SUFFICIENT OPTIMALITY CONDITIONS AND MOND-WEIR DUALITY FOR QUASIDIFFERENTIABLE OPTIMIZATION PROBLEMS WITH UNIVEX FUNCTIONS

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In the paper, a nonconvex quasidifferentiable optimization problem with the inequality constraints is considered. The concept of a univex function with respect to a convex compact set is introduced. Further, the sufficient optimality conditions and several duality results in the sense of Mond-Weir are established for the considered quasidifferentiable optimization problem under assumption that the functions constituting it are univex with respect to convex compact sets which are equal to Minkowski sum of their subdifferentials and superdifferentials.

Keywords: quasidifferentiable optimization problem; optimality conditions; duality; quasidifferentiable univex function with respect to convex compact set.

1. Introduction

Quasidifferential calculus were developed by Demyanov and Rubinov [8] and have been studied in more detail in [9]. Since then it has been developed extensively. A survey of results concerning this class of functions is presented in [10]. This is also a consequence of the fact that quasidifferential calculus plays an important role in nonsmooth analysis and optimization. Indeed, the concept of quasidifferentiability can be employed to study a wide range of theoretical and practical issues in many fields, for instance, in economics, engineering, mechanics, optimal control theory, etc. (see, [11], [13], [24], and others). Further, the class of quasidifferentiable functions is fairly broad. It contains not only convex, concave, and differentiable functions but also convex-concave, D.C. (i.e., difference of two convex), maximum, and other functions. In addition, it even includes some functions which are not locally Lipschitz continuous.

Optimality and duality results for quasidifferentiable optimization problems can be found in several works (see, for example, Eppler and Luderer [20], Demyanov and Rubinov [12], Gao [15], [16], Kuntz and Scholtes [18], Luderer and Rosiger [20], Polyakova [21], Shapiro [23], Uderzo [25], Ward [26], Xia et al. [28], and others). In most of the works mentioned above, the necessary

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optimality conditions have been proved for quasidifferentiable optimization problems only. However, there are a few papers in the literature in which the sufficient optimality conditions and duality results have been established for some classes of nonconvex quasidifferentiable optimization problems. Namely, Craven [6] established the sufficient optimality conditions for directionally differentiable optimization problems under cone-invexity hypotheses. In [7], Craven presented the sufficient optimality conditions and Wolfe duality results for directionally differentiable optimization problems under directional invexity hypotheses. Glover [17] proved the sufficiency of the presented necessary optimality conditions under assumptions that the objective function is directionally differentiable pseudo-invex and the constraints are directionally differentiable quasi-invex. Yin and Zhang [27] established sufficient optimality conditions for the considered quasidifferentiable optimization problem under generalized convexity. Gao [16] proved the sufficient optimality conditions for quasidifferentiable optimization problems under assumption that the objective function is directionally differentiable pseudoconvex and the constraint functions are directionally differentiable quasiconvex.

The aim of this paper is to prove the sufficient optimality conditions of the Lagrange multiplier type and several Mond-Weir duality results for a new class of nonconvex quasidifferentiable optimization problems with inequality constraints. However, our approach in proving the sufficiency of the Karush-Kuhn-Tucker necessary optimality conditions and Mond-Weir duality results for the considered quasidifferentiable optimization problem differs even from those ones mentioned above in which directionally differentiable generalized convex functions have been used. In this paper, we introduce a new concept of generalized convexity, namely, the notion of univexity with respect to a convex compact set. We use this notion in establishing the sufficient optimality conditions and duality theorems in the sense of Mond-Weir for the considered quasidifferentiable optimization problem involving univex functions with respect to convex compact sets which are equal to Minkowski sum of their subdifferentials and superdifferentials. The results established in the paper are illustrated by an example of a nonsmooth optimization problem with quasidifferentiable univex functions with respect to the same function η and with respect to convex compact sets which are equal to Minkowski sum of subdifferentials and superdifferentials functions constituting this extremum problem.

2. Preliminaries

Definition 2.1 A mapping $f : \mathbb{R}^n \to \mathbb{R}$ is said to be directionally differentiable at $u \in \mathbb{R}^n$ into a direction d if the limit

$f'(u;d) = \lim_{t \downarrow 0} \frac{f(u+td) - f(u)}{t}$ exists finite. It is said that f is directionally differentiable or semi-differentiable at u if its directional derivative $f'(u;d)$ exists finite for all $d \in \mathbb{R}^n$.

Definition 2.2 A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be quasidifferentiable at $u \in \mathbb{R}^n$ if f is directionally differentiable and there exists a ordered pair of convex compact sets $D_f(u) = [\underline{\partial} f(u), \overline{\partial} f(u)]$ such that

$$f'(u;d) = \max_{v \in \underline{\partial} f(u)} v^T d + \min_{w \in \overline{\partial} f(u)} w^T d, \tag{1}$$

where $\underline{\partial} f(u)$ and $\overline{\partial} f(u)$ are called subdifferential and superdifferential of f at u, respectively. Further, the ordered par of sets $D_f(u) = [\underline{\partial} f(u), \overline{\partial} f(u)]$ is called quasidifferential of the function f at u.

Let us note that the pair of sets constituting the quasidifferential to a function f at a certain point u is not unique, because if $D(u) = [\underline{\partial} f(u), \overline{\partial} f(u)]$ is a quasidifferential of f at $\bar{x}$, then, for any convex compact set V, the ordered pair of sets $[\underline{\partial} f(u) + V, \overline{\partial} f(u) - V]$ is also its quasidifferential.

Now, we introduce the definition of a univex function with respect to a convex compact subset of $\mathbb{R}^n$. The concept of a univex function with respect to a convex compact set generalizes the notion of a differentiable univex function, earlier given in the literature by Bector et al. [4].

Definition 2.3 Let $f : \mathbb{R}^n \to \mathbb{R}$, $u \in \mathbb{R}^n$ and $S_{f(u)}$ be an arbitrary convex compact subset of $\mathbb{R}^n$. If there exist functions $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with $b(x,u) \geq 0$ for all $x \in \mathbb{R}^n$, $\Phi : \mathbb{R} \to \mathbb{R}$ and $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that the inequality

$$b(x,u)\Phi(f(x) - f(u)) \geq w^T \eta(x,u), \ \forall w \in S_{f(u)} \tag{2}$$

holds for all $x \in \mathbb{R}^n$ ($x \neq u$), then f is said to be a (strictly) univex function at u on $\mathbb{R}^n$ with respect to the convex compact set $S_{f(u)}$ and with respect to Φ, b and η.

If f is defined on a nonempty set $X \subset \mathbb{R}^n$, $u \in X$ and inequality (2) is satisfied for all $x \in X$, then f is to be a (strictly) univex function at u on X with respect to convex compact set $S_{f(u)}$ and with respect to Φ, b and η.

If inequality (2) is satisfied for all $u \in \mathbb{R}^n$ with respect to a convex compact set S_f, then f is said to be a (strictly) univex function on $\mathbb{R}^n$ with respect to S_f and with respect to Φ, b and η.

Remark 2.1. In order to define an analogous class of (strictly) unicave functions, the direction of inequality (2) should be reversed.

Remark 2.2. Note that, in the case when f is a locally Lipschitz function and $S_{f(u)}$ is equal to the Clarke generalized gradient of f at u (see [5]), then we obtain the definition of a locally Lipschitz univex function.

Remark 2.3. Note that the definition of a locally Lipschitz univex function generalizes and extends many other definitions of nondifferentiable generalized convex functions. Indeed, if we assume that $S_{f(u)}$ is equal to the Clarke generalized gradient [5] of f at u, then, from Definition 2.3, there are the following special cases:

i) If $\Phi(a) \equiv a$ and $b(x,u) = 1$ for all $x \in X$ and $\eta(x,u) = x - u$, then we obtain the definition of a nonsmooth convex function.

ii) If $\Phi(a) \equiv a$ and $b_i(x,u) = 1$ for all $x \in X$, then we obtain the definition of a locally Lipschitz invex function (with respect to η) given by Reiland [22].

iii) If $\Phi(a) \equiv a$ and $\eta(x,u) = x - u$, then we obtain the definition of a locally Lipschitz b-convex function.

iv) If $\Phi(a) \equiv a$, then we obtain the definition of a locally Lipschitz b-invex function (with respect to η) (see Li et al. [19]).

v) If $\Phi(a) = \frac{1}{r}(e^a - 1)$ for a certain scalar $r \neq 0$ and $\eta(x,u) = x - u$ and $b(x,u) = 1$ for all $x \in X$, then we obtain the definition of a locally Lipschitz r-convex function (see Avriel [3], in the differentiable case).

vi) If $\Phi(a) = \frac{1}{r}(e^a - 1)$ for a certain scalar $r \neq 0$ and $b(x,u) = 1$, then we obtain the definition of a locally Lipschitz r-invex function (with respect to η) introduced by Antczak [1].

vii) If $\Phi(a) = \frac{1}{r}(e^a - 1)$ for a certain scalar $r \neq 0$, then we obtain the definition of a locally Lipschitz B-r-invex function (with respect to η) (see Antczak [2], in the differentiable case).

Example 2.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $f(x) = \exp\left(\left|x_1 + |x_2|\right|\right) - 1$. First, we show that f is a quasidifferentiable function at $\bar{x} = (0,0)$. Indeed, we have $f'(\bar{x};d) = \left|d_1 + |d_2|\right|$. Hence, it can be proved that $f'(\bar{x};d) = \max_{v \in co\{(0,0),(2,-2),(2,2)\}} v^T d + \min_{w \in co\{(-1,-1),(1,-1)\}} w^T d$, $\underline{\partial} f(\bar{x}) = co\{(0,0),(2,-2),(2,2)\}$ and $\overline{\partial} f(\bar{x}) = co\{(-1,1),(-1,-1)\}$. Hence, by Definition 2.2, it follows that f is a quasidifferentiable function at $\bar{x} = (0,0)$. Further, we have $S_{f(\bar{x})} = \underline{\partial} f(\bar{x}) + \overline{\partial} f(\bar{x}) = co\{(-1,1),(1,-1),(1,3),(-1,-1),(1,-3),(1,1)\}$. Now, let $b : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ be defined by $b(x,\bar{x}) = 4$ for all $x \in \mathbb{R}$, $\Phi : \mathbb{R} \to \mathbb{R}$ be a function defined by $\Phi(a) = \ln(a + 1)$ and

$\eta : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a vector-valued function $\eta(x,\bar{x}) = \begin{bmatrix} |x_1 + |x_2|| \\ -|x_1 + |x_2|| \end{bmatrix}$. Hence, by Definition 2.3, it can be proved that f is a quasidifferentiable univex function at $\bar{x}$ = (0,0) on $\mathbb{R}^2$ with respect to the convex compact set $S_{f(\bar{x})}$ and with respect to functions Φ, b and η defined above.

3. Optimality

In the paper, consider the following nonsmooth optimization problem:

$$f(x) \rightarrow \min$$
$$\text{s.t.} \quad g_j(x) \leq 0, \; j \in J = \{1,\dots,m\}, \qquad \text{(P)}$$
$$x \in \mathbb{R}^n,$$

where f: $\mathbb{R}^n \rightarrow \mathbb{R}$, $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j \in J$, are quasidifferentiable functions on $\mathbb{R}^n$. Thus, problem (P) may be referred as a quasidifferentiable optimization problem.

Let X:= { $x \in \mathbb{R}^n$: $g_j \leq 0$, $j \in J$} be the set of all feasible solutions in problem (P). Further, we denote by $J(\bar{x})$ the set of inequality constraint indexes that are active at point $\bar{x} \in X$, that is, $J(\bar{x}) := \{ j \in J : g_j(\bar{x}) = 0 \}$.

In [16], Gao presented the following necessary optimality conditions for nonsmooth optimization problems with inequality constraints in which the functions involved are quasidifferentiable.

Theorem 3.1. (Karush-Kuhn-Tucker type necessary optimality conditions). Let $\bar{x} \in X$ be an optimal solution for the considered quasidifferentiable optimization problem (P). Further, assume that f is quasidifferentiable at $\bar{x}$, with the quasidifferential $D_f(\bar{x}) = [\underline{\partial} f(\bar{x}), \overline{\partial} f(\bar{x})]$, each g_j, $j \in J$, is quasidifferentiable at $\bar{x}$, with the quasidifferential $D_{g_j}(\bar{x}) = [\underline{\partial} g_j(\bar{x}), \overline{\partial} g_j(\bar{x})]$. If the constraint qualification [20] is satisfied at $\bar{x}$ for problem (P), then, for any sets of $w_0 \in \overline{\partial} f(\bar{x})$ and $w_j \in \overline{\partial} g_j(\bar{x})$, $j \in J$, there exist the scalars $\bar{\lambda}_j(w) \geq 0$, $j \in J$, not all zero, such that

$$0 \in \underline{\partial} f(\bar{x}) + w_0 + \sum_{j=1}^{m} \bar{\lambda}_j(w)(\underline{\partial} g_j(\bar{x}) + w_j), \tag{3}$$

$$\bar{\lambda}_j(w) g_j(\bar{x}) = 0, \quad j \in J, \tag{4}$$

$$\bar{\lambda}_j(w) \geq 0, \quad j \in J, \tag{5}$$

where $\bar{\lambda}_1(w),\dots,\bar{\lambda}_m(w)$ are dependent on the specific choice of $w = (w_0, w_1,\dots,w_m)$.

Theorem 3.2. (Sufficient optimality conditions). Let $\bar{x}$ be a feasible solution in the considered optimization problem (P) and the Karush-Kuhn-Tucker type necessary optimality conditions (3) – (5) be satisfied at $\bar{x}$ with the quasidifferentials $D_f(\bar{x}) = [\underline{\partial} f(\bar{x}), \overline{\partial} f(\bar{x})]$, $D_{g_j}(\bar{x}) = [\underline{\partial} g_j(\bar{x}), \overline{\partial} g_j(\bar{x})]$, $j \in J$. Further, assume that f is quasidifferentiable univex function at $\bar{x}$ on X with respect to $S_{f(\bar{x})} = \underline{\partial} f(\bar{x}) + w_0$ (for any $w_0 \in \overline{\partial} f(\bar{x})$) and with respect to Φ_f, b_f and η, where $b_f(x, \bar{x}) > 0$ for all $x \in X$, $a < 0 \Rightarrow \Phi_f(a) < 0$ and, moreover, each g_j, $j \in J(\bar{x})$, is quasidifferentiable univex function at $\bar{x}$ on X with respect to $S_{g_j(\bar{x})} = \underline{\partial} g_j(\bar{x}) + w_j$ (for any $w_j \in \overline{\partial} g_j(\bar{x})$, $j \in J$,) and with respect to Φ_{g_j}, b_{g_j} and η, where $a \le 0 \Rightarrow \Phi_{g_j}(a) \le 0$, $j \in J(\bar{x})$. Then $\bar{x}$ is an optimal solution in the considered optimization problem (P).

Proof Assume that $\bar{x}$ is such a feasible point in problem (P) at which the Karush-Kuhn-Tucker type necessary optimality conditions (3) – (5) are satisfied with the quasidifferentials $D_f(\bar{x}) = [\underline{\partial} f(\bar{x}), \overline{\partial} f(\bar{x})]$, $D_{g_j}(\bar{x}) = [\underline{\partial} g_j(\bar{x}), \overline{\partial} g_j(\bar{x})]$, $j \in J$. This means that, for given sets of $w_0 \in \overline{\partial} f(\bar{x})$ and $w_j \in \overline{\partial} g_j(\bar{x})$, $j \in J$, there exist $\bar{\lambda}_0(w) \in R$ and $\bar{\lambda}(w) \in R^m$ such that the conditions (3) – (5) are satisfied. Hence, by the Karush-Kuhn-Tucker type necessary optimality condition (3), it follows that there exist $v_0 \in \underline{\partial} f(\bar{x})$ and $v_j \in \underline{\partial} g_j(\bar{x})$, $j \in J$, such that

$$0 = v_0 + w_0 + \sum_{j=1}^{m} \bar{\lambda}_j(w)(v_j + w_j). \tag{6}$$

By hypotheses, f is a quasidifferentiable univex function at $\bar{x}$ on $\mathbb{R}^n$ with respect to $S_{f(\bar{x})} = \underline{\partial} f(\bar{x}) + w_0$ and with respect to Φ_f, b_f and η, g_j, $j \in J(\bar{x})$, is quasidifferentiable univex function at $\bar{x}$ on X with respect to $S_{g_j(\bar{x})} = \underline{\partial} g_j(\bar{x}) + w_j$ and with respect to Φ_{g_j}, b_{g_j} and η. Hence, by Definition 2.3, the inequalities

$$b_f(x, \bar{x})\Phi_f(f(x) - f(\bar{x})) \ge \omega_0^T \eta(x, \bar{x}), \quad \forall \omega_0 \in S_{f(\bar{x})}, \tag{7}$$

$$b_{g_j}(x, \bar{x})\Phi_{g_j}(g_j(x) - g_j(\bar{x})) \ge \omega_j^T \eta(x, \bar{x}), \quad \forall \omega_j \in S_{g_j(\bar{x})},\ j \in J(\bar{x}) \tag{8}$$

hold for all $x \in X$. Since (7) and (8) are fulfilled for any sets $\omega_0 \in S_{f(\bar{x})}$ and $\omega_j \in S_{g_j(\bar{x})}$, $j \in J(\bar{x})$, respectively, by the definitions of $S_{f(\bar{x})}$ and $S_{g_j(\bar{x})}$, they are also fulfilled for $\omega_0 = v_0 + w_0 \in S_{f(\bar{x})}$ and $\omega_j = v_j + w_j \in S_{g_j(\bar{x})}$. Thus, (7) and (8) yield

$$b_f(x, \bar{x})\Phi_f(f(x) - f(\bar{x})) \ge (v_0^T + w_0^T)\eta(x, \bar{x}), \tag{9}$$

$$b_{g_j}(x, \bar{x})\Phi_{g_j}(g_j(x) - g_j(\bar{x})) \ge (v_j^T + w_j^T)\eta(x, \bar{x}), \quad j \in J(\bar{x}). \tag{10}$$

Using $x \in X$ and $\bar{x} \in X$ together with the definition of $J(\bar{x})$, we get $g_j(x) \le g_j(\bar{x})$, $j \in J(\bar{x})$. By assumption, it follows that

$$\Phi_{g_j}(g_j(x) - g_j(\overline{x})) \le 0,\ j \in J(\overline{x}). \tag{11}$$

By definition, $b_{g_j}(x,\overline{x}) \ge 0$, $j \in J(\overline{x})$, for all $x \in X$. Thus, (11) gives

$$b_{g_j}(x,\overline{x})\Phi_{g_j}(g_j(x) - g_j(\overline{x})) \le 0, \quad j \in J(\overline{x}). \tag{12}$$

Combining (10) and (12), we obtain

$$(v_j^T + w_j^T)\eta(x,\overline{x}) \le 0. \quad j \in J(\overline{x}). \tag{13}$$

Since $\overline{\lambda}_j(w) > 0$, $j \in J(\overline{x})$, and $\overline{\lambda}_j(w) = 0$, $j \notin J(\overline{x})$, therefore, (13) yields

$$\sum_{j=1}^{m} \overline{\lambda}_j(w)(v_j^T + w_j^T)\eta(x,\overline{x}) \le 0. \tag{14}$$

By (6) and (14), it follows that

$$(v_0^T + w_0^T)\eta(x,\overline{x}) \ge 0. \tag{15}$$

Combining (9) and (15), we obtain

$$b_f(x,\overline{x})\Phi_f(f(x) - f(\overline{x})) \ge 0. \tag{16}$$

By assumption, $b_f(x,\overline{x}) > 0$ for all $x \in X$ and $a < 0 \Rightarrow \Phi_f(a) < 0$. Thus, (16) implies that the inequality $f(x) \ge f(\overline{x})$ holds for all $x \in X$. This means that $\overline{x}$ is optimal in problem (P). Hence, the proof of this theorem is complete.

Example 3.1. Consider the following nonsmooth optimization problem:

$$f(x) = \ln\left(x_1^2 + x_2^2 + \left|x_1 - |x_2|\right| + 1\right) \to \min$$

$$\text{s.t.} \quad g_1(x) = \operatorname{arctg}\left(\left|x_1 + |x_2|\right|\right) \le 0, \tag{P1}$$

$$x \in \mathbb{R}^2.$$

Note that $X = \{ x \in \mathbb{R}^2 : \operatorname{arctg}(\left|x_1 + |x_2|\right|) \le 0 \}$ and $\overline{x} = (0,0)$ is a feasible solution in problem (P1). Further, it can be proved that f and g_1 are quasidifferentiable at $\overline{x}$. Indeed, we have $f_1'(\overline{x};d) = \left|d_1 - |d_2|\right|$ and, therefore,

$$f'((0,0);d) = \max_{v \in co\{(0,-2),(2,0),(0,2)\}} v^T d + \min_{w \in \{(0,-1),(0,1)\}} w^T d \ ,$$

where $\underline{\partial} f(0,0) = co\{(0,-2),(2,0),(0,2)\}$, $\overline{\partial} f(0,0) = \{(0,-1),(0,1)\}$. Hence, by Definition 2.2, f is a quasidifferentiable function at $\overline{x} = (0,0)$. Further, by Definition 2.1, we have $g_1'(\overline{x};d) = \left|d_1 + |d_2|\right|$ and, therefore,

$$g_1'(\overline{x};d) = \max_{v \in co\{(0,0),(-2,2),(2,2)\}} v^T d + \min_{w \in co\{(-1,-1),(1,-1)\}} w^T d,$$

where $\underline{\partial} g_1(\overline{x}) = co\{(0,0),(-2,2),(2,2)\}$ and $\overline{\partial} g_1(\overline{x}) = co\{(-1,-1),(1,-1)\}$. Hence, by Definition 2.2, g_1 is a quasidifferentiable function at $\overline{x} = (0,0)$.

It can be proved that the Karush-Kuhn-Tucker necessary optimality conditions are fulfilled at $\overline{x}$. Indeed, it can be shown that, for any sets of

$w_0 \in \overline{\partial} f(\overline{x})$ and $w_1 \in \overline{\partial} g_1(\overline{x})$, there exists $\overline{\lambda}_1(w) > 0$ such that the conditions (3) – (5) are satisfied. Since the Karush-Kuhn-Tucker necessary optimality conditions are fulfilled at $\overline{x}$, in order to prove optimality of $\overline{x}$ by Theorem 3.2, we have to show that f and g_1 are quasidifferentiable univex functions at $\overline{x}$ on X with respect to convex compact sets which are equal to Minkowski sum of their subdifferentials and superdifferentials and with respect to the same function η, but not necessarily with respect to the same functions b and Φ. We set $b_f(x, \overline{x}) = 3$, $\Phi_f(a) = \exp(a) - 1$, $S_{f(\overline{x})} = \underline{\partial} f(\overline{x}) + w_0$ (for any $w_0 \in \overline{\partial} f(\overline{x})$), $b_{g_1}(x, \overline{x}) = 4$, $\Phi_{g_1}(a) = \mathrm{tg}(a)$, $S_{g_1(\overline{x})} = \underline{\partial} g_1(\overline{x}) + w_1$ (for any $w_1 \in \overline{\partial} g_1(\overline{x})$), $\eta : X \times X \to \mathbb{R}^2$ be a vector-valued function defined by $\eta(x, \overline{x}) = \begin{bmatrix} \left| x_1 + |x_2| \right| \\ -\left| x_1 + |x_2| \right| \end{bmatrix}$. Then, by Definition 2.3, f is a quasidifferentiable function at $\overline{x}$ on X with respect to $S_{f(\overline{x})}$ and with respect to Φ_f, b_f, η and g_1 is a quasidifferentiable function at $\overline{x}$ on X with respect to $S_{g_1(\overline{x})}$ and with respect to Φ_{g_1}, b_{g_1}, η. Further, note that functions Φ_f and Φ_{g_1} satisfy conditions given in Theorem 3.2. Hence, since all hypotheses of Theorem 3.2 are fulfilled, $\overline{x}$ is an optimal solution in the considered nonsmooth optimization problem (P1).

3. Mond-Weir duality

In this section, for the considered quasidifferentiable optimization problem (P), we define its dual problem in the sense of Mond-Weir as follows:

$$f(y) \to \max$$

$$0 \in \underline{\partial} f(y) + w_0 + \sum_{j=1}^{m} \lambda_j(w)(\underline{\partial} g_j(y) + w_j),$$

$$\text{for any sets of } w_0 \in \overline{\partial} f(y) \text{ and } w_j \in \overline{\partial} g_j(y),\ j \in J, \quad \text{(D)} \qquad (17)$$

$$\lambda_j(w) g_j(y) \geq 0,\ j \in J, \qquad (18)$$

$$y \in R^n,\ \lambda_j(w) \geq 0,\ j \in J, \qquad (19)$$

where $\lambda(w) = (\lambda_1(w), ..., \lambda_m(w))$ are dependent on the specific choice of w = $(w_0, w_1, \ldots, w_k)$.

We denote by Z the set of all feasible solutions in Mond-Weir dual problem (D), that is, the set of (y, λ(w)) satisfying constraints (17) – (19). Further, we denote by Y = $pr_{\mathbb{R}^n} Z$ the projection of the set Z on $\mathbb{R}^n$.

Theorem 4.1. (Weak duality). Let x and (y,λ(w)) be any feasible solutions in the considered optimization problem (P) and its Mond-Weir dual problem (D), respectively. Further, assume that f is a quasidifferentiable univex function at y on

$X \cup Y$ with respect to $S_{f(y)} = \underline{\partial} f(y) + w_0$ and with respect to Φ_f, b_f and η, where $b_f(x,y) > 0$, $a < 0 \Rightarrow \Phi_f(a) < 0$, each $\lambda_j(w)g_j$, $j \in J(y)$, is a quasidifferentiable univex function at y on $X \cup Y$ with respect to $S_{g_j}(y) = \underline{\partial} g_j(y) + w_j$ and with respect to Φ_{g_j}, b_{g_j} and η, where $a < 0 \Rightarrow \Phi_{g_j}(a) \le 0$, $j \in J$. Then $f(x) \ge f(y)$.

Proof Let x and $(y,\lambda(w))$ be any feasible solutions in problem (P) and its Mond-Weir dual problem (D), respectively. This means that, for given sets of $w_0 \in \overline{\partial} f(y)$ and $w_j \in \overline{\partial} g_j(y)$, $j \in J$, there exist $\lambda(w) \in \mathbb{R}$ and $\mu(w) \in \mathbb{R}^m$ such that the constraints (17) – (19) are fulfilled. Suppose, contrary to the result, that

$$f(x) < f(y). \tag{20}$$

By hypotheses, f is a quasidifferentiable univex function at y on $X \cup Y$ with respect to $S_{f(y)} = \underline{\partial} f(y) + w_0$ and with respect to Φ_f, b_f and η, each g_j, $j \in J(y)$, is a quasidifferentiable univex function at y on $X \cup Y$ with respect to $S_{g_j(y)} = \underline{\partial} g_j(y) + w_j$ and with respect to Φ_{g_j}, b_{g_j} and η. Hence, by Definition 2.3, the following inequalities

$$b_f(x,y)\Phi_f(f(x) - f(y)) \ge \omega_0^T \eta(x,y), \quad \forall \omega_0 \in S_{f(y)}, \tag{21}$$

$$b_{g_j}(x,y)\Phi_{g_j}(\lambda_j(w)g_j(x) - \lambda_j(w)g_j(y)) \ge \lambda_j(w)\omega_j^T \eta(x,y), \ \forall \omega_j \in S_{g_j(y)}, j \in J(y) \tag{22}$$

hold. By assumption, $b_f(x,y) > 0$ and $a < 0 \Rightarrow \Phi_f(a) < 0$. Hence, (20) yields

$$b_f(x,y)\Phi_f(f(x) - f(y)) < 0. \tag{23}$$

Combining (21) and (23), by the definition of S_f, we get

$$\omega_0^T \eta(x,y) < 0, \quad \forall \omega_0 \in \underline{\partial} f(y) + w_0. \tag{24}$$

By $x \in X$, $y \in Y$ and the constraint (18) of dual problem (D), it follows that

$$\lambda_j(w)g_j(x) - \lambda_j(w)g_j(y) \le 0, \ j \in J. \tag{25}$$

By assumption, $a \le 0 \Rightarrow \Phi_{g_j}(a) \le 0$, $j \in J$. Since $b_{g_j}(x,y) > 0$, $j \in J$, therefore, by (25), we have

$$b_{g_j}(x,y)\Phi_{g_j}(\lambda_j(w)g_j(x) - \lambda_j(w)g_j(y)) \le 0, \ j \in J(y). \tag{26}$$

Combining (22) and (26), by the definition of $S_{g_j(y)}$, $j \in J$, we obtain

$$\sum_{j=1}^{m} \lambda_j(w)\omega_j^T \eta(x,y) \le 0, \quad \forall \omega_j \in \underline{\partial} g_j(y) + v_j, \ j \in J. \tag{27}$$

Thus, (22) and (30), we get

$$\left[\lambda_0(w)\,\omega_0^T + \sum_{j=1}^{m} \lambda_j(w)\omega_j^T\right]\eta(x,y) < 0, \quad \forall \omega_0 \in \underline{\partial} f(y) + w_0, \ \omega_j \in \underline{\partial} g_j(y) + w_j.$$

This means that there exist $v_0 \in \underline{\partial} f(y)$ and $v_j \in \underline{\partial} g_j(y)$, $j \in J$, such that

$$\left[v_0^T + w_0^T + \sum_{j=1}^{m} \lambda_j(w)(v_j^T + w_j^T)\right]\eta(x,y) < 0. \tag{28}$$

By the constraint (17) of dual problem (D), it follows that the following inequality

$$\left[v_0^T + w_0^T + \sum_{j=1}^{m} \lambda_j(w)(v_j^T + w_j^T)\right]\eta(x,y) = 0$$

holds, which is a contradiction to (28). This completes the proof of this theorem.

It turns out that, under stronger univexity hypothesis imposed on the objective function, it is possible to prove the stronger result.

Theorem 4.2. (Weak duality). Let x and $(y,\lambda(w))$ be any feasible solutions in the considered optimization problem (P) and its Mond-Weir dual problem (D), respectively. Further, assume that f is a quasidifferentiable strictly univex function at y on $X \cup Y$ with respect to $S_{f(y)} = \underline{\partial} f(y) + w_0$ and with respect to Φ_f, b_f and η, where $b_f(x,y) > 0$, $a < 0 \Rightarrow \Phi_f(a) < 0$, each g_j, $j \in J(y)$, is a quasidifferentiable univex function at y on $X \cup Y$ with respect to $S_{g_j(y)} = \underline{\partial} g_j(y) + w_j$ and with respect to Φ_{g_j}, b_{g_j} and η, where $a < 0 \Rightarrow \Phi_{g_j}(a) \leq 0$, $j \in J$. Then $f(x) > f(y)$.

Theorem 4.3. (Direct duality). Let $\overline{x}$ be an optimal solution in the considered optimization problem (P) and the constraint qualification [20] be satisfied at $\overline{x}$. Further, assume that, for any sets of $w_0 \in \overline{\partial} f(\overline{x})$, $w_j \in \overline{\partial} g_j(\overline{x})$, $j \in J$, there exists $\overline{\lambda}(w) = (\overline{\lambda}_1(w),\ldots,\overline{\lambda}_m(w)) \in \mathbb{R}^m$ depending on the specific choice of w = $(w_0,w_1,\ldots,w_m)$, such that $(\overline{x},\overline{\lambda}(w))$ is feasible in its Mond-Weir dual problem (D). Further, if all hypotheses of the weak duality theorem (Theorem 4.1) are fulfilled, then $(\overline{x},\overline{\lambda}(w))$ is optimal in Mond-Weir dual problem (D).

Proof By assumption, $\overline{x}$ is an optimal solution in the considered optimization problem (P) and the constraint qualification [20] is satisfied at $\overline{x}$. Further, we assume that, for any sets of $w_0 \in \overline{\partial} f(\overline{x})$ and $w_j \in \overline{\partial} g_j(\overline{x})$, $j \in J$, there exist scalars $\overline{\lambda}_j(w) \geq 0$, $j \in J$, not all zero, such that the Karush-Kuhn-Tucker necessary optimality conditions (3) – (5) are fulfilled at $\overline{x}$. Further, we assume that $(\overline{x},\overline{\lambda}(w))$ is feasible in Mond-Weir dual problem (D). If all hypotheses of the weak duality theorem (Theorem 4.1) are fulfilled, then $(\overline{x},\overline{\lambda}(w))$ is optimal in Mond-Weir dual problem (D).

Theorem 4.4. (Converse duality). Let $(\overline{y},\overline{\lambda}(w))$ be an optimal solution in Mond-Weir dual problem (D) and $\overline{y} \in X$. Further, assume that f is a quasidifferentiable univex function at $\overline{y}$ on $X \cup Y$ with respect to

$S_{f(\bar{y})} = \underline{\partial} f(\bar{y}) + w_0$ and with respect to Φ_f, b_f, η, where $b_f(x, \bar{y}) > 0$ for all x ∈ X, a < 0 ⇒ $\Phi_f(a) < 0$, each $\bar{\lambda}_j(w) g_j$, j ∈ J($\bar{y}$), is a quasidifferentiable univex function at $\bar{y}$ on X∪Y with respect to $S_{g_j(\bar{y})} = \underline{\partial} g_j(\bar{y}) + w_j$ and with respect to Φ_{g_j}, b_{g_j}, η, where a < 0 ⇒ $\Phi_{g_j}(a) \le 0$, j ∈ J. Then $\bar{y}$ is optimal in problem (P).

Proof Proof of this theorem follows directly from the weak duality theorem (Theorem 4.1).

5. Conclusions

In this paper, we have introduced a new concept of generalized convexity, namely the definition of a univex function with respect to a convex compact set. It turned out that it generalizes many other concepts of generalized convexity, previously defined in the literature. Further, in the paper, a nonconvex quasidifferentiable optimization problem with inequality constraints has been considered in which all functions constituting it are quasidifferentiable univex with respect to convex compact sets. Under the introduced concept generalized convexity, the sufficient optimality conditions and several duality results have been proved for such nonsmooth optimization problems. Namely, to prove the results mentioned above, the functions constituting the considered nondifferentiable optimization problems have been assumed to be quasidifferentiable univex with respect to convex compact sets which are Minkowski sum of their subdifferentials and superdifferentials.

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