THE COMMUTATOR OF TWO WEAKLY DECOMPOSABLE OPERATORS

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In this paper some results obtained by Colojoară and Foiaş in [8] *concerning the asymptotic behavior of the commutator of two decomposable operators are extended to weakly decomposable operators.*

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1. Introduction

In the following section, we survey very briefly several definitions and notations from the local spectral theory for bounded linear operators on Banach spaces, that led to the notions of decomposability and weakly decomposability, respectively. We hope that this short overview will be further useful even for the less experienced reader.

Throughout the paper, let $\mathbb{C}$ be the complex plane, and let $\mathbf{B}(X)$ denote the Banach algebra of all linear bounded operators on a complex Banach space X. Given an arbitrary operator $T \in \mathbf{B}(X)$ and I the identity operator on X, we use the following notations ([10], [13]):
$\rho(T) = \{\lambda \in \mathbb{C}; \lambda I - T$ is inversable in $\mathbf{B}(X\}$ the *resolvent set* of T;
$\sigma(T) = \mathbb{C} \setminus \rho(T)$ the *spectrum* of T;
$\Re(\lambda, T) = (\lambda I - T)^{-1}$, for $\lambda \in \rho(T)$, the *resolvent function* of T.

As usual, for $T \in \mathbf{B}(X)$ and Y a subspace of X (Y meaning a linear closed manifold of X), invariant to T, the restriction of T to Y is denoted by $T|Y$. A subspace Y of X is called *spectral maximal space* of $T \in \mathbf{B}(X)$ if Y is invariant to T and if $Z \subset Y$, for any other subspace Z of X invariant to T such that $\sigma(T|Z) \subset \sigma(T|Y)$ ([9]).

Following Dunford and Schwartz ([10]), an operator $T \in \mathbf{B}(X)$ has the *single-valued extension property* if, for every open set $D \subset \mathbb{C}$, the only analytic function $f : D \to X$ verifying the equation $(\lambda I - T)f(\lambda) \equiv 0$, for all $\lambda \in D$, is $f(\lambda) \equiv 0$.

For $T \in \mathbf{B}(X)$ having the single-valued extension property and for $x \in X$, $\rho_T(x)$ is called the *local resolvent set of T at the point x* and it is defined as the set of all elements $\xi \in \mathbb{C}$ for which there is an X-valued analytic function $\lambda \to x_T(\lambda)$ on an open neighborhood of ξ, satisfying the equation $(\lambda I - T)x_T(\lambda) \equiv x$. Obviously, the local analytic solution $x_T(\cdot)$ is unique, $\rho_T(x)$ is open and $\rho(T) \subset \rho_T(x)$. The *local spectrum of T at x* is the closed set defined as $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$ and the *local spectral subspaces of T* are

$$X_T(F) = \{x \in X; \sigma_T(x) \subset F\}, \text{ for all } F \subset \mathbb{C} \text{ arbitrary.}$$

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2. Preliminaries

Definition 2.1. ([9]) An operator $T \in \mathbf{B}(X)$ is said to be *weakly decomposa-ble* if the following conditions are fulfilled:

(1) T has the single-valued extension property and the local spectral space $X_T(F)$ is closed, for any $F \subset \mathbb{C}$ closed;
(2) for every finite open covering $\{G_i\}_{i=1}^n$ of the spectrum $\sigma(T)$, there is a system $\{Y_i\}_{i=1}^n$ of spectral maximal spaces of T such that:
   (i) $\sigma(T|Y_i) \subset G_i$, $1 \leq i \leq n$;
   (ii) $X = \overline{\sum_{i=1}^{n} Y_i}$.

Under the assumption that T satisfies only the condition (2) for which the assertion (ii) is replaced by $X = \sum_{i=1}^{n} Y_i$, then T is called *decomposable* ([9]).

Weakly contractions (see [9], Chap. VIII) and adjoints of decomposable operators (see [3]) can be presented as a very natural examples of weakly decomposable operators.

Remark 2.1. If $T \in \mathbf{B}(X)$ is a weakly decomposable operator, then the following properties hold:

1) $X_T(F)$ is spectral maximal space of T, for any $F \subset \mathbb{C}$ closed and

$$\sigma(T|X_T(F)) \subset F \cap \sigma(T)$$

2) any spectral maximal space Y of T can be viewed as

$$Y = X_T(\sigma(T|Y)).$$

(see [9], Proposition I.3.8. and Theorem II.1.5.).

Lemma 2.1. ([17]) *Let $T \in \boldsymbol{B}(X)$ be weakly decomposable. If there is no spectral maximal space $Y \neq \{0\}$ of T such that $\sigma(T|Y) \subset D$, where $D \subset \mathbb{C}$ is an open fixed set, then*

$$D \cap \sigma(T) = \emptyset.$$

Lemma 2.2. ([17]) *An operator $T \in \boldsymbol{B}(X)$ is weakly decomposable if and only if the following assertions hold true:*

(a) *T has the single-valued extension property and $X_T(F)$ is closed, for any $F \subset \mathbb{C}$ closed;*
(b) *for any finite open covering $\{G_i\}_{i=1}^n$ of $\sigma(T)$, the set Z of all $x \in X$, with $x = x_1 + x_2 + \cdots + x_n$, $\sigma_T(x_i) \subset G_i$, $1 \leq i \leq n$, is dense in X.*

Definition 2.2. ([8]) Let X, Y be two Banach spaces. For the operators $S \in \mathbf{B}(X)$ and $T \in \mathbf{B}(Y)$, the *commutator* of T and S is the operator

$$C(T, S) : \mathbf{B}(X, Y) \to \mathbf{B}(X, Y)$$

given by the equality

$$C(T, S)(A) = TA - AS, \text{ where } A \in \mathbf{B}(X, Y).$$

Evidently, $C(T, S)$ is a linear bounded operator, hence $C(T, S) \in \mathbf{B}(\mathbf{B}(X, Y))$.

Remark 2.2. ([8]) It is clear that, for every $A \in \mathbf{B}(X, Y)$ we have

$$C^n(T, S)(A) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{n-k} A S^k, \;\; n \geq 1, \text{ and}$$

$$C^{n+1}(T, S)(A) = TC^n(T, S)(A) - C^n(T, S)(A)S.$$

3. The commutator of two weakly decomposable operators

Lemma 3.1. *Let $S \in \boldsymbol{B}(X)$, $T \in \boldsymbol{B}(Y)$ be two operators having the single-valued extension property, and let $C(T,S) \in \boldsymbol{B}(\boldsymbol{B}(X,Y))$ be their commutator. If $A(\xi) : D \to \boldsymbol{B}(X,Y)$ is an analytic operatorial function (where $D \subset \mathbb{C}$ open) which satisfies the equality*

$$(\xi I - C(T,S))A(\xi) \equiv 0, \tag{1}$$

then

$$A(\xi)X_S(F) \subset Y_T(\xi + F), \quad \text{for } F \subset \mathbb{C}.$$

Proof. The equality (1) can be written

$$\xi A(\xi) - TA(\xi) + A(\xi)S \equiv 0.$$

Let $x \in X$ and let $\lambda \in \rho_S(x)$; then

$$(\lambda I - S)x_S(\lambda) = x, \text{ or}$$

$$Sx_S(\lambda) = \lambda x_S(\lambda) - x,$$

hence

$$\xi A(\xi)x_S(\lambda) - TA(\xi)x_S(\lambda) + A(\xi)Sx_S(\lambda) \equiv 0,$$

$$[(\xi + \lambda)I - T]A(\xi)x_S(\lambda) \equiv A(\xi)x. \tag{2}$$

From relation (2), if $x \in X_S(F)$ (and thus $\complement F \subset \rho_S(x)$), it results that $\xi + \lambda \in \rho_T(A(\xi)x)$ or, equivalently,

$$\sigma_T(A(\xi)x) \subset \complement\{\xi + \rho_S(x)\} = \xi + \sigma_S(x).$$

Consequently, if $x \in X_S(F)$, then $\sigma_T(A(\xi)x) \subset \xi + F$ and so that

$$A(\xi)X_S(F) \subset Y_T(F + \xi).$$

□

Theorem 3.1. *Let $S \in \boldsymbol{B}(X)$, $T \in \boldsymbol{B}(Y)$ be two weakly decomposable operators. Then their commutator $C(T,S) \in \boldsymbol{B}(\boldsymbol{B}(X,Y))$ has the single-valued extension property.*

Proof. Let $D \subset \mathbb{C}$ be an open set and $A(\xi)$ an analytic operatorial function in D such that

$$(\xi I - C(T,S))A(\xi) \equiv 0.$$

One can prove that $A(\xi) \equiv 0$. Let us consider in D two disjoint disks, namely δ_1 and δ_2, with the distance between them sufficiently big and having the radii sufficiently small. We put $F_j = \overline{\bigcup_{\xi \in \delta_j} (\xi + \sigma_S(x))}$, $j = 1,2$. If $x \in X$ has the local spectrum $\sigma_S(x)$ sufficiently small, then $F_1 \cap F_2 = \emptyset$. From Lemma 3.1, it results that

$$A(\xi)x \in Y_T(F_j),$$

for any $\xi \in \delta_j$, $j = 1,2$. But $A(\xi)x$ is an analytic function in D, hence by analytic extension we have $A(\xi)x \in Y_T(F_j)$, $j = 1,2$, for any $\xi \in D$ (because $Y_T(F_j)$ is closed, i.e. it is Banach space).

Therefore

$$A(\xi)x \in Y_T(F_1) \cap Y_T(F_2) = Y_T(F_1 \cap F_2) = Y_T(\emptyset),$$

hence $A(\xi)x = 0$, for any $\xi \in D$.

On the other hand, because S is weakly decomposable, according to Lemma 2.2, the set of those elements $x \in X$ of the form $x = x_1 + x_2 + \cdots + x_{n_\varepsilon}$, where $x_1, x_2, \ldots, x_{n_\varepsilon}$ have the local spectra containing in disks with radii smaller than any $\varepsilon > 0$, is dense in X.

For any $x \in X$ written as above, we have

$$A(\xi)x = \sum_{i=1}^{n_\varepsilon} A(\xi)x_i = 0 \ \ (\xi \in D),$$

hence $A(\xi) = 0$ on a set dense in X. The operators $A(\xi)$ being continuous for any $\xi \in D$, it results that $A(\xi) = 0$; therefore $C(T, S)$ has the single-valued extension property. □

Corollary 3.1. *Let $T \in \boldsymbol{B}(X)$ be a decomposable operator, let $T^* \in \boldsymbol{B}(X^*)$ and $T^{**} \in \boldsymbol{B}(X^{**})$ be the adjoint operator of T on the topological dual space X^*, respectively the second adjoint of T on the second dual space X^{**}. Then the commutators $C(T, T^*)$, $C(T, T^{**})$, $C(T^*, T^{**})$ etc. have the single-valued extension property.*

Corollary 3.2. *If $S \in \boldsymbol{B}(X)$ and $T \in \boldsymbol{B}(Y)$ are two decomposable operators, then the commutators $C(T^*, S^*)$, $C(T^{**}, S^{**})$, $C(T^*, S^{**})$ etc. have the single-valued extension property.*

Corollary 3.3. *Let H be a Hilbert space and let S, $T \in \boldsymbol{B}(H)$ be two weakly contractions. Then the commutators $C(T, S)$, $C(T^*, S^*)$ have the single-valued extension property.*

Theorem 3.2. *Let $S \in \boldsymbol{B}(X)$ be decomposable, let $T \in \boldsymbol{B}(Y)$ be weakly decomposable, and let $A \in \boldsymbol{B}(X, Y)$. Then the following properties are equivalent:*

(1) *$AX_S(F) \subset Y_T(F)$, for any $F \subset \mathbb{C}$ closed;*

(2) $\lim\limits_{n\to\infty} \|C^n(T, S)(A)\|^{\frac{1}{n}} = 0$.

Proof. (1) ⇒ (2). Let $\varepsilon > 0$ be sufficiently small and let us consider $\{G_i\}_{i=1}^{n_\varepsilon}$ a finite open covering of $\sigma(S)$, with $\operatorname{diam}(G_i) < \varepsilon$, where $\operatorname{diam}(G_i) = \sup\limits_{\lambda,\ \mu \in G_i} |\lambda - \mu|$ is the diameter of G_i. It follows from the decomposability of the operator S (see the last part of Definition 2.1) that there is a system $\{X_i\}_{i=1}^{n_\varepsilon}$ of spectral maximal spaces of S with the properties:

$$\sigma(S|X_i) \subset G_i, \ \ 1 \le i \le n_\varepsilon \ \text{ and } \ X = \sum_{i=1}^{n_\varepsilon} X_i.$$

We have $X_i = X_S(\sigma(S|X_i))$ ([9]) and let us denote $Y_i = Y_T(\sigma(S|X_i))$, $1 \le i \le n_\varepsilon$. T being weakly decomposable, from Remark 2.1, the spaces Y_i are spectral maximal spaces of T and $\sigma(T|Y_i) \subset \sigma(S|X_i) \subset G_i$, $1 \le i \le n_\varepsilon$.

Furthermore, for $\lambda_i \in G_i$, we denote the operators $S_i = (S - \lambda_i I)|X_i$ and $T_i = (T - \lambda_i I)|Y_i$. Evidently, $S_i \in \mathbf{B}(X_i)$ and $T_i \in \mathbf{B}(Y_i)$, $1 \le i \le n_\varepsilon$.

From the "theorem of spectral mapping" (see [10], Theorem 11, VII, §3), we deduce that

$$\sigma(S_i) = \{\lambda - \lambda_i; \lambda \in \sigma(S|X_i)\} \subset \{\lambda - \lambda_i; \lambda \in G_i\} \subset \{\lambda; |\lambda| \le \varepsilon\}$$

hence

$$\lim_{n\to\infty} \|S_i^n\|^{\frac{1}{n}} = \sup_{\lambda \in \sigma(S|X_i)} |\lambda| \le \varepsilon \ \ (1 \le i \le n_\varepsilon).$$

From the last relation, there is a constant $M'_\varepsilon > 0$ such that

$$\|S_i^n\| \le \varepsilon^n M'_\varepsilon, \ \text{ for every } n \ge 0 \text{ and } 1 \le i \le n_\varepsilon. \tag{3}$$

Using the same argument as above, because $\sigma(T|Y_i) \subset G_i$, there is a constant $M''_\varepsilon > 0$ such that

$$\|T_i^n\| \le \varepsilon^n M''_\varepsilon, \ \text{ for every } n \ge 0 \text{ and } 1 \le i \le n_\varepsilon. \tag{4}$$

The linear application $X_1 \oplus X_2 \oplus ... \oplus X_{n_\varepsilon} \ni x_1 \oplus x_2 \oplus ... \oplus x_{n_\varepsilon} \to \sum\limits_{i=1}^{n_\varepsilon} x_i \in X$ being continuous and surjective, from the theorem of closed graph, we may deduce that there is a constant $M \geq 0$ such that for every $x \in X$, there is $x_1 \oplus x_2 \oplus \cdots \oplus x_{n_\varepsilon} \in X_1 \oplus X_2 \oplus \cdots \oplus X_{n_\varepsilon}$, with $x = \sum_{i=1}^{n_\varepsilon} x_i$ and

$$\|x_1\| + \|x_2\| + \cdots + \|x_{n_\varepsilon}\| \leq M\|x\|. \tag{5}$$

According to Remark 2.2, one can easily observe that

$$\begin{aligned} C^n(T,S)(A)x_i &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (T-\lambda_i I)^{n-k} A (S-\lambda_i I)^k x_i = \\ &= C^n(T_i,S_i)(A)x_i \end{aligned} \tag{6}$$

for every $1 \leq i \leq n_\varepsilon$, with $\lambda_i \in G_i$.

If we apply the relations (2)-(6), it follows that

$$\begin{aligned} \|C^n(T,S)(A)x\| &= \left\| \sum_{i=1}^{n_\varepsilon} C^n(T,S)(A)x_i \right\| = \left\| \sum_{i=1}^{n_\varepsilon} C^n(T_i,S_i)(A)x_i \right\| = \\ &= \left\| \sum_{i=1}^{n_\varepsilon} \sum_{k=0}^{n} (-1)^k \binom{n}{k} T_i^{n-k} A S_i^k x_i \right\| \leq \\ &\leq \sum_{i=1}^{n_\varepsilon} \sum_{k=0}^{n} \binom{n}{k} \left\|T_i^{n-k}\right\| \cdot \|A\| \cdot \left\|S_i^k\right\| \cdot \|x_i\| \leq \\ &\leq M'_\varepsilon \cdot M''_\varepsilon \cdot \|A\| \cdot \sum_{i=1}^{n_\varepsilon} (\sum_{k=0}^{n} \binom{n}{k} \varepsilon^{n-k}\varepsilon^k) \cdot \|x_i\| = \\ &= M'_\varepsilon \cdot M''_\varepsilon \cdot \|A\| \cdot (2\varepsilon)^n \cdot \sum_{i=1}^{n_\varepsilon} \|x_i\| \leq M_\varepsilon \cdot (2\varepsilon)^n \|x\| \end{aligned}$$

where $M_\varepsilon = M'_\varepsilon \cdot M''_\varepsilon \cdot M \cdot \|A\|$; consequently

$$\|C^n(T,S)(A)\| \leq M_\varepsilon \cdot (2\varepsilon)^n \quad (n \geq 1),$$

i.e. $\limsup_{n\to\infty} \|C^n(T,S)(A)\|^{\frac{1}{n}} \leq 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it results that

$$\lim_{n\to\infty} \|C^n(T,S)(A)\|^{\frac{1}{n}} = 0.$$

(2) $\Rightarrow$ (1). Let us suppose that $x \in X_S(F)$, $F \subset \mathbb{C}$ closed. Then $(\lambda I - S)x_S(\lambda) \equiv x$, $\lambda \in \rho_S(x)$. Putting $y_T(\lambda) = \sum_{k=0}^{\infty} (-1)^k \frac{C^k(T,S)(A)}{k!} x_S^{(k)}(\lambda)$, then from Remark 2.2 we obtain $(\lambda I - T)y_T(\lambda) = A(\lambda I - S)x_S(\lambda) = Ax$. We proved that $\sigma_T(Ax) \subset \sigma_S(x)$; accordingly $Ax \in Y_T(F)$. □

Corollary 3.4. *Let $T \in \boldsymbol{B}(X)$ be decomposable and let $A \in \boldsymbol{B}(X, X^*)$. Then the following assertions are equivalent:*

(1) *$AX_T(F) \subset X^*_{T^*}(F)$, for any $F \subset \mathbb{C}$ closed;*
(2) $\lim_{n\to\infty} \|C^n(T^*,T)(A)\|^{\frac{1}{n}} = 0$.

Proof. These equivalences can be directly obtained by using Theorem 3.2 for the Banach spaces X and $Y = X^*$, and for the operators $S = T$ and $T = T^*$, because if T is decomposable, then T^* is weakly decomposable. □

Corollary 3.5. *Let $S, T \in \boldsymbol{B}(X)$ such that S is decomposable and T is weakly decomposable. If the following two assertions hold:*

1) $TS = ST$
2) $X_S(F) \subset X_T(F)$, *for any* $F \subset \mathbb{C}$ *closed,*

then T *is decomposable.*

Proof. If the conditions 1) and 2) are fulfilled, then by Theorem 3.2 for $A = I \in \mathbf{B}(X)$, it follows that

$$\lim_{n\to\infty} \|C^n(T,S)(I)\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\|(T-S)^{[n]}\right\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\|(S-T)^{[n]}\right\|^{\frac{1}{n}} = 0,$$

where $(T-S)^{[n]} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{n-k} S^k$; therefore T and S are spectral equivalent (see [9], Definition I.2.1). Since S is decomposable, from [9], Theorem II.2.1, we obtain that T is decomposable. □

4. **Conclusions**

This work generalizes properties of decomposable operators for weakly decomposable operators and it is a sequel of the articles [15] and [17]. These results can be used in the theory of differential equations and fractal theory through this generalization of decomposable operators.

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