpha \frac{1}{\int_\Omega \|\mathcal{A}_\omega(i,n)z(\omega)\|\, d\mu(\omega)}\\ &\leq K\sum_{i=n}^{2n}\frac{1}{i}\left(\frac{m}{i}\right)^d\left(\frac{i}{n}\right)^{d+\alpha+1} \frac{1}{\int_\Omega \|\mathcal{A}_\omega(i,n)z(\omega)\|\, d\mu(\omega)}\\ &\leq 2^{d+\alpha+1}K\sum_{i=n}^{m}\frac{1}{i}\left(\frac{m}{i}\right)^d \frac{1}{\int_\Omega \|\mathcal{A}_\omega(i,n)z(\omega)\|\, d\mu(\omega)}\\ &\leq \frac{2^{d+\alpha+1}KD}{\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\, d\mu(\omega)}.\end{aligned}$$

If $n \leq m < 2n$, then it follows from (1) that

$$\begin{aligned}\left(\frac{m}{n}\right)^d \frac{1}{\int_\Omega \|z(\omega)\|\, d\mu(\omega)} &\leq K\left(\frac{m}{n}\right)^{d+\alpha} \frac{1}{\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\, d\mu(\omega)}\\ &\leq \frac{2^{d+\alpha}K}{\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\, d\mu(\omega)}.\end{aligned}$$

From the above two cases, we conclude that (2) holds with $N = 2^{d+\alpha+1}KD$ and $v = d$. □

4. **Uniform polynomial instability in average**

In this section, we extend the techniques used in the previous section to the case of a uniform polynomial instability in average.

Theorem 4.1. *A cocycle $\mathcal{A}$ is uniformly polynomially unstable in average if and only if there exist $c > 1$ and $\lambda \in \mathbb{N}_{\geq 2}$ such that*

$$\int_\Omega \|\mathcal{A}_\omega(\lambda n, n)z(\omega)\|\, d\mu(\omega) \geq c\int_\Omega \|z(\omega)\|\, d\mu(\omega), \tag{9}$$

for all $(n, z) \in \mathbb{N}_{\geq 1} \times \mathcal{F}$.

Proof. Necessity. It is a simple verification that (9) holds for $\lambda = 1 + [N^{\frac{1}{v}}]$ and $c = \frac{\lambda^v}{N}$, where N, v are given by Definition 2.2(ii).

Sufficiency. Let $(m, n, z) \in \Delta \times \mathcal{F}$ and $p = max\left\{j \in \mathbb{N} : m \geq n\lambda^j\right\}$. Then we have $\lambda^p \leq \frac{m}{n} < \lambda^{p+1}$. By the relation (1), we have

$$\begin{aligned}\int_\Omega \left\|\mathcal{A}_\omega(\lambda^{p+1}n, n)z(\omega)\right\| d\mu(\omega) &= \int_\Omega \left\|\mathcal{A}_\omega(\lambda^{p+1}n, m)\mathcal{A}_\omega(m,n)z(\omega)\right\| d\mu(\omega)\\ &\leq K\left(\frac{\lambda^{p+1}n}{m}\right)^\alpha \int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\, d\mu(\omega) \qquad (10)\\ &\leq K\lambda^\alpha \int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\, d\mu(\omega).\end{aligned}$$

From (10) and (9) we obtain that

$$\begin{aligned}\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega) &\geq K^{-1}\lambda^{-\alpha}\int_\Omega \left\|\mathcal{A}_\omega(\lambda^{p+1}n,n)z(\omega)\right\|d\mu(\omega)\\ &\geq K^{-1}\lambda^{-\alpha}c^{p+1}\int_\Omega \|z(\omega)\|\,d\mu(\omega)\\ &= K^{-1}\lambda^{-\alpha}\left(\lambda^{p+1}\right)^v\int_\Omega \|z(\omega)\|\,d\mu(\omega)\\ &\geq K^{-1}\lambda^{-\alpha}\left(\frac{m}{n}\right)^v\int_\Omega \|z(\omega)\|\,d\mu(\omega)\\ &= N^{-1}\left(\frac{m}{n}\right)^v\int_\Omega \|z(\omega)\|\,d\mu(\omega),\end{aligned}$$

where $N = K\lambda^\alpha$ and $v = \frac{\ln c}{\ln\lambda}$. Hence, $\mathcal{A}$ is uniformly polynomially unstable in average. □

Theorem 4.2. *A cocycle $\mathcal{A}$ is uniformly polynomially unstable in average if and only if there exists a constant $L > 1$ such that*

$$L\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega) \geq \left(\ln\frac{m}{n}\right)\int_\Omega \|z(\omega)\|\,d\mu(\omega), \tag{11}$$

for all $(m,n,z)\in\Delta\times\mathcal{F}$.

Proof. Necessity. If $\mathcal{A}$ is uniformly polynomially unstable in average, then by Definition 2.2(ii), there are $N \geq 1$ and $v > 0$ such that (3) holds. By (3), we have

$$\begin{aligned}\left(\ln\frac{m}{n}\right)\int_\Omega \|z(\omega)\|\,d\mu(\omega) &\leq N\left(\ln\frac{m}{n}\right)\left(\frac{m}{n}\right)^{-v}\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega)\\ &= \frac{N}{v}\frac{\ln\left(\frac{m}{n}\right)^v}{\left(\frac{m}{n}\right)^v}\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega)\\ &\leq L\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega),\end{aligned}$$

where $L = 1 + \frac{N}{ve}$.

Sufficiency. Let c is an arbitrary constant belongs to $(1,\infty)$ and $\lambda = \left[e^{Lc}\right] + 1$. From (11) we have that

$$\int_\Omega \|\mathcal{A}_\omega(\lambda n,n)z(\omega)\|\,d\mu(\omega) \geq \frac{\ln\lambda}{L}\int_\Omega \|z(\omega)\|\,d\mu(\omega) \geq c\int_\Omega \|z(\omega)\|\,d\mu(\omega)$$

for all $(n,z)\in\mathbb{N}_{\geq 1}\times\mathcal{F}$. Now by Theorem 4.1, we conclude that $\mathcal{A}$ is uniformly polynomially unstable in average. □

Theorem 4.3. *Assume that $\mathcal{A}$ has uniform polynomial decay in average. Then it is uniformly polynomially unstable in average if and only if there exist $D > 1$ and $d > 0$ such that*

$$\sum_{i=n}^{m}\frac{1}{i}\left(\frac{m}{i}\right)^d\int_\Omega \|\mathcal{A}_\omega(i,n)z(\omega)\|\,d\mu(\omega) \leq D\int_\Omega \|\mathcal{A}_\omega(m,n)z(\omega)\|\,d\mu(\omega), \tag{12}$$

for all $(m,n,z)\in\Delta\times\mathcal{F}$.

Proof. Necessity. Assume that $\mathcal{A}$ is uniformly polynomially unstable in average, then by Definition 2.2(ii), there are $N \geq 1$ and $v > 0$ such that the relation (3) holds. Let $d \in (0,v)$.

From (3) and Proposition 2.1(ii) we have

$$\begin{aligned}
&\sum_{i=n}^{m}\frac{1}{i}\left(\frac{m}{i}\right)^{d}\int_{\Omega}\|\mathcal{A}_{\omega}(i,n)z(\omega)\|\,d\mu(\omega)\\
&\quad\le N\sum_{i=n}^{m}\frac{1}{i}\left(\frac{m}{i}\right)^{d}\left(\frac{m}{i}\right)^{-v}\int_{\Omega}\|\mathcal{A}_{\omega}(m,n)z(\omega)\|\,d\mu(\omega)\\
&\quad= Nm^{d-v}\int_{\Omega}\|\mathcal{A}_{\omega}(m,n)z(\omega)\|\,d\mu(\omega)\sum_{i=n}^{m}i^{v-d-1}\\
&\quad\le Nm^{d-v}\int_{\Omega}\|\mathcal{A}_{\omega}(m,n)z(\omega)\|\,d\mu(\omega)\cdot\frac{2m^{v-d}}{v-d}\\
&\quad=\frac{2N}{v-d}\int_{\Omega}\|\mathcal{A}_{\omega}(m,n)z(\omega)\|\,d\mu(\omega),
\end{aligned}$$

for all $(m,n,z)\in\Delta\times\mathcal{F}$. Hence, (12) holds with $D=\frac{2N}{v-d}+1$.

Sufficiency. Let $(m,n,z)\in\Delta\times\mathcal{F}$.

If $m\ge 2n$, then from (4) and (12) we have that

$$\begin{aligned}
\left(\frac{m}{n}\right)^{d}\int_{\Omega}\|z(\omega)\|\,d\mu(\omega)&=\frac{1}{n+1}\sum_{i=n}^{2n}\left(\frac{m}{n}\right)^{d}\int_{\Omega}\|z(\omega)\|\,d\mu(\omega)\\
&\le M\sum_{i=n}^{2n}\frac{1}{i}\left(\frac{m}{i}\right)^{d}\left(\frac{i}{n}\right)^{d+\beta+1}\int_{\Omega}\|\mathcal{A}_{\omega}(i,n)z(\omega)\|\,d\mu(\omega)\\
&\le 2^{d+\beta+1}M\sum_{i=n}^{m}\frac{1}{i}\left(\frac{m}{i}\right)^{d}\int_{\Omega}\|\mathcal{A}_{\omega}(i,n)z(\omega)\|\,d\mu(\omega)\\
&\le 2^{d+\beta+1}MD\int_{\Omega}\|\mathcal{A}_{\omega}(m,n)z(\omega)\|\,d\mu(\omega).
\end{aligned}$$

If $n\le m<2n$, then it follows from (4) that

$$\begin{aligned}
\left(\frac{m}{n}\right)^{d}\int_{\Omega}\|z(\omega)\|\,d\mu(\omega)&\le M\left(\frac{m}{n}\right)^{d+\beta}\int_{\Omega}\|\mathcal{A}_{\omega}(m,n)z(\omega)\|\,d\mu(\omega)\\
&\le 2^{d+\beta}M\int_{\Omega}\|\mathcal{A}_{\omega}(m,n)z(\omega)\|\,d\mu(\omega).
\end{aligned}$$

From the above two cases, we conclude that (3) holds with $N=2^{d+\beta+1}MD$ and $v=d$. ☐

Theorem 4.4. *Assume that $\mathcal{A}$ has uniform polynomial decay in average. Then it is uniformly polynomially unstable in average if and only if there exist $D>1$ and $d>0$ such that*

$$\sum_{i=n}^{\infty}\frac{1}{i}\left(\frac{i}{n}\right)^{d}\frac{1}{\int_{\Omega}\|\mathcal{A}_{\omega}(i,n)z(\omega)\|\,d\mu(\omega)}\le\frac{D}{\int_{\Omega}\|z(\omega)\|\,d\mu(\omega)},\tag{13}$$

for all $(n,z)\in\mathbb{N}_{\ge 1}\times(\mathcal{F}\setminus\{0\})$.

Proof. Necessity. Assume that $\mathcal{A}$ is uniformly polynomially unstable in average, then by Definition 2.2(ii), there are $N\ge 1$ and $v>0$ such that the relation (3) holds. Let $d\in(0,v)$.

From (3) and Proposition 2.1(i) we have

$$\begin{aligned}\sum_{i=n}^{\infty}\frac{1}{i}\left(\frac{i}{n}\right)^{d}\frac{1}{\int_{\Omega}\|\mathcal{A}_{\omega}(i,n)z(\omega)\|\,d\mu(\omega)} &\leq N\sum_{i=n}^{\infty}\frac{1}{i}\left(\frac{i}{n}\right)^{d}\left(\frac{i}{n}\right)^{-v}\frac{1}{\int_{\Omega}\|z(\omega)\|\,d\mu(\omega)}\\ &\leq Nn^{v-d}\frac{2^{v-d+1}}{(v-d)n^{v-d}}\frac{1}{\int_{\Omega}\|z(\omega)\|\,d\mu(\omega)}\\ &=\frac{2^{v-d+1}N}{v-d}\frac{1}{\int_{\Omega}\|z(\omega)\|\,d\mu(\omega)},\end{aligned}$$

for all $(n,z)\in\mathbb{N}_{\geq 1}\times(\mathcal{F}\setminus\{0\})$. Hence, (13) holds with $D=\frac{2^{v-d+1}N}{v-d}+1$.

Sufficiency. Let $(m,n,z)\in\Delta\times(\mathcal{F}\setminus\{0\})$.

If $(m,n)\in\Delta$ with $m\geq 2n$, then from (4) and (13) we have that

$$\begin{aligned}&\left(\frac{m}{n}\right)^{d}\frac{1}{\int_{\Omega}\|\mathcal{A}_{\omega}(m,n)z(\omega)\|\,d\mu(\omega)}\\ &=\frac{1}{m-[m/2]+1}\sum_{i=[m/2]}^{m}\left(\frac{m}{n}\right)^{d}\frac{1}{\int_{\Omega}\|\mathcal{A}_{\omega}(m,i)\mathcal{A}_{\omega}(i,n)z(\omega)\|\,d\mu(\omega)}\\ &\leq\frac{2M}{m}\sum_{i=[m/2]}^{m}\left(\frac{m}{n}\right)^{d}\left(\frac{m}{i}\right)^{\beta}\frac{1}{\int_{\Omega}\|\mathcal{A}_{\omega}(i,n)z(\omega)\|\,d\mu(\omega)}\\ &\leq 2M\sum_{i=[m/2]}^{m}\frac{1}{i}\left(\frac{i}{n}\right)^{d}\left(\frac{m}{[m/2]}\right)^{d+\beta}\frac{1}{\int_{\Omega}\|\mathcal{A}_{\omega}(i,n)z(\omega)\|\,d\mu(\omega)}\\ &\leq 2^{2(d+\beta)+1}M\sum_{i=n}^{\infty}\frac{1}{i}\left(\frac{i}{n}\right)^{d}\frac{1}{\int_{\Omega}\|\mathcal{A}_{\omega}(i,n)z(\omega)\|\,d\mu(\omega)}\\ &\leq\frac{2^{2(d+\beta)+1}MD}{\int_{\Omega}\|z(\omega)\|\,d\mu(\omega)}.\end{aligned}$$

If $(m,n)\in\Delta$ with $n\leq m<2n$, then it follows from (4) that

$$\begin{aligned}\left(\frac{m}{n}\right)^{d}\frac{1}{\int_{\Omega}\|\mathcal{A}_{\omega}(m,n)z(\omega)\|\,d\mu(\omega)} &\leq M\left(\frac{m}{n}\right)^{d+\beta}\frac{1}{\int_{\Omega}\|z(\omega)\|\,d\mu(\omega)}\\ &\leq\frac{2^{d+\beta}M}{\int_{\Omega}\|z(\omega)\|\,d\mu(\omega)}.\end{aligned}$$

From the above two cases, we conclude that (3) holds with $N=2^{2(d+\beta)+1}MD$ and $v=d$. □

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REFERENCES

[1] *L. Arnold*, Random Dynamical Systems, Springer-Verlag, Berlin, 1998.

[2] *M. A. Alhalawa and D. Dragičević*, New conditions for (non)uniform behaviour of linear cocycles over flows, J. Math. Anal. Appl., **473**(2019), No. 1, 367-381.

[3] *L. Barreira, D. Dragičević and C. Valls*, Admissibility for exponential dichotomies in average, Stoch. Dyn., **15**(2015), No. 3, 1550014.

[4] *L. Barreira, D. Dragičević and C. Valls*, Exponential dichotomies in average for flows and admissibility, Publ. Math. Debrecen, **89**(2016), No. 4, 415-439.

[5] *L. Barreira and C. Valls*, Robustness of exponential dichotomies in mean, Stochastic Process. Appl., **124**(2014), No. 12, 4244-4265.

[6] *L. Barreira and C. Valls*, Polynomial growth rates, Nolinear Anal., **71**(2009), No. 11, 5208-5219.

[7] *L. Barreira and C. Valls*, Stable manifolds for perturbations of exponential dichotomies in mean, Stoch. Dyn., **18**(2018), No. 3, 1850022.

[8] *R. Boruga*, Polynomial stability in average for cocycles of linear operators, Theory Appl. Math. Comput. Sci., **9**(2019), No. 1, 8-13.

[9] *R. Datko*, Uniform asymptotic stability of evolutionary processes in a Banach space, SIAM J. Math. Anal., **3**(1972), No. 3, 428-445.

[10] *D. Dragičević*, A version of a theorem of R. Datko for stability in average, Systems Control Lett., **96**(2016), 1-6.

[11] *P. V. Hai*, On the polynomial stability of evolution families, Appl. Anal., **95**(2016), No. 6, 1239-1255.

[12] *P. V. Hai*, Polynomial behavior in mean of stochastic skew-evolution semiflows, arXiv, (2019), 1902.04214.

[13] *P. V. Hai*, Polynomial expansiveness and admissibility of weighted Lebesgue spaces, Czech. Math. J., **71**(2021), No. 1, 111-136.

[14] *D. Stoica*, Uniform exponential dichotomy of stochastic cocycles, Stochastic Process. Appl., **120**(2010), No. 10, 1920-1928.

[15] *T. M. Személy Fülöp, M. Megan and D. I. Borlea*, On uniform stability with growth rates of stochastic skew-evolution semiflows in Banach spaces, Axioms, **10**(2021), No. 3, 182.

[16] *T. Yue*, Datko type characterizations for exponential instability in average of cocycles, Hiroshima Math. J., **54**(2024), No. 2, 219-232.

[17] *T. Yue*, On uniform instability in mean of stochastic skew-evolution semiflows, Glas. Mat. Ser. III, **59**(2024), No. 1, 107-123.

[18] *T. Yue*, Barbashin type characterizations for the uniform polynomial stability and instability of evolution families, Georgian Math. J., **29**(2022), No. 6, 953-966.