CONTINUOUS SOLUTIONS OF PARAMETRIC COMPLETELY GENERALIZED MIXED IMPLICIT QUASI-VARIATIONAL INCLUSIONS

Yangqing Qiu*[1], Nan Li[2], Liwei Liu[3]

In this paper, by using the selection technique of set-valued mappings, under suitable conditions, we prove that the solution set of the parametric completely generalized mixed implicit quasi-variational inclusion problem (PCG-MIQVIP) involving set-valued mappings is nonempty closed convex, and the PCG-MIQVIP has continuous solutions with respect to the parameters. Our results are supplement and complement for many known results in this field.

Keywords: Parametric completely generalized mixed implicit quasi-variational inclusion, Continuous solutions Continuous selections of set-valued mappings, Sensitivity analysis, Resolvent operator.

MSC2020: 47H 05. 47J 20. 47J 25. 49J 40. 49J 53.

1. Introduction

In recent years, by utilizing novel and innovative techniques, variational inequalities have been extended in different directions. Many scholars have done a lot of researches on various classes of parametric variational inclusions involving set-valued mappings, for example [1, 2, 3, 4, 5, 6]. Their research results focus on the existence of solutions, the closeness of the solution set, the continuity and Lipschitz continuity of the solution set with respect to the parameter. Very recently, the sensitivity analysis of the solution mappings of variational inequalities was performed in [7, 8]. Several quantitative semicontinuity properties of the solution mapping to parameterized set-valued inclusions are studied in [9]. And continuous solutions of perturbed evolution inclusions [10], iterative algorithms of variational inclusions [11, 12, 13] are established. However, their research results do not mention the convexity of the solution set and the existence of (Lipschitz) continuous solutions. Under what circumstances do the parametric set-valued variational inclusion problems have (Lipschitz) continuous solutions? It is still an open question.

Throughout the paper, unless otherwise stated, let H be a real Hilbert space, Ω be a nonempty open subset of H in which the parameter λ takes values, $C(H)$ be the family of all nonempty compact subsets of H. Let $N : H \times H \times H \times \Omega \to H$, $W : H \times H \times \Omega \to H$, $m, i, j : H \times \Omega \to H$ and $h : H \to H$ be single-valued

[1] School of Mathematics Physics and Statistics, Shanghai Polytechnic University, 201209 Shanghai, China, e-mail: yqqiu@sspu.edu.cn

[2] School of Mathematics Physics and Statistics, Shanghai Polytechnic University, 201209 Shanghai, China

[3] Department of Mathematics, Nanchang University, 330000 Nanchang, China

mappings. Let A, B, C, D, E, F and $G : H \times \Omega \to C(H)$ be set-valued mappings. Let $M : H \times H \times \Omega \to 2^H$ be a set-valued mapping such that for each given $(f, \lambda) \in H \times \Omega$, $M(\cdot, f, \lambda) : H \to 2^H$ is a h-maximal monotone mapping with $(G(H, \lambda) - m(H, \lambda)) \cap \text{dom } M(\cdot, f, \lambda) \neq \emptyset$. The following parametric completely generalized mixed implicit quasi-variational inclusion problem (PCGMIQVIP) is introduced by Ding [1]:

for each $(\lambda, w) \in \Omega \times H$, find $x = x(\lambda) \in H$, $a = a(x, \lambda) \in A(i(x, \lambda), \lambda)$, $b = b(x, \lambda) \in B(x, \lambda)$, $c = c(x, \lambda) \in C(x, \lambda)$, $d = d(x, \lambda) \in D(x, \lambda)$, $e = e(x, \lambda) \in E(x, \lambda)$, $f = f(x, \lambda) \in F(x, \lambda)$, $g = g(x, \lambda) \in G(x, \lambda)$ such that

$$w \in W(j(e, \lambda), a, \lambda) - N(b, c, d, \lambda) + M(g - m(x, \lambda), f, \lambda). \tag{1}$$

For a suitable choice of the mappings, Ding [1] points out that the PCGMIQVIP (1) includes a number of (parametric) generalized quasi-variational inclusions studied by many authors as special cases [14]. In Ding [1], by applying resolvent operator technique of h-maximal monotone mapping and the property of fixed points of set-valued contractive mappings, the solution set of the PCGMIQVIP (1) is proved to be nonempty and closed, the continuity and Lipschitz continuity of the solution set with respect to the parameter are established. And the following proposition holds.

Theorem 1.1. *([1], Theorem 3.1 and Theorem 3.3) The following statements are equivalent:*

(i) *For each $(\lambda, w) \in \Omega \times H$, (x, a, b, c, d, e, f, g) $(x \in H)$ is a solution of the PCGMIQVIP (1);*

(ii) *(x, a, b, c, d, e, f, g) $(x \in H)$ satisfies*

$$g = m(x, \lambda) + R^h_{M(\cdot, f, \lambda), \rho}[h(g - m(x, \lambda)) - \rho W(j(e, \lambda), a, \lambda) + \rho N(b, c, d, \lambda) + \rho w],$$

where $\rho > 0$ is a constant, $R^h_{M(\cdot, f, \lambda), \rho} = (h + \rho M)^{-1} : H \to H$ is the resolvent operator of the h-maximal monotone mapping M;

(iii) *The set-valued mapping $Q : H \times \Omega \to 2^H$ defined by*

$$Q(x, \lambda) = \bigcup_{a \in A(i(x,\lambda),\lambda), b \in B(x,\lambda), c \in C(x,\lambda), d \in D(x,\lambda), e \in E(x,\lambda), f \in F(x,\lambda), g \in G(x,\lambda)} [x - (g - m(x, \lambda)) + R^h_{M(\cdot, f, \lambda), \rho}(h(g - m(x, \lambda)) - \rho W(j(e, \lambda), a, \lambda) + \rho N(b, c, d, \lambda) + \rho w)] \tag{2}$$

has a fixed point $x = x(\lambda)$.

By Theorem 1.1, the solution set $S(\lambda)$ of the PCGMIQVIP (1) can be denoted as

$$S(\lambda) = \{u = u(\lambda) \in H | u \in Q(u, \lambda)\}.$$

The main aim of this paper is continued to study the behavior and sensitivity analysis of the solution set $S(\lambda)$ of the PCGMIQVIP (1). We concern the following aspects:

(1) We further discuss the convexity of the solution set $S(\lambda)$.

(2)We further examine the existence of (Lipschitz) continuous solutions of the PCGMIQVIP (1), including the minimal selection, the Chebyshev selection and the barycentric selection.

2. Preliminaries

In this section, we recall some concepts. And for brief, we will no longer describe the definitions and lemmas involved in [1].

Let K be a subset of a metric space X, the closed ball and the open ball of the radius $r > 0$ around K in X are denoted by

$$B(K, r) = \{x \in X | d(x, K) \leq r\}$$

and

$$C(K, r) = \{x \in X | d(x, K) < r\},$$

respectively.

Definition 2.1. [15, 16] *Let X and Y be metric spaces, $F : X \to 2^Y$ be a set-valued mapping. Then F is called to be*

(1) (a) *upper semi-continuous at $x_0 \in dom(F)$, if for any neighbourhood V of $F(x_0)$, there exists a neighbourhood U of x_0, such that for any $x \in U$, $F(x) \subset V$.*

(b) *lower semi-continuous at $x_0 \in dom(F)$, if for any $y \in F(x_0)$ and for any $\{x_n\} \subset dom(F)$, $x_n \to x_0$, there exists $y_n \in F(x_n)$, $y_n \to y$ as $n \to \infty$.*

(c) *continuous at $x_0 \in dom(F)$, if it is both upper semi-continuous and lower semi-continuous at x_0.*

(d) *continuous on $dom(F)$, if it is continuous at every point $x \in dom(F)$.*

(2) *Hausdorff lower semi-continuous at $x_0 \in dom(F)$, if for any $\varepsilon > 0$, there exists $\rho > 0$, such that*

$$F(x_0) \subset B(F(x), \varepsilon), \qquad \forall x \in B(x_0, \rho).$$

(3) *L-Lipschitz continuous, if there exists a constant $L > 0$ such that for any $x, y \in dom(F)$,*

$$F(x) \subset B(F(y), L\|x - y\|)$$

or

$$F(x) \subset C(F(y), L\|x - y\|).$$

(4) *L-H-Lipschitz continuous, if there exists a constant $L > 0$ such that*

$$H(Fx, Fy) \leq L\|x - y\|, \quad \forall x, y \in dom(F).$$

(5) *convex, if for any $x, y \in dom(F)$, $\forall t \in [0, 1]$, there exists*

$$tF(x) + (1 - t)F(y) \subset F(tx + (1 - t)y).$$

Remark 2.1. (i) *Similarly, according to Definition 2.1 (5), $F : X_1 \times X_2 \to 2^{X_3}$ is convex, if for any $x_i, y_i \in X_i, i = 1, 2$, $\forall t \in [0, 1]$, there exists*

$$tF(x_1, x_2) + (1 - t)F(y_1, y_2) \subset F(tx_1 + (1 - t)y_1, tx_2 + (1 - t)y_2).$$

(ii) *If F is Hausdorff lower semi-continuous, then F is lower semi-continuous. (see* [17], *P_{17})*

Definition 2.2. *Let X and Y be two Banach spaces, $F : X \to 2^Y$ be a set-valued mapping with nonempty images.*

(1) *(*[15], *P_{355}, Definition 9.1.1) A single-valued mapping $f : X \to Y$ is called a selection of F, if for every $x \in X$, $f(x) \in F(x)$. If f is also continuous, then f is called a continuous selection of F.*

(2)*([15],P_{360}) If Y is a Hilbert space, or more generally, a reflexive strictly convex space, and F has closed convex images, then the single-valued mapping $m : X \to Y$ defined by*

$$m(x) \doteq m(F(x)) = \{u \in F(x) | \|u\| = min_{y \in F(x)} \|y\|\}$$

is called the minimal selection of F.

(3) *([16], P_{74}, Definition 1) Let Y be a Hilbert space. Assume that for any $x \in dom(F)$, $F(x)$ has bounded values, r_c and $c(F(x))$ are the Chebyshev radius and Chebyshev center of $F(x)$, respectively. Then the single-valued mapping $c : X \to Y$ defined by*

$$c(x) \doteq c(F(x)) = \bigcap_{\rho > r_c} \{k | F(x) \subset B(k, \rho), k \in Y\},$$

is called the Chebyshev selection of F.

(4) *([16], P_{77}) Let $Y = \mathbb{R}^n$, $F(x)$ be a compact convex body, i.e., a compact convex set with nonempty interior, $m_n(B_1(F(x)))$ be the n-dimensional Lebesgue measure of $B_1(F(x))$. The single-valued mapping $b : X \to \mathbb{R}^n$ defined by*

$$b(x) \doteq b(B_1(F(x))) = \frac{1}{m_n(B_1(F(x)))} \int_{B_1(F(x))} x dm_n, \ \forall x \in dom(F),$$

is called the barycenter selection of F.

Remark 2.2. *If the solution set $S(\lambda)$ of the PCGMIQVIP (1) is nonempty, $u(\lambda)$ is a selection of $S(\lambda)$, then obviously, $u(\lambda)$ is a solution of the PCGMIQVIP (1).*

Lemma 2.1. *([16], P_{70}, Theorem 1) Let X be a metric space and Y be a Hilbert space, $F : X \to 2^Y$ be a continuous mapping with closed convex values. Then the mapping $x \to m(F(x))$ is continuous, where $m(F(x))$ is the minimal selection of F .*

Lemma 2.2. *([17], P_{47}, Theorem 2.3.7) Let X be a normed space and Y be a Hilbert space, $F : X \to 2^Y$ be a mapping with bounded closed convex values. If F is upper semi-continuous and Hausdorff lower semi-continuous, then the single-valued mapping $f(x) = c(F(x))$ is continuous, where $c(F(x))$ is the Chebyshev selection of $F(x)$.*

Lemma 2.3. *([16], P_{77}, Theorem 1) Let X be a metric space, $Y = \mathbb{R}^n$, $F : X \to 2^Y$ be a Lipschitz continuous mapping with compact convex values. Assume moreover that for some $R > 0$, $F(x) \subset B(0, R)$, for every x in X. Then the single-valued mapping $b(x) = b(B_1(F(x)))$ is Lipschitz continuous, where $b(B_1(F(x)))$ is the barycenter selection of $F(x)$.*

Lemma 2.4. *Let $F : X \to 2^Y$ be a L-H-Lipschitz continuous set-valued mapping. Then F is upper semi-continuous and Hausdorff lower semi-continuous.*

Proof. Since $F : X \to 2^Y$ is L-H-Lipschitz continuous, $\forall x_1, x_2 \in \mathrm{dom}(F)$, we have

$$H(F(x_1), F(x_2)) = \max\{\sup_{u \in F(x_1)} d(u, F(x_2)), \sup_{v \in F(x_2)} d(F(x_1), v)\} \leq L\|x_1 - x_2\|,$$

where $d(u, K) = \inf_{v \in K} \|u - v\|$. Then

$$F(x_2) \subset B(F(x_1), L\|x_1 - x_2\|).$$

On the one hand, for $\forall\varepsilon > 0$, let $V = C(F(x_1), \varepsilon)$, a neighbourhood of $F(x_1)$, we take $\delta = \frac{\varepsilon}{L}$ and $U = \{x | \|x - x_1\| < \delta, x \in \mathrm{dom}(F)\}$, then for any $x \in U$, we have

$$H(F(x_1), F(x)) \leq L\|x_1 - x\| < L \cdot \frac{\varepsilon}{L} = \varepsilon. \tag{3}$$

This implies that

$$F(x) \subset C(F(x_1), \varepsilon).$$

That is, $F(x) \subset V$. By Definition 2.1 (1), we conclude $F(x)$ is upper semi-continuous at x_1. Because of the arbitrariness of $x_1 \in \mathrm{dom}(F)$, $x \mapsto F(x)$ is upper semi-continuous on $\mathrm{dom}(F)$.

On the other hand, by (3), we can obtain

$$F(x_1) \subset C(F(x), \varepsilon).$$

That is,

$$F(x_1) \subset \bigcap_{x \in U} \{C(F(x), \varepsilon)\}.$$

By Definition 2.1 (2), $F(x)$ is Hausdorff lower semi-continuous at $x = x_1$. By the arbitrariness of $x_1 \in \mathrm{dom}(F)$, $x \mapsto F(x)$ is Hausdorff lower semi-continuous on $\mathrm{dom}(F)$. □

3. **Main results**

In this section, we study the convexity of the solution set $S(\lambda)$ and the existences of continuous solutions for the PCGMIQVIP (1).

We list the following hypotheses for convenience.

(H_1) $N : H \times H \times H \times \Omega \to H$, $W : H \times H \times \Omega \to H$, $m, i, j : H \times \Omega \to H$ and $h : H \to H$ are linear single-valued mappings.

(H_2) $A, B, C, D, E, F, G : H \times \Omega \to C(H)$ and $M : H \times H \times \Omega \to 2^H$ are convex set-valued mappings.

(H_3) A, B, C, D, E, F and G are Lipschitz continuous in the first argument with constants L_A, L_B, L_C, L_D, L_E, L_F and L_G, respectively. Moreover G is δ-strongly monotone in the first argument.

(H_4) m, i and j are Lipschitz continuous in the first argument with constants L_m, L_i and L_j, respectively.

(H_5) h is r-strongly monotone and L_h-Lipschitz continuous.

(H_6) N is γ-relaxed Lipschitz continuous in the first argument with respect to B and σ-pseudo-contractive in the second argument with respect to C. N is Lipschitz continuous in the first, second and third arguments with constants L_{N1}, L_{N2} and L_{N3}, respectively.

(H_7) W is Lipschitz continuous in the first and second arguments with constants L_{W1} and L_{W2}, respectively.

(H_8) M is a mapping such that for each given $(f, \lambda) \in H \times \Omega$, $M(\cdot, f, \lambda) : H \to 2^H$ is a h-maximal monotone mapping satisfying $(G(H, \lambda) - m(H, \lambda)) \cap \mathrm{dom}\, M(\cdot, f, \lambda) \neq \emptyset$.

(H_9) There exists a constant $\mu > 0$ such that

$$\|R^h_{M(\cdot,x,\lambda),\rho}(z) - R^h_{M(\cdot,y,\lambda),\rho}(z)\| \leq \mu\|x - y\|, \quad \forall (x, y, z, \lambda) \in H \times H \times H \times \Omega.$$

(H_{10}) There exists a constant $\rho > 0$ such that

$$\left|\rho - \frac{\gamma - \sigma - rq(1-k)}{p^2 - q^2}\right| < \frac{\sqrt{[\gamma - \sigma - rq(1-k)]^2 - (p^2 - q^2)(1 - r^2(1-k)^2)}}{p^2 - q^2},$$

or

$$\begin{aligned} t(\rho) =& \frac{1}{r}[\sqrt{1 - 2\rho(\gamma - \sigma) + \rho^2(L_{N1}L_B + L_{N2}L_C)^2} + \rho(L_{N3}L_D + L_{W1}L_jL_E \\ &+ L_{W2}L_iL_A)], \\ \theta_1 =& k + t(\rho) < 1, \end{aligned}$$

where

$$\begin{aligned} &k = \left(1 + \frac{1}{r}\right)\left(\sqrt{1 - 2\delta + L_G^2} + L_m\right) + \frac{L_G + L_M}{r}\sqrt{1 - 2r + L_h^2} + \mu L_F < 1, \\ &p = L_{N1}L_B + L_{N2}L_C > L_{N3}L_D + L_{W1}L_jL_E + L_{W2}L_iL_A = q, \\ &\gamma > \sigma + rq(1-k) + \sqrt{(p^2 - q^2)(1 - r^2(1-k)^2)}. \end{aligned}$$

(H_{11}) For any $x \in H$, $\lambda \mapsto A(x,\lambda)$, $\lambda \mapsto B(x,\lambda)$, $\lambda \mapsto C(x,\lambda)$, $\lambda \mapsto D(x,\lambda)$, $\lambda \mapsto E(x,\lambda)$, $\lambda \mapsto F(x,\lambda)$, $\lambda \mapsto G(x,\lambda)$, $\lambda \mapsto h(x,\lambda)$, $\lambda \mapsto m(x,\lambda)$, $\lambda \mapsto i(x,\lambda)$ and $\lambda \mapsto j(x,\lambda)$, are Lipschitz continuous (or continuous) with constants $l_A, l_B, l_C, l_D, l_E, l_F, l_G, l_h, l_m, l_i$ and l_j, respectively. For any $a, b, c, d, f, z \in H$, $\lambda \mapsto N(b,c,d,\lambda)$, $\lambda \mapsto W(a,c,\lambda)$ and $\lambda \mapsto R^h_{M(\cdot,f,\lambda),\rho}(z)$ are Lipschitz continuous (or continuous) with constants l_N, l_W and l_R, respectively. And suppose

$$\begin{aligned} \theta_2 = &\left(1 + \frac{L_h}{r}\right)(l_G + l_m) + \frac{\rho}{r}[L_{W1}(L_jl_E + l_j) + L_{W2}(L_AL_il_i + L_Al_i + l_A) \\ &+ l_W + L_{N1}l_B + L_{N2}l_c + L_{N3}l_D + l_N] + \mu l_F + l_R. \end{aligned}$$

(H_{12}) For any $u \in$ dom $Q(\cdot,\lambda)$, there exists a constant $R > 0$ such that $Q(u,\lambda) \subset B(0,R)$.

Theorem 3.1. *Assume that $A, B, C, D, E, F, G, M, N, m, i, j, h$ and W are the same as which in the PCGMIQVIP (1).*

If $S(\lambda) \neq \emptyset$, hypotheses ($H_1$) *and* ($H_2$) *hold, then for each $\lambda \in \Omega$, the solution set $S(\lambda)$ of the PCGMIQVIP (1) is a convex set.*

Proof. First, we assume that (H_1) holds and for $i = 1, 2$, $x_i, u_i \in H$, $f(x_i) \in F(x_i,\lambda)$, $g(x_i) \in G(x_i,\lambda)$ satisfy

$$x_i = x_i - g(x_i) + m(x_i,\lambda) + R^h_{M(\cdot,f(x_i),\lambda),\rho}(u_i).$$

That is,

$$g(x_i) - m(x_i,\lambda) = R^h_{M(\cdot,f(x_i),\lambda),\rho}(u_i).$$

Now for any $t \in [0,1]$, we prove that

$$\begin{aligned} &t(g(x_1) - m(x_1,\lambda)) + (1-t)(g(x_2) - m(x_2,\lambda)) \\ &\quad = R^h_{M(\cdot,tf(x_1)+(1-t)f(x_2),\lambda),\rho}(tu_1 + (1-t)u_2). \end{aligned} \tag{4}$$

As a matter of fact, let

$$g(x) - m(x,\lambda) = R^h_{M(\cdot,f(x),\lambda),\rho}(u), \quad \forall (x, u \in H, g(x) \in G(x,\lambda), f(x) \in F(x,\lambda)).$$

Then by the definition of the resolvent operator, we have

$$u \in h(g(x) - m(x, \lambda)) + \rho M(g(x) - m(x, \lambda), f(x), \lambda).$$

Therefore, by virtue of (H_1) and (H_2), we obtain

$$\begin{aligned} tu_1 + (1-t)u_2 \in\ & h(t(g(x_1) - m(x_1, \lambda)) + (1-t)(g(x_2) - m(x_2, \lambda))) \\ & + t\rho M(g(x_1) - m(x_1, \lambda), f(x_1), \lambda) \\ & + (1-t)\rho M(g(x_2) - m(x_2, \lambda), f(x_2), \lambda) \\ \subset\ & h(tg(x_1) + (1-t)g(x_2) - m(tx_1 + (1-t)x_2, \lambda)) \\ & + \rho M(tg(x_1) + (1-t)g(x_2) - m(tx_1 + (1-t)x_2, \lambda), \\ & \quad tf(x_1) + (1-t)f(x_2), \lambda) \\ =\ & h(\cdot) + \rho M(\cdot, tf(x_1) + (1-t)f(x_2), \lambda)(tg(x_1) + (1-t)g(x_2) \\ & - m(tx_1 + (1-t)x_2, \lambda)). \end{aligned}$$

Again, by the definition of the resolvent operator, we have

$$tg(x_1) + (1-t)g(x_2) - m(tx_1 + (1-t)x_2, \lambda) = R^h_{M(\cdot, tf(x_1)+(1-t)f(x_2), \lambda), \rho}(tu_1 + (1-t)u_2).$$

That is, (4) holds.

Secondly, let $x_1, x_2 \in S(\lambda)$, $t \in [0,1]$, we shall establish $tx_1 + (1-t)x_2 \in S(\lambda)$. For $i = 1, 2$, $x_i \in Q(x_i, \lambda)$, by (2), there exist $a_i \in A(i(x_i, \lambda), \lambda)$, $b_i \in B(x_i, \lambda)$, $c_i \in C(x_i, \lambda)$, $d_i \in D(x_i, \lambda)$, $e_i \in E(x_i, \lambda)$, $f_i \in F(x_i, \lambda)$ and $g_i \in G(x_i, \lambda)$ such that

$$\begin{aligned} x_i =\ & x_i - (g_i - m(x_i, \lambda)) + R^h_{M(\cdot, f(x_i), \lambda), \rho}[h(g_i - m(x_i, \lambda)) \\ & - \rho W(j(e_i, \lambda), a_i, \lambda) + \rho N(b_i, c_i, d_i, \lambda) + \rho\omega]. \end{aligned}$$

By (H_1) and (H_2), for any $t \in [0,1]$, find $\lambda \in \Omega$ such that

$$ta_1 + (1-t)a_2 \in A(i(tx_1 + (1-t)x_2, \lambda), \lambda), \tag{5}$$

$$tb_1 + (1-t)b_2 \in B(tx_1 + (1-t)x_2, \lambda), \tag{6}$$

$$tc_1 + (1-t)c_2 \in C(tx_1 + (1-t)x_2, \lambda), \tag{7}$$

$$td_1 + (1-t)d_2 \in D(tx_1 + (1-t)x_2, \lambda), \tag{8}$$

$$te_1 + (1-t)e_2 \in E(tx_1 + (1-t)x_2, \lambda), \tag{9}$$

$$tf_1 + (1-t)f_2 \in F(tx_1 + (1-t)x_2, \lambda), \tag{10}$$

and

$$tg_1 + (1-t)g_2 \in G(tx_1 + (1-t)x_2, \lambda). \tag{11}$$

By (H_1), (4)-(11), and the definition of $Q(x, \lambda)$, we have

$$\begin{aligned} tx_1 + (1-t)x_2 =\ & tx_1 + (1-t)x_2 - [tg_1 + (1-t)g_2 - m(tx_1 + (1-t)x_2, \lambda)] \\ & + R^h_{M(\cdot, tf_1+(1-t)f_2, \lambda), \rho}[h(tg_1 + (1-t)g_2 - m(tx_1 + (1-t)x_2, \lambda)) \\ & - \rho W(j(te_1 + (1-t)e_2, \lambda), ta_1 + (1-t)a_2, \lambda) \\ & + \rho N(tb_1 + (1-t)b_2, tc_1 + (1-t)c_2, td_1 + (1-t)d_2, \lambda) + \rho\omega] \\ & \in Q(tx_1 + (1-t)x_2, \lambda). \end{aligned}$$

Therefore, $tx_1 + (1-t)x_2 \in S(\lambda)$, and hence $S(\lambda)$ is a convex set. □

Using the selection technique of set-valued mappings, we discuss the existence of continuous solutions for the PCGMIQVIP (1).

Theorem 3.2. *Assume that $A, B, C, D, E, F, G, M, N, m, i, j, h$ and W are the same as which in the PCGMIQVIP (1) such that hypotheses (H_1)-(H_{11}) hold. Then*

(1) For each fixed $\lambda \in \Omega$, $S(\lambda)$ is a nonempty closed convex set.

(2) The PCGMIQVIP (1) has a continuous solution $x(\lambda) = m(S(\lambda))$, where $m(S(\lambda))$ is the minimal selection of $S(\lambda)$.

(3) If the hypothese (H_{12}) holds, then the PCGMIQVIP (1) has a continuous solution $x(\lambda) = c(S(\lambda))$, where $c(S(\lambda))$ is the Chebyshev selection of $S(\lambda)$.

(4) If $H = \mathbb{R}^n$ and the hypothese (H_{12}) is satisfied, then the PCGMIQVIP (1) has a Lipschitz continuous solution $x(\lambda) = b(B_1(S(\lambda)))$, where $b(B_1(S(\lambda)))$ is the barycenter selection of $S(\lambda)$.

Proof. (1) Since the hypotheses (H_3)-(H_{10}) are the same as those in Theorem 3.3 of Ding [1], $S(\lambda)$ is a nonempty closed set. Again by Theorem 3.1, $S(\lambda)$ is a convex set. Therefore, $S(\lambda)$ is a nonempty closed convex set.

(2) Through hypotheses (H_3)-(H_{11}) and the proof of Theorem 3.4 in Ding [1], the mapping $\lambda \mapsto S(\lambda)$ is $\frac{\theta_2}{1-\theta_1}$-$H$-Lipschitz continuous set-valued mapping from Ω to H, i.e.,

$$H(S(\lambda), S(\overline{\lambda})) \leq \frac{\theta_2}{1-\theta_1}\|\lambda - \overline{\lambda}\|, \quad \forall \lambda, \overline{\lambda} \in \Omega, \tag{12}$$

where θ_1 and θ_2 are given by (H_{10}) and (H_{11}), respectively.

By virtue of (12), $\forall \lambda, \overline{\lambda} \in \Omega$, we have

$$S(\lambda) \subset B\left(S(\overline{\lambda}), \frac{\theta_2}{1-\theta_1}\|\lambda - \overline{\lambda}\|\right).$$

Therefore, from Theorem 3.1, $\lambda \mapsto S(\lambda)$ is Lipschitz continuous set-valued mapping with closed convex values. By Lemma 2.1, the minimal selection $m(S(\lambda))$ of $S(\lambda)$ is continuous, and hence the PCGMIQVIP (1) has a continuous solution $x(\lambda) = m(S(\lambda))$.

(3) By (12) and Lemma 2.4, the mapping $\lambda \mapsto S(\lambda)$ is upper semi-continuous and Hausdorff lower semi-continuous. By (H_{12}) and the conclusion of (1), $\lambda \mapsto S(\lambda)$ is a set-valued mapping with bounded closed convex values. By Lemma 2.2, the Chebyshev selection $c(S(\lambda))$ of $S(\lambda)$ is continuous, and hence the PCGMIQVIP (1) has a continuous solution $x(\lambda) = c(S(\lambda))$.

(4) By (12), (H_{12}) and the conclusion of (1), the mapping $\lambda \mapsto S(\lambda)$ is a Lipschitz continuous set-valued mapping with bounded closed convex values. By Lemma 2.3, the barycenter selection $b(B_1(S(\lambda)))$ of $S(\lambda)$ is Lipschitz continuous, and hence the PCGMIQVIP (1) has a Lipschitz continuous solution $x(\lambda) = b(B_1(S(\lambda)))$. □

Remark 3.1. *Examples given by Aubin and Cellina* [16] *show that even if the mapping is set-valued Lipschitz continuous, its minimal selection and Chebyshev selection are not necessarily Lipschitz continuous.*

Example 3.1. *Let $H = R^n$ be a Euclidean space, Ω be a nonempty open subset of R^n in which the parameter λ takes values, $G : R^n \times \Omega \to 2^{R^n}$ be a set-valued mapping with bounded closed convex values and $G(\theta) = \theta$. Given $I = [0, 1], 1 < p < +\infty$, and $D = \overline{B_\rho(\theta)} \subset L^p(I, R^n)$ $(\rho > 0)$, for $\forall g(x, \lambda) \in L^p(I, R^n, \Omega)$, the set-valued mapping*

$M(\cdot,\lambda): R^n \to 2^{R^n}$ is defined by

$$M(g(x(\lambda),\lambda),\lambda) = \int_0^\lambda G(x(t),t)dt$$
$$= \left\{u : u(\lambda) = \int_0^\lambda g(x(t),t)dt, g(x(t),t) \in G(x(t),t), g(x,t) \in L^p(I,R^n,\Omega)\right\}.$$

Let $w=0$, $W=0$, $N=B=I$, $C=D=F=0$, $m=0$ in the PCGMIQVIP (1), then the PCGMIQVIP (1) reduces to the following integral inclusion: for each $\lambda\in\Omega$, find $x=x(\lambda)\in R^n$ such that

$$0 \in \int_0^\lambda G(x(t),t)dt - x(\lambda). \tag{13}$$

If G is L_G-Lipschitz continuous in the first argument, and $L_G \le p^{\frac{1}{p}}$, obviously, (13) satisfies the conditions of Theorem 3.2, then the solution set of (13) is nonempty closed convex, and it has a continuous minimal selection, a continuous Chebyshev selection and a Lipschitz continuous barycenter selection in D, respectively.

4. **Conclusions**

We continued to study the convexity of the solution set of the parametric completely generalized mixed implicit quasi-variational inclusion problem (PCGMIQVIP) proposed by Ding (J. Comput. Appl. Math. 182 (2005)252-269). Further, using the selection technique of set-valued mappings, we obtained the existence of (Lipschitz) continuous solutions, i.e., the minimal selection, the Chebyshev selection and the barycenter selection of the solution set are (Lipschtiz) continuous.

REFERENCES

[1] *X.P. Ding*, Parametric completely generalized mixed implicit quasi-variational inclusions involving h-maximal monotone mappings, J. Comput. Appl. Math., **182**(2005), 252-269.

[2] *X.P. Ding, Z.B. Wang*, Sensitivity analysis for a system of parametric generalized mixed quasi-variational inclusions involving (k,η)-monotone mappings, Appl. Math. Comput.,**214**(2009), 318-327.

[3] *K.R. Kazmi, F.A. Khan*, Sensitivity analysis for parametric generalized implicit quasi-variational-like inclusions involving P-η-accretive mappings, J. Math. Anal. Appl., **337**(2008), 1198-1210.

[4] *H.Y. Lan*, Sensitivity analysis for generalized nonlinear parametric (A,η,m)-maximal monotone operator inclusion systems with relaxed cocoercive type operators, Nonlinear Anal., **74**(2011), 386-395.

[5] *M.A. Noor*, Sensitivity analysis framework for general quasi-variational inclusions, Comput. Math. Appl., **44**(2002), 1175-1181.

[6] *N.J. Huang, H.Y. Lan, Y.J. Cho*, Sensitivity analysis for nonlinear generalized mixed implicit equilibrium problems with non-monotone set-valued mappings, J. Comput. Appl. Math., **196**(2006), 608-618.

[7] *S. Adly, L. Bourdin*, Sensitivity analysis of variational inequalities via twice epi-differentiability and proto-differentiability of the proximity operator, SIAM J. on Optimiz., **28**(2018),1699-1725.

[8] *S. Adly, R.T. Rockafellar*, Sensitivity analysis of maximally monotone inclusions via the proto-differentiability of the resolvent operator, Math. Program., **2**(2020).

[9] *A. Uderzo*, On the quantitative solution stability of parameterized set-valued inclusions, Set-Valued Var. Anal., **3**(2021).

[10] *A.A. Tolstonogov*, BV continuous solutions of an evolution inclusion with maximal monotone operator and nonconvex-valued perturbation existence theorem, Set-Valued Var. Anal., **29**(4)(2021),29-60.

[11] *R. Monteiro, B.F. Svaiter*, Iteration-complexity of a newton proximal extragradient method for monotone variational inequalities and inclusion problems, SIAM J. on Optimiz., **22**(3)(2012),914-935.

[12] *Y.L. Yu, K. Chen, L.J. Zhu*, An approximation scheme for solving quasi-variational inclusions and quasimonotone variational inequalities, U.P.B. Sci. Bull., Series A, **84**(3)(2022),41-52.

[13] *F.A. Khan, M. Iqbal, T. Ram, H.I.A. Mohammed*, Mixed variational-like inclusion involving yosida approximation operator in banach spaces, Mathematics, **10**(2022),4067.

[14] *X.P. Ding*, Sensitivity analysis for generalized nonlinear implicit quasi-variational inclusions, Appl. Math. Lett., **17**(2004), 225-235.

[15] *J.P. Aubin, H. Frankowska*, Set-Valued Analysis, Birkhauser, 1990.

[16] *J.P. Aubin, A. Cellina*, Differential Inclusions: Set-Valued Maps and Viability Theory, Springer-Verlag, Berlin, 1984.

[17] *L. Li, C.X. Wu*, Set-Valued Analysis, Science and Technology Press, Beijing, 2003.