ENERGY-CONSERVING SCHEMES FOR THE TIME-DEPENDENT INCOMPRESSIBLE MAGNETOHYDRODYNAMICS FLOWS

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In this work we consider the energy conservation property of the magnetohydrodynamics problem without the divergence constraint strongly enforced. Based on the various trilinear forms, we verify the energy conservation of the solutions generated by the variational formulation, semi-discrete Crank-Nicolson-type scheme with respect to time and full-discrete linearized Crank-Nicolson-type finite element scheme. Finally, numerical experiment is provided to verify the theoretical findings of the presented scheme.

Keywords: magnetohydrodynamic flows, finite element method, energy conservation, Crank-Nicolson-type scheme

MSC2010: 65M 60, 65M 12.

1. Introduction

The incompressible magnetohydrodynamic (MHD) flows have many important applications in astrophysics, geophysics, aerodynamics and so on, and describe the law of motion of a conductive fluid in an electromagnetic field. In this paper we consider the governing equations, time-dependent incompressible MHD equations, of the MHD flows. These governing equations are formed by coupling the incompressible Navier-Stokes equation in fluid mechanics and the Maxwell equation in electromagnetism under the influence of external forces and currents.

Given a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, and for a final time $T > 0$, find the velocity field $\mathbf{u} : [0, T] \times \Omega \to \mathbb{R}^d$, the pressure $p : [0, T] \times \Omega \to \mathbb{R}$ and the magnetic field $\mathbf{H} : [0, T] \times \Omega \to \mathbb{R}^d$ satisfying [4, 5]

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + s\mathbf{H}\times \mathrm{curl}\mathbf{H} + \nabla p = \mathbf{f}, & \text{in } \Omega\times(0,T], \\ \mathrm{div}\mathbf{u} = 0, & \text{in } \Omega\times(0,T], \\ s\mathbf{H}_t + \sigma^{-1}\mathrm{curl}\mathrm{curl}\mathbf{H} - s\mathrm{curl}(\mathbf{u}\times\mathbf{H}) = \sigma^{-1}\mathrm{curl}\mathbf{g}, & \text{in } \Omega\times(0,T], \\ \mathrm{div}\mathbf{H} = 0, & \text{in } \Omega\times(0,T], \end{cases} \tag{1}$$

where $\mathbf{f}$ is the external force term and $\mathbf{g}$ denotes the known current satisfying $(\mathbf{n}\times\mathbf{g})|_{S_T} = 0$. Here $\mathbf{n}$ represents the unit outer normal of $\partial\Omega$ and $S_T := \partial\Omega \times [0, T]$. The three constant physical parameters ν, s and σ are the kinematic viscosity, the magnetic permeability and

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the electric conductivity, respectively. The above equations are equipped with the following homogeneous boundary

$$\mathbf{u}|_{S_T} = 0,\ (\mathbf{H}\cdot\mathbf{n})|_{S_T} = 0,\ (\mathbf{n}\times \mathrm{curl}\mathbf{H})|_{S_T} = 0\ , \tag{2}$$

and the initial conditions [14]

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}),\ \mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}), \qquad \text{in } \Omega, \tag{3}$$

with $\mathrm{div}\mathbf{u}_0 = 0$ and $\mathrm{div}\mathbf{H}_0 = 0$.

The purpose of this paper is to preserve the energy conservation law for the MHD problem (1)-(3) in the numerical schemes. As is known, the energy conservation is an important physical law. However, in typical numerical schemes, this conservation law is lost or a time-step condition is needed to make sure it is conserved. In fact, this lost in Galerkin discretizations of the Navier-Stokes equations is well-known, and a fix for this problem by using the rotation and skew-symmetric forms for the nonlinear convection term has been shown [16, 9, 10, 6]. In particular, for the incompressible Navier-Stokes equations, beyond just energy, Charnyi et al. [2] have studied some conservation properties of the solutions generated by the variational formulation based on the different formulation of the nonlinear term, and semi-discrete Crank-Nicolson scheme [18] with respect to time. Furthermore, based on the EMA conserving formulation, they have considered the conservation properties of two linearized methods in [3]. A skew-symmetrized linearization conserves energy, but the Newton linearization does not ensure that the energy is accurately conserved.

Hence, for the time-dependent incompressible MHD flows, it is also important to find energy-conserving schemes. At the time of writing, there are numerous works devoted to the development of energy-conserving schemes for the MHD problem. Some fully discrete schemes introduced in [15] are analyzed. The numerical solutions to these schemes satisfy the perturbed discrete energy law. In [7, 8], the energy conservation is preserved at the discrete level for some nonlinear schemes. Liu and Wang [11] have proposed the MAC-Yee scheme for the incompressible MHD equations, which preserves the energy identity exactly. Furthermore, they have studied a class of simple and efficient numerical scheme for the considered equations with coordinate symmetry [12]. With proper discretization of the nonlinear terms, the schemes preserve both the energy and helicity identities numerically. In particular, Case et al. [1] have demonstrated the conversation law of three physical quantity for solving incompressible MHD equations by using the methods which require strongly solenoidal constraints. The energy conservation law of the solutions generated by the full-discrete nonlinear Crank-Nicolson finite element scheme is proved.

By considering some various formulations of the nonlinear term, we mainly study the energy conservation law of the solutions generated by the variational formulations, semi-discrete Crank-Nicolson-type scheme [13] with respect to time and full-discrete linearized Crank-Nicolson-type finite element scheme of the MHD problem (1)-(3), even if the schemes do not enforce the divergence constraint strongly. We find that the energy conservation law of the symmetric, rotation and energy formulations is observed in the form of variation, semi-discretization with respect to time and full-discretization where the nonlinear terms are linearized.

2. Preliminaries

Let us give the notations which will be used in this paper. $(\cdot,\cdot)$ and $\|\cdot\|$ denote $L^2(\Omega)$ inner product and norm on the domain Ω. For $1 \le p \le \infty$, $\|\cdot\|_{L^p(\Omega)}$ denotes $L^p(\Omega)$ norm and $\|\cdot\|_{W^{m,p}(\Omega)}$ refers to $W^{m,p}(\Omega)$ norm for $m \in \mathbb{N}^+$. For $p = 2$, the Sobolev space $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$ which is equipped with the norm $\|\cdot\|_m$. Besides, for a function space X on Ω, $L^p(0,t;X)$ is the space of all functions defined on $\Omega \times (0,t], t \in (0,T]$ for which the norm

$$\|\cdot\|_{L^p(0,t;X)} = \left(\int_0^t \|\cdot\|_X^p \mathrm{dt}\right)^{\frac{1}{p}},\ p \in [1,\infty)$$

is finite. The velocity, magnetic and pressure spaces are respectively introduced as

$$\mathbf{X} = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v}|_{\partial\Omega} = 0\}, \quad \mathbf{W} = \{\mathbf{B} \in H^1(\Omega)^d : \mathbf{B}\cdot\mathbf{n}|_{\partial\Omega} = 0\},$$

and

$$Q = \{q \in L^2(\Omega) : (1,q) = 0\}.$$

Then, we define the continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $\mathbf{X}\times\mathbf{X}$ and $\mathbf{X}\times Q$, respectively, by

$$a(\mathbf{u},\mathbf{v}) = \nu(\nabla\mathbf{u},\nabla\mathbf{v}),\ \forall \mathbf{u},\mathbf{v} \in \mathbf{X},$$

$$d(\mathbf{v},q) = (\mathrm{div}\mathbf{v},q),\ \forall \mathbf{v} \in \mathbf{X},\ q \in Q,$$

and a trilinear term on $\mathbf{X}\times\mathbf{X}\times\mathbf{X}$ by

$$b(\mathbf{u},\mathbf{w},\mathbf{v}) = ((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v}),\ \forall \mathbf{u},\mathbf{w},\mathbf{v} \in \mathbf{X}.$$

By using the above notations, the week formulation of the time dependent incompressible MHD problem (1)-(3) reads as follow: Find $(\mathbf{u},p,\mathbf{H}) \in L^2(0,T;\mathbf{X}) \times L^2(0,T;Q) \times L^2(0,T;\mathbf{W})$ satisfying

$$(\mathbf{u}_t,\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + s(\mathbf{H}\times\mathrm{curl}\mathbf{H},\mathbf{v}) - d(\mathbf{v},p) + d(\mathbf{u},q) = (\mathbf{f},\mathbf{v}), \tag{4}$$

$$s(\mathbf{H}_t,\mathbf{B}) + \sigma^{-1}(\mathrm{curl}\mathbf{H},\mathrm{curl}\mathbf{B}) - s(\mathbf{u}\times\mathbf{H},\mathrm{curl}\mathbf{B}) = \sigma^{-1}(\mathbf{g},\mathrm{curl}\mathbf{B}), \tag{5}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}),\ \mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}), \tag{6}$$

for all $(\mathbf{v},q,\mathbf{B}) \in \mathbf{X}\times Q\times\mathbf{W}$ and $t \in (0,T]$.

3. Energy conservation for the MHD equations

Denote the symmetric part of $\nabla\mathbf{u}$ by $\mathbf{D}(\mathbf{u}) := \frac{\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm{T}}}{2}$ and $NL(\mathbf{u},\mathbf{u}) := (\mathbf{u}\cdot\nabla)\mathbf{u}$ in (4). Then, we define various formulations of $NL(\mathbf{u},\mathbf{u})$ as follows:

skew-symmetric form: $NL_s(\mathbf{u},\mathbf{u}) := \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{2}(\mathrm{div}\mathbf{u})\mathbf{u}$,

rotational form: $NL_r(\mathbf{u},\mathbf{u}) := (\mathrm{curl}\mathbf{u})\times\mathbf{u}$,

energy-conserving form: $NL_e(\mathbf{u},\mathbf{u}) := 2\mathbf{D}(\mathbf{u})\mathbf{u} + (\mathrm{div}\mathbf{u})\mathbf{u}$.

The same definitions can be found in [16, 2]. In fact, if the divergence constraint div$\mathbf{u} = 0$ holds pointwise, then all the above forms are equivalent. When we use rotational and energy-conserving forms to replace the nonlinear term in(4), the pressure is modified. For the rotational form, the modified pressure is the Bernoulli pressure $p_r = p + \frac{1}{2}|\mathbf{u}|^2$, and for the energy-conserving form, the pressure is modified with a negative sign $p_e = p - \frac{1}{2}|\mathbf{u}|^2$.

For the rotational form, it follows from

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = (\mathrm{curl}\mathbf{u}) \times \mathbf{u} + \nabla \frac{1}{2}|\mathbf{u}|^2, \ \forall \mathbf{u} \in H^1(\Omega)^d;$$

and for the energy-conserving form, it follows from

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = 2\mathbf{D}(\mathbf{u})\mathbf{u} - \nabla \frac{1}{2}|\mathbf{u}|^2, \ \forall \mathbf{u} \in H^1(\Omega)^d.$$

Moreover, for all $\mathbf{u} \in \mathbf{X}$, taking the inner product of $NL(\mathbf{u}, \mathbf{u})$ with $\mathbf{v} \in \mathbf{X}$, we deduce

$$\begin{aligned}(NL_s(\mathbf{u}, \mathbf{u}), \mathbf{v}) &= (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) + \frac{1}{2}((\mathrm{div}\mathbf{u})\mathbf{u}, \mathbf{v}) \\ &= b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \frac{1}{2}(-b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - b(\mathbf{u}, \mathbf{v}, \mathbf{u})) \\ &= \frac{1}{2}b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \frac{1}{2}b(\mathbf{u}, \mathbf{v}, \mathbf{u}), \end{aligned} \tag{7}$$

and

$$\begin{aligned}(NL_r(\mathbf{u}, \mathbf{u}), \mathbf{v}) &= ((\mathrm{curl}\mathbf{u}) \times \mathbf{u}, \mathbf{v}) \\ &= ((\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathrm{T}})\mathbf{u}, \mathbf{v}) \\ &= b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - b(\mathbf{v}, \mathbf{u}, \mathbf{u}), \end{aligned} \tag{8}$$

as well as

$$\begin{aligned}(NL_e(\mathbf{u}, \mathbf{u}), \mathbf{v}) &= (2\mathbf{D}(\mathbf{u})\mathbf{u}, \mathbf{v}) + ((\mathrm{div}\mathbf{u})\mathbf{u}, \mathbf{v}) \\ &= ((\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}})\mathbf{u}, \mathbf{v}) + ((\mathrm{div}\mathbf{u})\mathbf{u}, \mathbf{v}) \\ &= b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \mathbf{u}, \mathbf{u}) + ((\mathrm{div}\mathbf{u})\mathbf{u}, \mathbf{v}) \\ &= -b(\mathbf{u}, \mathbf{v}, \mathbf{u}) + b(\mathbf{v}, \mathbf{u}, \mathbf{u}). \end{aligned} \tag{9}$$

3.1. Energy conservation of the variational formulation

To prove the conservation of energy for the variational formulation of the MHD equations, we first define the energy

$$E := \frac{1}{2}\int_\Omega (\mathbf{u} \cdot \mathbf{u} + s\mathbf{H} \cdot \mathbf{H})\mathrm{d}\mathbf{x}.$$

Next, let us verify the energy conservation for the variational formulation. Setting $\mathbf{v} = \mathbf{u}$ and $q = p$ in (4), and $\mathbf{B} = \mathbf{H}$ in (5), we get

$$(\mathbf{u}_t, \mathbf{u}) + (NL(\mathbf{u}, \mathbf{u}), \mathbf{u}) + a(\mathbf{u}, \mathbf{u}) + s(\mathbf{H} \times \mathrm{curl}\mathbf{H}, \mathbf{u}) = (\mathbf{f}, \mathbf{u}), \tag{10}$$

$$s(\mathbf{H}_t, \mathbf{H}) + \sigma^{-1}(\mathrm{curl}\mathbf{H}, \mathrm{curl}\mathbf{H}) - s(\mathbf{u} \times \mathbf{H}, \mathrm{curl}\mathbf{H}) = \sigma^{-1}(\mathbf{g}, \mathrm{curl}\mathbf{H}). \tag{11}$$

Adding (10) and (11) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2+\frac{s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{H}\|^2+(NL(\mathbf{u},\mathbf{u}),\mathbf{u})+\nu\|\nabla\mathbf{u}\|^2 \\ +\sigma^{-1}\|\mathrm{curl}\mathbf{H}\|^2=(\mathbf{f},\mathbf{u})+\sigma^{-1}(\mathbf{g},\mathrm{curl}\mathbf{H}), \tag{12}$$

by using [5]

$$(\mathbf{b}\times\mathrm{curl}\mathbf{B},\mathbf{v})=(\mathbf{v}\times\mathbf{b},\mathrm{curl}\mathbf{B}),\ \forall\mathbf{b},\mathbf{B},\mathbf{v}\in H^1(\Omega)^d. \tag{13}$$

In light of (7)-(9), we notice that

$$(NL_s(\mathbf{u},\mathbf{u}),\mathbf{u})=(NL_r(\mathbf{u},\mathbf{u}),\mathbf{u})=(NL_e(\mathbf{u},\mathbf{u}),\mathbf{u})=0. \tag{14}$$

Thus, the energy will be preserved for $\nu=0$, $\sigma=\infty$ and $\mathbf{f}=0$, for the skew-symmetric, rotational and energy-conserving forms whenever div$\mathbf{u}\neq 0$.

3.2. Energy conservation of the semi-discrete Crank-Nicolson-type scheme

As is known, some temporal discretizations, backward Euler and BDF2, are known to dissipate energy by their treatment of the time derivative terms. However, the Crank-Nicolson scheme, a temporal discretization, is known to be energy conserving. Hence, we consider the energy conservation of the semi-discrete Crank-Nicolson-type scheme with respect to time for the MHD problem (1)-(3). Here, we show the Galerkin formulation together with Crank-Nicolson-type time-stepping, and the skew-symmetric, rotational and energy-conserving forms are applied for the nonlinear term.

Let $t_n=n\Delta t$ for $0\leqslant n\leqslant N$ and $N=\frac{T}{\Delta t}$, where $\Delta t>0$ is the time-step size. Now, we design the following semi-discrete scheme based on the Crank-Nicolson-type approximation for the MHD equations.

Step I: Find $(\mathbf{u}^1,p^1,\mathbf{H}^1)\in\mathbf{X}\times Q\times\mathbf{W}$ satisfying

$$\left(\frac{\mathbf{u}^1-\mathbf{u}^0}{\Delta t},\mathbf{v}\right)+\left(NL(\frac{\mathbf{u}^1+\mathbf{u}^0}{2},\frac{\mathbf{u}^1+\mathbf{u}^0}{2}),\mathbf{v}\right)+a\left(\frac{\mathbf{u}^1+\mathbf{u}^0}{2},\mathbf{v}\right) \\ +s\left(\frac{\mathbf{H}^1+\mathbf{H}^0}{2}\times\mathrm{curl}\frac{\mathbf{H}^1+\mathbf{H}^0}{2},\mathbf{v}\right)-d(\mathbf{v},p^1)+d\left(\frac{\mathbf{u}^1+\mathbf{u}^0}{2},q\right)=\left(\mathbf{f}(t_{\frac{1}{2}}),\mathbf{v}\right), \tag{15}$$

$$s\left(\frac{\mathbf{H}^1-\mathbf{H}^0}{\Delta t},\mathbf{B}\right)+\sigma^{-1}\left(\mathrm{curl}\frac{\mathbf{H}^1+\mathbf{H}^0}{2},\mathrm{curl}\mathbf{B}\right) \\ -s\left(\frac{\mathbf{u}^1+\mathbf{u}^0}{2}\times\frac{\mathbf{H}^1+\mathbf{H}^0}{2},\mathrm{curl}\mathbf{B}\right)=\sigma^{-1}\left(\mathbf{g}(t_{\frac{1}{2}}),\mathrm{curl}\mathbf{B}\right), \tag{16}$$

for all $(\mathbf{v},q,\mathbf{B})\in\mathbf{X}\times Q\times\mathbf{W}$. Here, $\mathbf{f}(t_{\frac{1}{2}})=\frac{\mathbf{f}(t_0)+\mathbf{f}(t_1)}{2}$ and $\mathbf{g}(t_{\frac{1}{2}})=\frac{\mathbf{g}(t_0)+\mathbf{g}(t_1)}{2}$. Note that $\mathbf{u}^0=\mathbf{u}_0$ and $\mathbf{H}^0=\mathbf{H}_0$.

Step II: For $n \geqslant 1$, given $(\mathbf{u}^{n-1}, p^{n-1}, \mathbf{H}^{n-1})$, $(\mathbf{u}^n, p^n, \mathbf{H}^n) \in \mathbf{X} \times Q \times \mathbf{W}$, find $(\mathbf{u}^{n+1}, p^{n+1}, \mathbf{H}^{n+1}) \in \mathbf{X} \times Q \times \mathbf{W}$ satisfying

$$
\begin{aligned}
&\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n-1}}{2\Delta t}, \mathbf{v}\right) + \left(NL(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}, \frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}), \mathbf{v}\right) - d(\mathbf{v}, p^{n+1}) \\
&+ d\left(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}, q\right) + s\left(\frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2} \times \mathrm{curl}\frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2}, \mathbf{v}\right) \\
&+ a\left(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}, \mathbf{v}\right) = (\mathbf{f}(t_n), \mathbf{v}), \qquad (17) \\
&s\left(\frac{\mathbf{H}^{n+1}-H^{n-1}}{2\Delta t}, \mathbf{B}\right) + \sigma^{-1}\left(\mathrm{curl}\frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2}, \mathrm{curl}\mathbf{B}\right) \\
&- s\left(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2} \times \frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2}, \mathrm{curl}\mathbf{B}\right) = \sigma^{-1}(\mathbf{g}(t_n), \mathrm{curl}\mathbf{B}) \qquad (18)
\end{aligned}
$$

for all $(\mathbf{v}, q, \mathbf{B}) \in \mathbf{X} \times Q \times \mathbf{W}$.

In the following part of this subsection, we shall verify that the energy is conserving for our scheme. Define the discrete energy:

$$
E^n := \frac{1}{2}\left(\|\mathbf{u}^{n+1}\|^2 + \|\mathbf{u}^n\|^2 + s(\|\mathbf{H}^{n+1}\|^2 + \|\mathbf{H}^n\|^2)\right).
$$

Now, setting $(\mathbf{v}, q) = (\frac{\mathbf{u}^1+\mathbf{u}^0}{2}, p^1)$ and $\mathbf{B} = \frac{\mathbf{H}^1+\mathbf{H}^0}{2}$ in (15) and (16), respectively, it follows that

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|\mathbf{u}^1\|^2 - \|\mathbf{u}^0\|^2) + \left(NL(\frac{\mathbf{u}^1+\mathbf{u}^0}{2}, \frac{\mathbf{u}^1+\mathbf{u}^0}{2}), \frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right) + \nu\left\|\nabla\frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right\|^2 \\
&+ s\left(\frac{\mathbf{H}^1+\mathbf{H}^0}{2} \times \mathrm{curl}\frac{\mathbf{H}^1+\mathbf{H}^0}{2}, \frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right) = \left(\mathbf{f}(t_{\frac{1}{2}}), \frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right), \qquad (19) \\
&\frac{s}{2\Delta t}(\|\mathbf{H}^1\|^2 - \|\mathbf{H}^0\|^2) + \sigma^{-1}\left\|\mathrm{curl}\frac{\mathbf{H}^1+\mathbf{H}^0}{2}\right\|^2 \\
&- s\left(\frac{\mathbf{u}^1+\mathbf{u}^0}{2} \times \frac{\mathbf{H}^1+\mathbf{H}^0}{2}, \mathrm{curl}\frac{\mathbf{H}^1+\mathbf{H}^0}{2}\right) = \sigma^{-1}\left(\mathbf{g}(t_{\frac{1}{2}}), \mathrm{curl}\frac{\mathbf{H}^1+\mathbf{H}^0}{2}\right). \qquad (20)
\end{aligned}
$$

Combining (19) and (20) and applying (13), we have

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\mathbf{u}^1\|^2 + \frac{s}{2\Delta t}\|\mathbf{H}^1\|^2 + \left(NL(\frac{\mathbf{u}^1+\mathbf{u}^0}{2}, \frac{\mathbf{u}^1+\mathbf{u}^0}{2}), \frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right) \\
&+ \nu\left\|\nabla\frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right\|^2 + \sigma^{-1}\left\|\mathrm{curl}\frac{\mathbf{H}^1+\mathbf{H}^0}{2}\right\|^2 = \frac{1}{2\Delta t}\|\mathbf{u}^0\|^2 + \frac{s}{2\Delta t}\|\mathbf{H}^0\|^2 \\
&+ \left(\mathbf{f}(t_{\frac{1}{2}}), \frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right) + \sigma^{-1}\left(\mathbf{g}(t_{\frac{1}{2}}), \mathrm{curl}\frac{\mathbf{H}^1+\mathbf{H}^0}{2}\right). \qquad (21)
\end{aligned}
$$

On the other hand, choosing $(\mathbf{v},q)=(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2},p^{n+1})$ and $\mathbf{B}=\frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2}$ in (17) and (18), respectively, we obtain

$$\begin{aligned}&\frac{1}{4\Delta t}(\|\mathbf{u}^{n+1}\|^2-\|\mathbf{u}^{n-1}\|^2)+\left(NL(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2},\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}),\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right)\\&+\nu\left\|\nabla\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right\|^2+s\left(\frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2}\times\mathrm{curl}\frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2},\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right)\\=&\left(\mathbf{f}(t_n),\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right),\end{aligned}\tag{22}$$

$$\begin{aligned}&\frac{s}{4\Delta t}(\|\mathbf{H}^{n+1}\|^2-\|\mathbf{H}^{n-1}\|^2)+\sigma^{-1}\left\|\mathrm{curl}\frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2}\right\|^2\\&-s\left(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\times\frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2},\mathrm{curl}\frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2}\right)\\=&\sigma^{-1}\left(\mathbf{g}(t_n),\mathrm{curl}\frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2}\right).\end{aligned}\tag{23}$$

Combining (22) with (23) and summing the ensuing equation over $n=1,2,\cdots,$ $N-1$, we obtain

$$\begin{aligned}&\frac{1}{4\Delta t}(\|\mathbf{u}^N\|^2+\|\mathbf{u}^{N-1}\|^2-\|\mathbf{u}^1\|^2-\|\mathbf{u}^0\|^2)+\nu\sum_{n=1}^{N-1}\left\|\nabla\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right\|^2\\&+\sum_{n=1}^{N-1}\left(NL(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2},\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}),\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right)+\frac{s}{4\Delta t}(\|\mathbf{H}^N\|^2\\&+\|\mathbf{H}^{N-1}\|^2-\|\mathbf{H}^1\|^2-\|\mathbf{H}^0\|^2)+\sigma^{-1}\sum_{n=1}^{N-1}\left\|\mathrm{curl}\frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2}\right\|^2\\=&\sum_{n=1}^{N-1}\left(\mathbf{f}(t_n),\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right)+\sum_{n=1}^{N-1}\sigma^{-1}\left(\mathbf{g}(t_n),\mathrm{curl}\frac{\mathbf{H}^{n+1}+\mathbf{H}^{n-1}}{2}\right).\end{aligned}\tag{24}$$

Multiplying (21) and (24) with Δt and $2\Delta t$, respectively, and adding the ensuing equations, we get

$$\begin{aligned}&\frac{1}{2}(\|\mathbf{u}^N\|^2+\|\mathbf{u}^{N-1}\|^2)+\frac{s}{2}(\|\mathbf{H}^N\|^2+\|\mathbf{H}^{N-1}\|^2)+\nu\Delta t\left\|\nabla\frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right\|^2\\&+2\Delta t\sum_{n=1}^{N-1}\left(NL(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2},\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}),\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right)\\&+\Delta t\left(NL(\frac{\mathbf{u}^1+\mathbf{u}^0}{2},\frac{\mathbf{u}^1+\mathbf{u}^0}{2}),\frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right)+\sigma^{-1}\Delta t\left\|\mathrm{curl}\frac{\mathbf{H}^1+\mathbf{H}^0}{2}\right\|^2\\&+\nu\Delta t\sum_{n=1}^{N-1}\|\nabla(\mathbf{u}^{n+1}+\mathbf{u}^{n-1})\|^2+\sigma^{-1}\Delta t\sum_{n=1}^{N-1}\|\mathrm{curl}(\mathbf{H}^{n+1}+\mathbf{H}^{n-1})\|^2\\=&\|\mathbf{u}^0\|^2+s\|\mathbf{H}^0\|^2+\Delta t\left(\mathbf{f}(t_{\frac{1}{2}}),\frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right)+\sigma^{-1}\Delta t\left(\mathbf{g}(t_{\frac{1}{2}}),\mathrm{curl}\frac{\mathbf{H}^1+\mathbf{H}^0}{2}\right)\\&+\Delta t\sum_{n=1}^{N-1}(\mathbf{f}(t_n),\mathbf{u}^{n+1}+\mathbf{u}^{n-1})+\Delta t\sum_{n=1}^{N-1}\sigma^{-1}(\mathbf{g}(t_n),\mathrm{curl}(\mathbf{H}^{n+1}+\mathbf{H}^{n-1})).\end{aligned}\tag{25}$$

Further, according to (7)-(9), there holds

$$\begin{aligned}&\left(NL_s(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2},\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}),\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right)\\ =&\left(NL_r(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2},\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}),\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right)\\ =&\left(NL_e(\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2},\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}),\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right)=0,\end{aligned}$$

as well as

$$\begin{aligned}&\left(NL_s(\frac{\mathbf{u}^1+\mathbf{u}^0}{2},\frac{\mathbf{u}^1+\mathbf{u}^0}{2}),\frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right)=\left(NL_r(\frac{\mathbf{u}^1+\mathbf{u}^0}{2},\frac{\mathbf{u}^1+\mathbf{u}^0}{2}),\frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right)\\ =&\left(NL_e(\frac{\mathbf{u}^1+\mathbf{u}^0}{2},\frac{\mathbf{u}^1+\mathbf{u}^0}{2}),\frac{\mathbf{u}^1+\mathbf{u}^0}{2}\right)=0.\end{aligned}$$

Hence, one gets

$$\frac{1}{2}(\|\mathbf{u}^N\|^2+\|\mathbf{u}^{N-1}\|^2)+\frac{s}{2}(\|\mathbf{H}^N\|^2+\|\mathbf{H}^{N-1}\|^2)=\|\mathbf{u}^0\|^2+s\|\mathbf{H}^0\|^2, \tag{26}$$

when $\nu=0$, $\sigma=\infty$ and $\mathbf{f}=0$. Due to

$$\|\mathbf{u}^1\|^2+s\|\mathbf{H}^1\|^2=\|\mathbf{u}^0\|^2+s\|\mathbf{H}^0\|^2, \tag{27}$$

which results from (21), the energy is preserved for the presented semi-discrete scheme (15)-(18) with NL taken as NL_s, NL_r and NL_e. We recall that the divergence constraint is not strongly enforced.

3.3. Energy conservation of the full-discrete linearized Crank-Nicolson-type scheme

In this subsection, we introduce a spatial discretization of the time-discrete MHD equations (15)-(18) , where the nonlinear terms are linearized, by using the mixed finite element method.

To begin with, we take $\mathbf{X}_h\subset\mathbf{X}$, $Q_h\subset Q$, and $\mathbf{W}_h\subset\mathbf{W}$ as the conforming finite element spaces under a regular partition π_h of Ω with the largest diameter h for π_h. Furthermore, the finite element space pair $\mathbf{X}_h\times Q_h$ is assumed to satisfy the usual discrete inf-sup condition or LBB_h condition for the stability of the discrete pressure: there is a constant α independent of the mesh size h such that

$$\inf_{q_h\in Q_h}\sup_{\mathbf{v}_h\in\mathbf{X}_h}\frac{(q_h,\mathrm{div}\mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|\|q_h\|}\geq\alpha>0.$$

Next, we define $(\mathbf{u}_h^n,p_h^n,\mathbf{H}_h^n)$ to be a full-discrete approximation of the solution $(\mathbf{u}(t_n)$, $p(t_n),\mathbf{H}(t_n))$ of (1). The full-discrete linearized Crank-Nicolson-type scheme is as follows:

Step I: Find $(\mathbf{u}_h^1, p_h^1, \mathbf{H}_h^1) \in \mathbf{X}_h \times Q_h \times \mathbf{W}_h$, such that for all $(\mathbf{v}, q, \mathbf{B}) \in \mathbf{X}_h \times Q_h \times \mathbf{W}_h$,

$$\begin{aligned}
&\left(\frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\Delta t}, \mathbf{v}\right) + \left(NL(\mathbf{u}_h^0, \frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}), \mathbf{v}\right) + a\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, \mathbf{v}\right) \\
&+ s\left(\mathbf{H}_h^0 \times \mathrm{curl}\frac{\mathbf{H}_h^1 + H_h^0}{2}, \mathbf{v}\right) - d(\mathbf{v}, p_h^1) + d\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, q\right) = \left(\mathbf{f}(t_{\frac{1}{2}}), \mathbf{v}\right), \qquad (28)\\
&s\left(\frac{\mathbf{H}_h^1 - \mathbf{H}_h^0}{\Delta t}, \mathbf{B}\right) + \sigma^{-1}\left(\mathrm{curl}\frac{\mathbf{H}_h^1 + \mathbf{H}_h^0}{2}, \mathrm{curl}\mathbf{B}\right) + \sigma^{-1}\left(\mathrm{div}\frac{\mathbf{H}_h^1 + \mathbf{H}_h^0}{2}, \mathrm{div}\mathbf{B}\right) \\
&- s\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2} \times \mathbf{H}_h^0, \mathrm{curl}\mathbf{B}\right) = \sigma^{-1}\left(\mathbf{g}(t_{\frac{1}{2}}), \mathrm{curl}\mathbf{B}\right). \qquad (29)
\end{aligned}$$

Step II: For $n \geqslant 1$, given $(\mathbf{u}_h^{n-1}, p_h^{n-1}, \mathbf{H}_h^{n-1}), (\mathbf{u}_h^n, p_h^n, \mathbf{H}_h^n) \in \mathbf{X}_h \times Q_h \times \mathbf{W}_h$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}, \mathbf{H}_h^{n+1}) \in \mathbf{X}_h \times Q_h \times \mathbf{W}_h$, such that for all $(\mathbf{v}, q, \mathbf{B}) \in \mathbf{X}_h \times Q_h \times \mathbf{W}_h$,

$$\begin{aligned}
&\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n-1}}{2\Delta t}, \mathbf{v}\right) + \left(NL(\mathbf{u}_h^*, \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}), \mathbf{v}\right) + a\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \mathbf{v}\right) \\
&- d(\mathbf{v}, p_h^{n+1}) + d\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, q\right) + s\left(\mathbf{H}_h^* \times \mathrm{curl}\frac{\mathbf{H}_h^{n+1} + \mathbf{H}_h^{n-1}}{2}, \mathbf{v}\right) \\
&= (\mathbf{f}(t_n), \mathbf{v}), \qquad (30)\\
&s\left(\frac{\mathbf{H}_h^{n+1} - \mathbf{H}_h^{n-1}}{2\Delta t}, \mathbf{B}\right) + \sigma^{-1}\left(\mathrm{curl}\frac{\mathbf{H}_h^{n+1} + \mathbf{H}_h^{n-1}}{2}, \mathrm{curl}\mathbf{B}\right) \\
&+ \sigma^{-1}\left(\mathrm{div}\frac{\mathbf{H}_h^{n+1} + \mathbf{H}_h^{n-1}}{2}, \mathrm{div}\mathbf{B}\right) - s\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2} \times \mathbf{H}_h^*, \mathrm{curl}\mathbf{B}\right) \\
&= \sigma^{-1}(\mathbf{g}(t_n), \mathrm{curl}\mathbf{B}), \qquad (31)
\end{aligned}$$

where $\mathbf{u}_h^* = \frac{3}{2}\mathbf{u}_h^n - \frac{1}{2}\mathbf{u}_h^{n-1}$ and $\mathbf{H}_h^* = \frac{3}{2}\mathbf{H}_h^n - \frac{1}{2}\mathbf{H}_h^{n-1}$.

Then, according to (7)-(9), the linearized trilinear term $NL(\cdot,\cdot)$ in (28) can be written as

$$\begin{aligned}
\left(NL_s(\mathbf{u}_h^0, \frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}), \mathbf{v}\right) &= \frac{1}{2}b\left(\mathbf{u}^0, \frac{\mathbf{u}^1 + \mathbf{u}^0}{2}, \mathbf{v}\right) - \frac{1}{2}b\left(\mathbf{u}^0, \mathbf{v}, \frac{\mathbf{u}^1 + \mathbf{u}^0}{2}\right), \\
\left(NL_r(\mathbf{u}_h^0, \frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}), \mathbf{v}\right) &= b\left(\frac{\mathbf{u}^1 + \mathbf{u}^0}{2}, \mathbf{u}^0, \mathbf{v}\right) - b\left(\mathbf{v}, \mathbf{u}^0, \frac{\mathbf{u}^1 + \mathbf{u}^0}{2}\right), \\
\left(NL_e(\mathbf{u}_h^0, \frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}), \mathbf{v}\right) &= b\left(\mathbf{v}, \frac{\mathbf{u}^1 + \mathbf{u}^0}{2}, \mathbf{u}^0\right) - b\left(\frac{\mathbf{u}^1 + \mathbf{u}^0}{2}, \mathbf{v}, \mathbf{u}^0\right),
\end{aligned}$$

and the linearized trilinear term $NL(\cdot,\cdot)$ in (30) can be written as

$$\begin{aligned}
\left(NL_s(\mathbf{u}_h^*, \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}), \mathbf{v}\right) &= \frac{1}{2}b\left(\mathbf{u}^*, \frac{\mathbf{u}^{n+1} + \mathbf{u}^{n-1}}{2}, \mathbf{v}\right) - \frac{1}{2}b\left(\mathbf{u}^*, \mathbf{v}, \frac{\mathbf{u}^{n+1} + \mathbf{u}^{n-1}}{2}\right), \\
\left(NL_r(\mathbf{u}_h^*, \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}), \mathbf{v}\right) &= b\left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^{n-1}}{2}, \mathbf{u}^*, \mathbf{v}\right) - b\left(\mathbf{v}, \mathbf{u}^*, \frac{\mathbf{u}^{n+1} + \mathbf{u}^{n-1}}{2}\right), \\
\left(NL_e(\mathbf{u}_h^*, \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}), \mathbf{v}\right) &= b\left(\mathbf{v}, \frac{\mathbf{u}^{n+1} + \mathbf{u}^{n-1}}{2}, \mathbf{u}^*\right) - b\left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^{n-1}}{2}, \mathbf{v}, \mathbf{u}^*\right).
\end{aligned}$$

Note that

$$\begin{aligned}&\left(NL_s(\mathbf{u}_h^*, \frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2}), \frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2}\right)\\ =&\left(NL_r(\mathbf{u}_h^*, \frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2}), \frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2}\right)\\ =&\left(NL_e(\mathbf{u}_h^*, \frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2}), \frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2}\right)=0, \end{aligned}\tag{32}$$

and

$$\begin{aligned}&\left(NL_s(\mathbf{u}_h^0, \frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2}), \frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2}\right)=\left(NL_r(\mathbf{u}_h^0, \frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2}), \frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2}\right)\\ =&\left(NL_e(\mathbf{u}_h^0, \frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2}), \frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2}\right)=0, \end{aligned}\tag{33}$$

if we choose $\mathbf{v}=\frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2}$ and $\mathbf{v}=\frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2}$ in (30) and (28), respectively.

Moreover, letting $(\mathbf{v},q,\mathbf{B})$ be $\Delta t(\frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2}, p_h^1, \frac{\mathbf{H}_h^1+\mathbf{H}_h^0}{2})$ and $2\Delta t(\frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1}}{2}, p_h^{n+1}, \frac{\mathbf{H}_h^{n+1}+\mathbf{H}_h^{n-1}}{2})$ in (28), (29) and (30), (31), respectively, adding the ensuing equations and using (32) and (33), we arrive at

$$\begin{aligned}&\frac{1}{2}(\|\mathbf{u}_h^N\|^2+\|\mathbf{u}_h^{N-1}\|^2)+\frac{s}{2}(\|\mathbf{H}_h^N\|^2+\|\mathbf{H}_h^{N-1}\|^2)+\nu\Delta t\left\|\nabla\frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2}\right\|^2\\ &+\sigma^{-1}\Delta t\left\|\mathrm{curl}\frac{\mathbf{H}_h^1+\mathbf{H}_h^0}{2}\right\|^2+\sigma^{-1}\Delta t\left\|\mathrm{div}\frac{\mathbf{H}_h^1+\mathbf{H}_h^0}{2}\right\|^2\\ &+\nu\Delta t\sum_{n=1}^{N-1}\|\nabla(\mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1})\|^2+\sigma^{-1}\Delta t\sum_{n=1}^{N-1}\|\mathrm{curl}(\mathbf{H}_h^{n+1}+\mathbf{H}_h^{n-1})\|^2\\ &+\sigma^{-1}\Delta t\sum_{n=1}^{N-1}\|\mathrm{div}(\mathbf{H}_h^{n+1}+\mathbf{H}_h^{n-1})\|^2\\ =&\|\mathbf{u}_h^0\|^2+s\|\mathbf{H}_h^0\|^2+\Delta t\left(\mathbf{f}(t_{\frac{1}{2}}), \frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2}\right)+\sigma^{-1}\Delta t\left(\mathbf{g}(t_{\frac{1}{2}}), \mathrm{curl}\frac{\mathbf{H}_h^1+\mathbf{H}_h^0}{2}\right)\\ &+\Delta t\sum_{n=1}^{N-1}(\mathbf{f}(t_n), \mathbf{u}_h^{n+1}+\mathbf{u}_h^{n-1})+\Delta t\sum_{n=1}^{N-1}\sigma^{-1}(\mathbf{g}(t_n), \mathrm{curl}(\mathbf{H}_h^{n+1}+\mathbf{H}_h^{n-1})). \end{aligned}\tag{34}$$

Finally, if $\nu=0$, $\sigma=\infty$ and $\mathbf{f}=0$, we have

$$\frac{1}{2}(\|\mathbf{u}_h^N\|^2+\|\mathbf{u}_h^{N-1}\|^2)+\frac{s}{2}(\|\mathbf{H}_h^N\|^2+\|\mathbf{H}_h^{N-1}\|^2)=\|\mathbf{u}_h^0\|^2+s\|\mathbf{H}_h^0\|^2. \tag{35}$$

Besides, from the same argument as applied to obtain (27),

$$\|\mathbf{u}_h^1\|^2+s\|\mathbf{H}_h^1\|^2=\|\mathbf{u}_h^0\|^2+s\|\mathbf{H}_h^0\|^2$$

holds. Hence, the energy is preserved for the presented full-discrete scheme (28)-(31) with NL taken as NL_s, NL_r and NL_e that does not enforce the divergence constraint strongly.

4. Numerical experiment

For verifying the theoretical analysis results, a numerical experiment was presented to test the energy conservation of the full-discrete linearized Crank-Nicolson-type scheme based on the skew-symmetric, rotational and energy-conserving forms for the nonlinear term in this part.

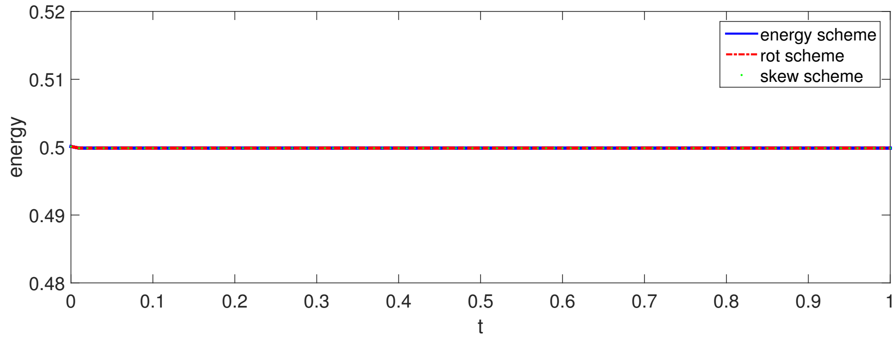

Fig. 1. The plot of time versus energy for the three formulations

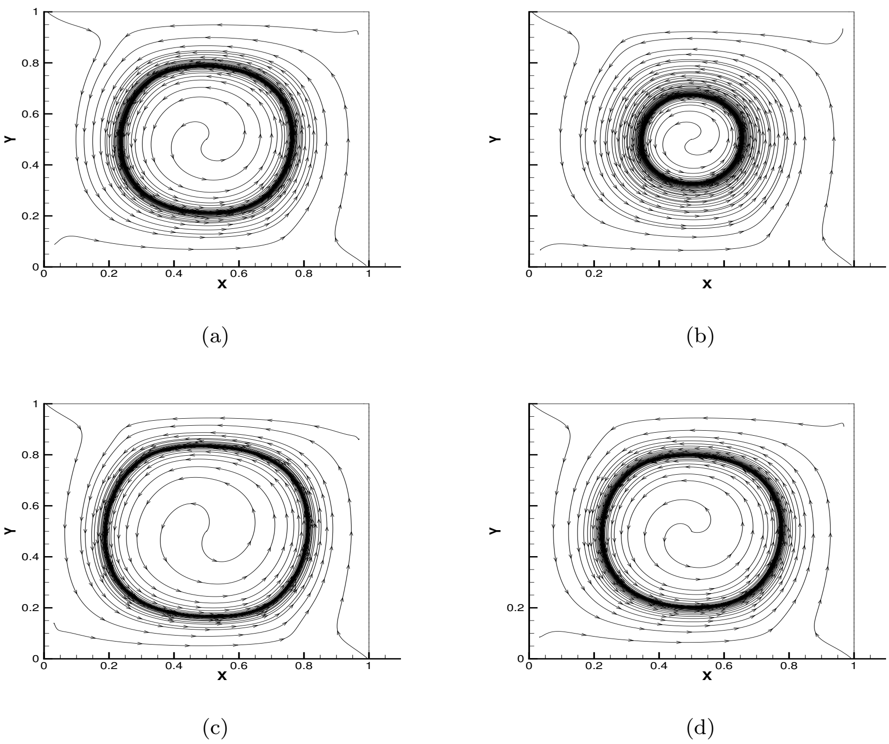

Fig. 2. Velocity streamlines: the skew-symmetric scheme (a); the energy-conserving scheme (b); the rotational scheme (c) and the scheme in [20] (d)

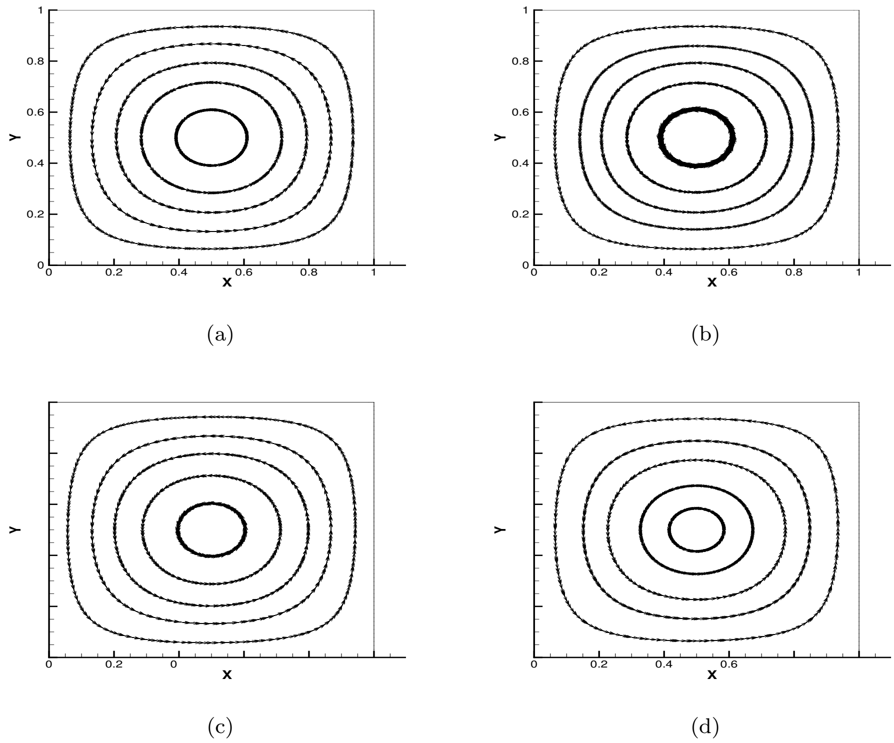

Fig. 3. Magnetic streamlines: the skew-symmetric scheme (a); the energy-conserving scheme (b); the rotational scheme (c) and the scheme in [20] (d)

For the numerical test, we consider the problem (1) on a unit square domain $[0,1]^2$ with the use of the finite element pair $(\mathbf{P}1b, P1, \mathbf{P}1b)$ [17] for the velocity field/pressure/magnetic field. In order to verify the property of energy conservation, we set $\sigma^{-1}=0, \nu=0$ and $\mathbf{f}=0$. Besides, we choose the magnetic permeability $s=1$ and the initial values are taken as [19]

$$\mathbf{u}_0(\mathbf{x}) = (x^2(x-1)^2y(y-1)(2y-1), -y^2(y-1)^2x(x-1)(2x-1)),$$
$$\mathbf{B}_0(\mathbf{x}) = (\sin(\pi x)\cos(\pi y), -\sin(\pi y)\cos(\pi x)).$$

We take $\Delta t = 0.01$, $h = \frac{1}{64}$ and apply the full-discrete scheme (28)-(31) to get the energy at the final time $T = 1$. Figure 1 presents the numerical results obtained by the skew-symmetric, rotational, and energy-conserving forms. Conformation of the energy conservation property of the schemes is shown.

In addition, in Figures 2 and 3, the profiles for the velocity streamlines and magnetic streamlines obtained by our schemes and the proposed scheme in [20] are presented. Compared our numerical solutions from Figures 2 and 3 with the solutions obtained by the proposed scheme in [20], we can find that the numerical results of these schemes almost coincide.

5. **Conclusions**

In this paper, we have presented some energy-conserving schemes for the time-dependent incompressible MHD problem based on the various nonlinear terms including the skew-symmetric, rotation and energy-conserving formulations. The absorbing point of these schemes lie in persevering energy without the divergence constraint strongly enforced in the form of variation, semi-discretization with respect to time and the full-discrete linearized Crank-Nicolson-type finite element scheme. The theoretical results are verified by the numerical test.

Acknowledgements

We would like to express our sincere gratitude to the editor and anonymous referees for their helpful suggestions on the quality improvement of our present paper. This work is supported by the Natural Science Foundation of China (grant number 11861067).

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