PERIODIC SOLUTION FOR VIBRATION OF EULER-BERNOULLI BEAMS SUBJECTED TO AXIAL LOAD USING DTM AND HA

Saeed KARIMIAN[1], Mohammadreza AZIMI[2]

This paper is concerned with analytical approximate solutions, to the nonlinear vibration of Euler-Bernoulli beams subjected to the axial loads. Hamiltonian Approach (HA) and Differential Transformation Method (DTM) are applied to solve the nonlinear differential equation cause in current problem and consequently the relationship between the natural frequency and the initial amplitude is obtained in an analytical form. To verify the accuracy of the present approach, illustrative examples are provided and compared with Exact Solution. The procedure yields rapid convergence with respect to the exact integral solution.

Keywords: Nonlinear oscillation, Euler-Bernoulli beam, Hamiltonian Approach, Differential Transformation Method.

1. Introduction

Beams have wide applications in Engineering Structures and many Structures such as long span bridges, Aircraft wings, flexible satellite can be modeled to as flexible beams [1]. Large amplitude vibration of beams usually leads to nonlinearity. The nonlinear vibrations of free classical Euler Bernoulli beams have been investigated in pervious researches both numerically and analytically [2 ,3].

Nonlinear problems such as vibration of beams can be solved numerically and analytically, but obtaining analytical solution for nonlinear systems is very important due to limitations of numerical methods [4]. With the rapid development of nonlinear science, there appears an ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems. In the recent years, many asymptotic techniques including the Energy Balance Method (EBM) [5], Adomian Decomposition Method (ADM) [6], Hamiltonian Approach (HA) [7], Differential transformation method (DTM) [8, 9], Parametrized Perturbation Method (PPM) [10], Amplitude Frequency Formulation (AFF) [11] and Variational Iteration Method (VIM) [12] have been developed to construct many types of approximate solutions of nonlinear differential equations.

The Aim of this paper was to determine the periodic solutions to nonlinear Euler Bernoulli beam subjected to axial load by applying HA and DTM . By

---

[1] Assistant Prof., Faculty of Engineering, Tariat Modares University (TMU), Iran, e-mail: saeed_karimian@yahoo.com.
[2] Msc., Faculty of Engineering, Tarbiat Modares University (TMU), Iran, e-mail: m_r_azimi1991@yahoo.com

comparing the analytical results with exact solutions, we illustrated the high accuracy of these methods.

2. Mathematical Formulation

Euler-Bernoulli beam theory assumes that plane cross sections, normal to the natural axis before deformation, continue to remain plane and continue to remain normal to the neutral axis and do not undergo any strain in their planes [3]. The nonlinear partial differential equation of the beam when the effects of mid-plane stretching are not negligible, can be written as follows:

$$EI\frac{\partial^4 \hat{w}}{\partial \hat{x}^4} + m\frac{\partial^2 \hat{w}}{\partial \hat{t}^2} - N_0\frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{EA}{2L}\int_0^L \left(\frac{\partial \hat{w}}{\partial \hat{x}}\right)^2 dx \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} = 0, \tag{1}$$

Where E is the Young's modulus of elasticity of the beam material, I is the second moment of area of the cross section with respect to the bending axis, $\hat{w}$ is the beam deflection, m is the longitudinal density, L is the length of the beam, $\hat{t}$ is the time, A is the cross sectional area of the beam and N_0 is the pretension of the beam.

For convenience, the following non-dimensional variables are used:

$$x = \frac{\hat{x}}{L}, \quad w = \frac{\hat{w}}{r}, \quad t = \sqrt{\frac{mL^4}{\beta^4 EI}}, \quad r = \sqrt{\frac{I}{A}}, \tag{2}$$

Assuming $W(x,t) = \phi(x)\psi(t)$ where $\phi(x)$ is the first eigenmode of the beam [9] and applying the Galerkin method, the equation of motion is obtained as following form:

$$\ddot{\varphi} + \omega_0^2\varphi + \alpha\varphi^3 = 0 \qquad \varphi(0) = \frac{W_{max}}{L}, \quad \dot{\varphi}(0) = 0, \tag{3}$$

In equation.3 ω_0^2 and α are defined as follows:

$$\omega_0^2 = \frac{1}{\beta^4 F_1}\left(F_3 + \frac{N_0}{EI}\left(\frac{L}{r}\right)^2 F_2\right) \qquad \alpha = \frac{1}{2\beta^4 F_1}\left(\frac{L}{r}\right)^2 F_2^2, \tag{4}$$

Where

$$F_1 = \int_0^1 \phi^2 dx, \quad F_2 = \int_0^1 \phi'^2 dx, \quad F_3 = \int_0^1 \phi'' dx, \tag{5}$$

The nonlinear ordinary differential equation in equation.3 is the governing nonlinear vibration of Euler- Bernoulli beams.

3. Solution Procedure

In this section the solution procedure of nonlinear vibration of Euler Bernoulli beam subjected to axial load by using DTM and HA will be presented.

3.1. Basic Idea of DTM

Let $x(t)$ be analytic in a domain D and let $t = t_i$ represents any point in D. The function $x(t)$ is then represented by one power series whose center is located at t_i. The Taylor series expansion function of $x(t)$ is in the form of:

$$x(t) = \sum_{k=0}^{\infty} \frac{(t - t_i)}{k!} \left[\frac{d^k x(t)}{dt^k} \right]_{t=t_i} \quad \forall t \in D , \tag{6}$$

The differential transformation of the function $x(t)$ can be defined as following form [8]:

$$X(k) = \sum_{k=0}^{\infty} \frac{H^k}{k!} \left[\frac{d^k x(t)}{dt^k} \right]_{t=0} , \tag{7}$$

Where $x(t)$ is the original function and $X(k)$ is the transformed function. The differential inverse transform of $X(k)$ is defined as follows:

$$x(t) = \sum_{k=0}^{\infty} \left(\frac{t}{H!} \right)^k X(k), \tag{8}$$

It is clear that the concept of the differential transformation is based upon the Taylor series expansion. The values of function $X(k)$ at values of argument k are referred to as discrete, i.e. $X(0)$ is known as the zero discrete, $X(1)$ as the first discrete, etc. the function is expressed by a finite series and equation.8 can be written as:

$$x(t) = \sum_{k=0}^{n} \left(\frac{t}{H!} \right)^k X(k), \tag{9}$$

Mathematical operations performed by DTM are listed in Table.1.

Table 1

Some of the basic operations of Differential Transformation Method

Original Function	Transformed Function
$x(t) = \alpha f(t) \pm \beta g(t)$	$X(k) = \alpha F(k) \pm \beta G(k)$
$x(t) = f(t) g(t)$	$X(k) = \sum_{l=0}^{k} F(l) G(k - l)$
$x(t) = \dfrac{df(t)}{dt}$	$X(k) = (k + 1) F(k + 1)$
$x(t) = \dfrac{d^2 f(t)}{dt^2}$	$X(k) = (k + 1)(k + 2) F(k + 2)$

3.2. Basic Idea of HA

Consider a simple second order conservative oscillator with odd-nonlinearity in the form:

$$\ddot{u} = f(u), \quad u(0) = A, \quad \dot{u}(0) = 0, \tag{10}$$

where a dot denotes differentiation with respect to t and $f(-u) = -f(u)$. The Hamiltonian of the nonlinear oscillation equation can be written as following form:

$$H = \frac{\dot{u}^2}{2} + F(u) = cons., \tag{11}$$

In equation.11 $\frac{\dot{u}^2}{2}$ and $F(u)$ denote kinetic and potential energy, respectively. Therefore, we can write:

$$\frac{\partial H}{\partial A} = 0, \tag{12}$$

We can introduce $\tilde{H}(u)$ as follows:

$$\tilde{H}(u) = \int_0^{\frac{T}{4}} \left(\frac{\dot{u}^2}{2} + F(u) \right) dt = \frac{TH}{4}, \tag{13}$$

So,

$$\frac{\partial}{\partial A} \left(\frac{\partial \tilde{H}}{\partial T} \right) = 0, \tag{14}$$

Or,

$$\frac{\partial}{\partial A} \left(\frac{\partial \tilde{H}}{\partial (1/\omega)} \right) = 0, \tag{15}$$

Consequently, the approximate frequency can be found from equation.15.

3. 3. Application of DTM

Taking the differential transform of equation.3

$$(k+1)(k+2)\Phi(k+2) + \omega_0^2 \Phi(k)$$
$$+ \alpha \sum_{m=0}^{k} \Phi(k-m) \sum_{l=0}^{m} \Phi(l)\Phi(m-l) = 0 \tag{16}$$

From boundary conditions in equation.3 , that we have it in point $t = 0$ and exerting transformation:

$$\Phi(0) = A \qquad \Phi(1) = 0, \tag{17}$$

We will have:

$$\Phi(2) = -\frac{\omega_0^2}{2} - \frac{\alpha}{2} \qquad \Phi(3) = 0$$

$$\Phi(4) = -\frac{\omega_0^4}{24} + \frac{\alpha\omega_0^2}{6} + \frac{\alpha^2}{8} \qquad \Phi(5) = 0$$

$$\Phi(6) = -\frac{\omega_0^6}{720} - \frac{5\alpha\omega_0^4}{144} - \frac{17\alpha^2\omega_0^2}{240} - \frac{3\alpha^3}{80} \qquad \Phi(7) = 0$$

$$\Phi(8) = \frac{\omega_0^8}{40320} + \frac{13\alpha\omega_0^6}{2520} + \frac{47\alpha^2\omega_0^4}{2240} + \frac{3\alpha^3\omega_0^2}{112} + \frac{7\alpha^4}{640}$$

(18)

The above process is continuous. Substituting equation.18 into the main equation based on DTM, it can be obtained that the closed form of the solutions is:

$$\Phi(t) = \Phi(0) \times t^0 + \Phi(1) \times t^1 + \Phi(2) \times t^2 + \cdots,$$

(19)

The approximate analytical solution can be easily yield.

3. 4. Application of HA

Considering the following equation:

$$\ddot{\phi} + \omega_0^2\phi + \alpha\phi^3 = 0$$

$$\phi(0) = A, \quad \dot{\phi}(0) = 0$$

(20)

Its Hamiltonian can be easily obtained:

$$H = -\frac{\dot{u}^2}{2} + \frac{\omega_0^2}{2}u^2 + \frac{\alpha}{4}u^4$$

(21)

Inserting equation.21 with respect to t from 0 to $T/4$, we can yield:

$$\tilde{H} = \int_0^{\frac{T}{4}} \left(-\frac{\dot{u}^2}{2} + \frac{\omega_0^2}{2}u^2 + \frac{\alpha}{4}u^4 \right) dt$$

(22)

Considering the initial conditions, we assume that the solution can be expressed as:

$$\phi(t) = A\cos(\omega t)$$

(22)

Substituting it into equation.3, we have:

$$\tilde{H} = \int_0^{\frac{T}{4}} \left(\frac{1}{2}A^2\sin(\omega t)^2\omega^2 + \frac{\omega_0^2}{2}A^2\cos(\omega t)^2 + \frac{\alpha}{4}A^4\cos(\omega t)^4 \right)$$

(23)

Therefore,

$$\tilde{H} = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} A^2 \sin(\varphi)^2 \omega + \frac{\omega_0^2}{2\omega} A^2 \cos(\varphi)^2 + \frac{\alpha}{4\omega} A^4 \cos(\varphi)^4 \right) d\varphi$$

$$= \frac{1}{64} \frac{\pi \left(8\omega_0^2 A^2 + 3A^2 \alpha + 8A^2 \omega^2 \right)}{\omega} \tag{24}$$

According to the Hamiltonian approach, we set:

$$\frac{\partial}{\partial A} \left(\frac{\partial \tilde{H}}{\partial \left(1/\omega \right)} \right) = -\frac{1}{4} A\pi\omega_0^4 + \frac{1}{4} A\pi\omega_0^2 + \frac{3}{16} A^3 \pi \alpha \tag{25}$$

So, we can easily obtain the following approximate frequency-amplitude relationship:

$$\omega_{HA} = \frac{1}{2} \sqrt{3\alpha A^2 + 4\omega_0^2} \tag{26}$$

The approximate analytical solution using Hamiltonian Approach is:

$$\phi_{HA} = A \cos\left(\left(\frac{1}{2} \sqrt{3\alpha A^2 + 4\omega_0^2} \right) t \right) \tag{27}$$

5. Results and Discussions

We compared the results of the present analysis based on DTM with an exact integral solution and absolute error, as demonstrated in Table.2 for case of

$\alpha = 0.5$, $\omega_0^2 = 0.5$ and $A = 1$.

A very interesting agreement between the results is observed too, which confirms the excellent validity of the Differential Transformation Method (DTM).

Table 2

Comparison of DTM results and Exact Solution when $\alpha = 0.5$, $\omega_0^2 = 0.5$, $A = 1$

t	Exact	DTM	Absolute Error
0	1.000000	1.000000	0.000000
0.25	0.969071	0.972781	0.003709
0.5	0.879941	0.892604	0.012663
0.75	0.742260	0.763835	0.02157
1	0.568569	0.593484	0.024915
1.25	0.371179	0.390821	0.019646
1.5	0.160398	0.166891	0.006491
1.75	-0.055622	-0.066131	0.010507
2	-0.269864	-0.295551	0.025686

To illustrate and verify the accuracy of Hamiltonian Approach (HA), the comparison between the approximate period (T_{HA}) with the exact one [13] (T_{Exact}) are tabulated in Table.3 for various values of α and ω_0^2 when $A = 1$.

Table 3

Comparison between HA results and exact solution for various α, ω_0^2

α	ω_0^2	T_{exact}	T_{HA}
	0.5	7.9534	7.9476
0.25	1	6.3006	6.2831
	2	4.7381	4.7497
	0.5	6.7292	6.7170
0.5	1	5.6213	5.6198
	2	4.4498	4.4429
	0.5	5.9178	5.9238
0.75	1	5.1256	5.1301
	2	4.1903	4.1887
	0.5	5.3505	5.3583
1	1	4.7612	4.7496
	2	3.9811	3.9738

In figure.1 the approximate dimensionless deflection of Euler Bernoulli beam for each dimensionless time have been compared to exact solution for case of $\alpha = 0.15$, $\omega_0^2 = 0.5$ and $A = 3$. as it can be illustrated in fig. 1 the exact solutions have a good agreement with approximate analytical results.
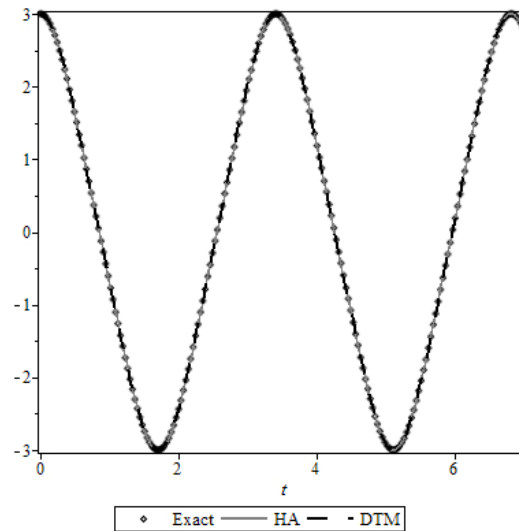

Fig. 1. Analytical solutions and exact ones for case of $\alpha = 0.15$, $\omega_0^2 = 0.5$, $A = 3$

6. Conclusions

In this paper, we have used analytical methods called Differential Transformation Method (DTM) and Hamiltonian Approach (HA) to determine the approximate solution of Euler Bernoulli beam nonlinear vibration which is subjected to the axial force.

The comparison between analytical approximate results and exact integral solutions assures us about the convenience and accuracy of the solution procedure. Some patterns are also given to illustrate the effectiveness and convenience of the methodologies.

R E F E R E N C E S

[1]. *S. P. Timoshenko, D. H. Young, W. Weaver,* Vibration Problems in Engineering, , John Willey & Sons, 1974.

[2]. *T. Pirbodaghi, M.T. Ahmadian, M. Fesanghary*, "On the Homotopy Analysis Method for Nonlinear Vibration of Beams",Mechanics Research Communications, **vol. 36**, no. 2, 2009, pp. 143-148.

[3]. *B. Ozturk,* " Free Vibration Analysis of Beam on Elastic Foundation by Variational Iteration Method", International Journl of Nonlinear SCience and Numerical Simulation, **vol.10,** no.10, 2009, pp.1255-1262.

[4]. *G. Adomian,* "A Review of the Decomposition Mthod and Some Recent Results for Nonlinear Equation ", Mathematicla Computation and Modelling, **vol.13**, no.7, 1992, pp. 17-43.

[5]. *D.D. Ganji, M. Azimi, M. Mostofi,* "Energy Balance Method and Amplitude Frequency Formulation Based of Strongly Nonlinear Oscillators, Indian Journal of Pure & Applied Physics, **vol.50**, no.11, 2012, pp. 670-675.

[6]. *J-C. Hsu, H-Y. Lai, C-K. Chen,* "An Innovative Eigenvalue Problem Solve For Fre Vibration of Timoshenko Beams By Using the Adomian Decomposition Method", Journal of Sound and Vibration, **vol.325**, 2009, pp.451-470.

[7]. *Lan Xu,* "A Hamiltonian Approach for a Plasma Physics Problem", Advances in Nolinear Dynamics, **vol.**61, no.8, 2011, pp.1909-1911

[8]. *D. D. Ganji, M. Azimi,* "Application of DTM on MHD Jeffery Hamel Flow with Nanoparticle", U.P.B. Scientific Bulletin, **vol.75**, no.1, 2013, pp. 223-230.

[9]. *S. Chen, Ch. Kuang Chen,* "Application of Differential Transformation Method to the Free Vibrations of Strongly Nonlinear Oscillators", **vol.10**, no.2, 2009, pp.881-888.

[10]. *A. Barari, H.D. Khaliji, M. Ghadimi, G. Domairry,* "Nonlinear Vibration of Euler Bernoulli Beams" , Latin American Journal of Solids and Structures, **vol.8**, 2011, pp.139-148.

[11]. *D.D. Ganji, M. Azimi,* "Application of Max Min Approach and Amplitude Frequency Formulation to Nonlinear Oscillation Systems", U.P.B. Scientific Bulletin, **vol.74**, no.3, 2012, pp. 131-140.

[12]. *Y. Liu, C.S. Gurram,* "The Use of He's Variational Iteration Method for Obtaining the Free Vibration of an Euler Bernoulli Beam" , Mathematical and Computer Modelling, **vol.50**, 2009, pp.1545-1552.

[13]. *V. Marinca, N. Herisanu,* "A modified iteration perturbation method for some nonlinear oscillation problems", Acta Mechanica, **vol.184**, 2006, pp.231-242.