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MONOTONICITY PROPERTIES RELATED TO SOME GAMMA FUNCTION ESTIMATES

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The aim of this paper is to establish some inequalities to some approximation formulas for gamma function that are stronger than other classical such as Ramanujan formula. The monotonicity and convexity of corresponding functions are demonstrated. However the proving of complete monotonicity of these functions is left as an open problem.

Keywords: Asymptotic approximation, asymptotic expansions, Gamma function, Digamma function

1. Introduction

The problem of approximating the gamma function defined (for $x > 0$) by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

Is of great interest in many branches of science and consequently it was widely studied in the recent past. See [2]-[17] and all reference therein. Undoubtedly, the most known formula is Stirling's formula, *e.g.*, [1],[18]

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x.$$

Burnside formula [2]

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{e}\right)^{x+\frac{1}{2}}$$

and Gosper's formula [5]

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \sqrt{1+\frac{1}{6x}}$$

are slightly better. However, Ramanujan formula, *e.g.*, [17]

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \cdot \sqrt[6]{1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}}$$

are much accurate. In the recent past, Nemes [16] proposed the following two formulas:

$$\Gamma(x+1) \sim \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{15x^2}\right)^{\frac{5x}{4}} \quad (1)$$

and

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$$\Gamma(x+1) \sim \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{12x^2-\frac{1}{10}}\right)^x \qquad (2)$$

which give much better results than Ramanujan formula. The advantage of formula (1)-(2) is also that they are of a simple form.

2. The Results

Usually to an approximation of the form $f(x) \sim g(x)$ (in the sense that $\frac{f(x)}{g(x)}$ tends to 1, as x approaches infinity) the following function is associated:

$$\Phi(x) = \ln \frac{f(x)}{g(x)}$$

If Φ is (completely) monotone, then sharp inequalities related to the approximation $f(x) \sim g(x)$ can be established. In this sense, we present the following results.

Theorem 1. *Let*

$$F(x) = \ln \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{15x^2}\right)^{\frac{5x}{4}}}$$

Then : a) F is strictly decreasing on [2,∞). b) F is strictly convex on [2,∞).

Theorem 2. *Let*

$$G(x) = \ln \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{12x^2-\frac{1}{10}}\right)^x}$$

Then : a) G is strictly decreasing on [2,∞). b) G is strictly convex on [2,∞).

The following double inequality obtained by truncation the classical asymptotic series of the gamma function is valid for every > 0 :

$$u(x) < \ln \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}} < v(x),$$

where

$$u(x) = \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} - \frac{691}{360360x^{11}} \qquad (3)$$

and

$$v(x) = \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9}. \qquad (4)$$

The function $\psi(x) = \Gamma'(x)/\Gamma(x)$ is called digamma function, while its derivative ψ' is the trigamma function. The following inequality obtained by truncation the asymptotic series of the trigamma function is valid for every $x > 0$:

$$\psi'(x) > \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}. \tag{5}$$

Proof of Theorem 1. By using the recurrence formula $(x+1) = \psi(x) + 1/x$, we deduce that

$$F'(x) = \psi(x) - \ln x + \frac{1}{2x} - \frac{5}{4}\ln\left(1 + \frac{1}{15x^2}\right) + \frac{5}{2(15x^2+1)}$$

and

$$F''(x) = \psi'(x) - \frac{1}{x} - \frac{1}{2x^2} + \frac{5}{2x(15x^2+1)} - \frac{75x}{(15x^2+1)^2}.$$

By (5), we deduce

$$F''(x) > \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{2x(15x^2+1)} - \frac{75x}{(15x^2+1)^2}$$
$$= \frac{1}{210x^9(15x^2+1)}$$
$$\times [950(x-2)^6 + 11400(x-2)^5 + 55568(x-2)^4$$
$$+140544(x-2)^3 + 193427(x-2)^2 + 135756(x-2) + 37061] > 0.$$

Hence $F'' > 0$ on $[2,\infty)$, so F is convex on $[2,\infty)$.
Now F' is strictly increasing, with $\lim_{x\to\infty} F'(x) = 0$. As a result, $F' < 0$, so F is strictly decreasing on $[2,\infty)$.

Proof of Theorem 2. By using the recurrence formula $(x+1) = \psi(x) + 1/x$, we deduce that

$$G'(x) = \psi(x) - \ln x + \frac{1}{2x} - ln\left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right)$$
$$+ \frac{24x^2}{\left(12x^2 + \frac{9}{10}\right)\left(12x^2 - \frac{1}{10}\right)}$$

and

$$G''(x) = \psi'(x) - \frac{1}{x} - \frac{1}{2x^2} - \frac{18x(-320x^2 + 4800x^4 + 9)}{25\left(12x^2 + \frac{9}{10}\right)^2\left(12x^2 - \frac{1}{10}\right)^2}.$$

By (5), we get

$$G''(x) > \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} - \frac{18x(-320x^2 + 4800x^4 + 9)}{25\left(12x^2 + \frac{9}{10}\right)^2\left(12x^2 - \frac{1}{10}\right)^2}$$

$$= \frac{1}{210x^9(120x^2-1)^2(40x^2+3)^2}$$

$$\times$$
$$[94760000(x-2)^{10}+1895200000(x-2)^{9}+16910297600(x-2)^{8}$$
$$+88625561600(x-2)^{7}+$$
$$301964208955(x-2)^{6}+698258097860(x-2)^{5}$$
$$+1108223852437(x-2)^{4}+1189597541096(x-2)^{3}$$
$$+824020977173(x-2)^{2}+331080529684(x-2)$$
$$+58169457989]>0.$$

Hence $G''>0$ on $[2,\infty)$, so G is convex on $[2,\infty)$.
Now G' is strictly increasing, with $\lim_{x\to\infty}G'(x)=0$. As a result, $G'<0$, so G is strictly decreasing on $[2,\infty)$.

Corollary 3. *The following sharp inequalities are valid for every* $x\geq 2$*:*

$$\alpha\left(\frac{x}{e}\right)^{x}\sqrt{2\pi x}\left(1+\frac{1}{15x^{2}}\right)^{\frac{5x}{4}}<\Gamma(x+1)\leq\beta\left(\frac{x}{e}\right)^{x}\sqrt{2\pi x}\left(1+\frac{1}{15x^{2}}\right)^{\frac{5x}{4}},\quad(6)$$

where the constants $\alpha=1$ *and*

$$\beta=\frac{1800\sqrt{915}}{226981\sqrt{\pi}}e^{2}=1.000017441\,...$$

are sharp.

Proof. As the function F is strictly decreasing, by exponentiating we deduce the following inequality for every $x\geq 2$:

$$\exp\{F(\infty)\}<\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^{x}\sqrt{2\pi x}\left(1+\frac{1}{15x^{2}}\right)^{\frac{5x}{4}}}\leq\exp\{F(2)\}.$$

The conclusion follows from the fact that

$$F(2)=\ln\left(\frac{1800\sqrt{915}}{226981\sqrt{\pi}}e^{2}\right).$$

Corollary 4. *The following sharp inequalities are valid for every* $x\geq 2$*:*

$$\delta\left(\frac{x}{e}\right)^{x}\sqrt{2\pi x}\left(1+\frac{1}{12x^{2}-\frac{1}{10}}\right)^{x}<\Gamma(x+1)\leq\eta\left(\frac{x}{e}\right)^{x}\sqrt{2\pi x}\left(1+\frac{1}{12x^{2}-\frac{1}{10}}\right)^{x},\quad(7)$$

where the constants $\delta=1$ *and*

$$\eta=\frac{229441}{956484\sqrt{\pi}}e^{2}=1.000016912\,...$$

are sharp.

Proof. As the function G is strictly decreasing, by exponentiating we deduce the following inequality for every $x\geq 2$:

$$\exp\{G(\infty)\} < \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{12x^2-\frac{1}{10}}\right)^x} \leq \exp\{G(2)\}.$$

The conclusion follows from the fact that

$$G(2) = \ln\left(\frac{229441}{956484\sqrt{\pi}}e^2\right).$$

The strongness of inequalities (6)-(7) follows also from the fact that the differences between sharp constants α, β, respectively δ, η are very small in absolute value.

Corollary 1-2 present constants bounds for

$$\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{15x^2}\right)^{\frac{5x}{4}}} \tag{8}$$

and

$$\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{12x^2-\frac{1}{10}}\right)^x}. \tag{9}$$

In the sequel we resent new bounds of (8)-(9) which are expressions in x^{-k} that tends to zero as x approaches infinity.

Theorem 5. *The following inequalities are valid for every $x \geq 2$:*

$$\exp\left\{\frac{19}{28350x^5}-\frac{167}{283500x^7}\right\} < \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{15x^2}\right)^{\frac{5x}{4}}}$$

$$< \exp\left\{\frac{19}{28350x^5}\right\}.$$

Proof. We have to proof that $a > 0$ and $b < 0$, where

$$a(x) = \ln\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}} - \frac{5x}{4}\ln\left(1+\frac{1}{15x^2}\right) - \left(\frac{19}{28350x^5}-\frac{167}{283500x^7}\right)$$

and

$$b(x) = \ln\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}} - \frac{5x}{4}\ln\left(1+\frac{1}{15x^2}\right) - \frac{19}{28350x^5}.$$

By using (3)-(4), we get $a(x) > p(x)$ and $b(x) < q(x)$, where

$$p(x) = u(x) - \frac{5x}{4}\ln\left(1+\frac{1}{15x^2}\right) - \left(\frac{19}{28350x^5}-\frac{167}{283500x^7}\right)$$

and

$$q(x) = u(x) - \frac{5x}{4}\ln\left(1+\frac{1}{15x^2}\right) - \frac{19}{28350x^5}.$$

Thus

$$p''(x) = \frac{1}{20270250x^{13}(15x^2+1)}$$
$$\times [345380490(x-2)^6 + 4144565880(x-2)^5 + 19614489268(x-2)^4 + 46394157344(x-2)^3 + 56138769807(x-2)^2 + 30236631356(x-2) + 3756240073] > 0.$$

and

$$q''(x) = -\frac{1}{2970x^{11}(15x^2+1)^2}$$
$$\times [22044(x-2)^6 + 264528(x-2)^5 + 1274974(x-2)^4 + 3145712(x-2)^3 + 4139925(x-2)^2 + 2680532(x-2) + 621331] < 0.$$

Finally, p is strictly convex, q is strictly concave on $[2,\infty)$, with $p(\infty) = q(\infty) = 0,\, so\ p > 0\ and\ q < 0\ on\ [2,\infty)$. The proof is now completed.

Theorem 6. The following inegalities are valid for every ≥ 2 :

$$\exp\left\{\frac{2369}{3628800x^5} - \frac{21313}{36288000x^7}\right\} < \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{12x^2-\frac{1}{10}}\right)}$$
$$< \exp\left\{\frac{2369}{3628800x^5}\right\}.$$

Proof. We have to proof that $c > 0$ and $d < 0$, where

$$c(x) = \ln\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x\sqrt{2\pi x}} - x\ln\left(1+\frac{1}{12x^2-\frac{1}{10}}\right) - \left(\frac{2369}{36328800x^5} - \frac{21313}{36288000x^7}\right)$$

and

$$d(x) = \ln\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x\sqrt{2\pi x}} - x\ln\left(1+\frac{1}{12x^2-\frac{1}{10}}\right) - \frac{2369}{36328800x^5}.$$

By using (3)-(4), we get $c(x) > r(x)$ and $d(x) < s(x)$, where

$$r(x) = u(x) - x\ln\left(1+\frac{1}{12x^2-\frac{1}{10}}\right) - \left(\frac{2369}{36328800x^5} - \frac{21313}{36288000x^7}\right)$$

and

$$s(x) = v(x) - x\ln\left(1+\frac{1}{12x^2-\frac{1}{10}}\right) - \frac{2369}{36328800x^5}$$

Thus

$$r''(x) = \frac{R(x-2)}{12972960000x^{13}(120x^2-1)^2(40x^2+3)^2} > 0$$

and

$$s''(x) = -\frac{S(x-2)}{190080x^{11}(120x^2-1)^2(40x^2+3)^2} < 0,$$

where R and S are polynomials of degree 10 with all coefficients positive.
Finally, r is strictly convex, s is strictly concave on $[2,\infty)$, with $r(\infty) = s(\infty) = 0$, so $r > 0$ and $s < 0$ on $[2,\infty)$. The proof is now completed.
By a completely monotonic function z on$(0,\infty)$ we mean that z has derivatives of all orders and

$$(-1)^n z^n(x) \geq 0, \tag{10}$$

for every integer $n \geq 0$ and $x \in (0,\infty)$. Completely monotonic functions involving gamma function are of great interest in the problem of approximating gamma and polygamma functions. Sharp bounds for these functions can be established.

We proved in Theorem 1-2 that functions F and G satisfy (10) for $n = 0, 1$. However, we propose as an open problem that proof of the fact that F and G are completely monotonic.

3. Further possible extensions

In this final section we show how to above theoreticals results can be extended. As a constant $\frac{1}{10}$ appears in (9), an idea is to consider a new term $\frac{b}{x^2}$ in (8) to obtain the approximation

$$\Gamma(x+1) \sim \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1+\frac{1}{15x^2+\frac{b}{x^2}}\right)^{\frac{5x}{4}}. \tag{11}$$

In order to find the value of b that provides the most accurate approximation (11), we use the method first presented in [11]. This method was proven to be also a strong tool for constructing asymptotic expansions or to accelerating some convergences and it was highly used by other authors in the recent past. See, e.g., [3]-[4], [6]-[9]. The relative error sequence is

$$\omega_n = \ln\Gamma(n+1) - n\ln n + n - \frac{1}{2}\ln 2\pi n - \frac{5n}{4}\ln\left(1+\frac{1}{15n^2+\frac{b}{n^2}}\right).$$

We are interested in the case when ω_n is of higher possible speed of convergence. But ω_n is of convergence rate $n^{-(k-1)}$ when $\omega_n - \omega_{n+1}$ converges to zero as n^{-k}. As

$$\omega_n - \omega_{n+1} = \left(\frac{1}{36}b + \frac{19}{5670}\right)\frac{1}{n^6} - \left(\frac{1}{12}b + \frac{19}{1890}\right)\frac{1}{n^7} + O\left(\frac{1}{n^8}\right),$$

The fastest sequence ω_n is obtained when the coefficient of $\frac{1}{n^6}$ vanishes, that is $b = -\frac{38}{315}$. Now (11) becomes

$$\Gamma(x+1)\sim\left(\frac{x}{e}\right)^{x}\sqrt{2\pi x}\left(1+\frac{1}{15x^{2}+\frac{38}{315x^{2}}}\right)^{\frac{5x}{4}}.$$

Then we prove the monotonicity and convexity of the function

$$H(x)=\ln\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^{x}\sqrt{2\pi x}\left(1+\frac{1}{15x^{2}-\frac{38}{315x^{2}}}\right)^{\frac{5x}{4}}}.$$

Using the method from the previous sections, we deduced that H is increasing and concave on $[2,\infty)$. From the monotonicity of H (more precisely $H(2)\leq H(x)\leq H(\infty)$) we got the following

Theorem 7. The following inequality is valid for every real $x\geq 2$:

$$v\cdot\left(\frac{x}{e}\right)^{x}\sqrt{2\pi x}\left(1+\frac{1}{15x^{2}-\frac{38}{315x^{2}}}\right)^{\frac{5x}{4}}\leq\Gamma(x+1)$$
$$<\omega\cdot\left(\frac{x}{e}\right)^{x}\sqrt{2\pi x}\left(1+\frac{1}{15x^{2}-\frac{38}{315x^{2}}}\right)^{\frac{5x}{4}},$$

where the constants

$$v=\frac{1427403961}{226687113682124}\sqrt{37781}\frac{\sqrt{38411}}{\sqrt{\pi}}e^{2}=0.9999968306\ldots$$

and $\omega=1.000000$ *are sharp.*

Personal computations we made lead us to the conclusion that $-H$ should be completely monotonic. However, the rigorous proof of the complete monotonicity of $-H$ we leave as an open problem.

The results stated in this paper are part of the problem of estimating the gamma function for large values of the argument. A possible subject of interest for the future is the behavior of functions F,G,H on intervals near origin. However, we are convinced that different method should be exploited, since the inequalities used in this paper give good results only for large values of the variable.

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