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WEIGHTED TRANSLATION PRE-FRAME OPERATORS ON THE SPACE OF VECTOR-VALUED WEAKLY MEASURABLE FUNCTIONS

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In this paper, we first characterize the boundedness of the weighted translation pre-frame operator, namely, $T_{\omega,\varphi,f}$ and its adjoint. Then, we identify the relation between the adjoint of $T_{\omega,\varphi,f}$ and the translation frame operator which is denoted by $S_{\omega,\varphi,f}$. Basically, all results are obtained by using the conditional expectation properties.

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1. Introduction

Frames were developed as a powerful tool in signal processing. The frame in a Hilbert space was defined by Duffin and Schaeffer [2] in 1952, for investigating some deep questions in non-harmonic Fourier series. A discrete frame is a countable family of elements in a separable Hilbert space, which allows stable and not necessarily unique decomposition of arbitrary elements in an expansion of frame elements. In this paper, $\mathcal{H}$ refers to a Hilbert space over $\mathbb{C}$, and the closed unit ball of $\mathcal{H}$ is denoted by $\mathcal{H}_1$.
Let (X, Σ, μ) be a complete σ-finite measure space and suppose that φ is a measurable transformation $(i.e., \varphi^{-1}\Sigma \subseteq \Sigma)$ from X into X such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ, that is, φ is non-singular. Let h_φ be the Radon-Nikodym derivative $\frac{d\mu\circ\varphi^{-1}}{d\mu}$ and we always assume that h_φ is almost everywhere finite-valued or, equivalently that $\mathcal{A} := \varphi^{-1}\Sigma \subseteq \Sigma$ is a σ-finite subalgebra. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. Recall that a measurable function f is said to be bounded away from zero, if there exists $\delta > 0$ such that $|f(x)| \geq \delta$ for almost $x \in X$. For any non-negative Σ-measurable functions f as well as for any $f \in L^2(\Sigma)$, corresponds a measure $\upsilon_f(B) = \int_B f d\mu$ for all $B \in \mathcal{A}$, which is absolutely continuous with respect to μ. Then, by the Radon-Nikodym theorem, there exists a unique non-negative $\mathcal{A}$-measurable function $E(f)$ such that

$$\int_B E(f)d\mu = \int_B f d\mu, \quad for\ all\ B \in \mathcal{A}.$$

Hence, we obtain an operator E from $L^2(\Sigma)$ onto $L^2(\mathcal{A})$, which is called the conditional expectation operator associated with the σ-finite subalgebra $\mathcal{A}$. The role of this operator is important and we list here some of its useful properties:
(i) If f is an $\mathcal{A}$-measurable function, then $E(fg) = fE(g)$.
(ii) If $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$.
(iii)$|E(f)|^p \leq E(|f|^p)$.
(iv) $E(|f|^2) = |E(f)|^2$ if and only if f $\mathcal{A}$-measurable.
It is easy to show that for each non-negative Σ-measurable function f or for each $f \in L^2(\Sigma)$,

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there exists a unique Σ-measurable function g with $\sigma(g) \subseteq \sigma(h_\varphi)$ provided that $E(f) = g\circ\varphi$. We then write $g = E(f) \circ \varphi^{-1}$ though we make no assumptions regarding the invertibility of φ. For more details see [6], [8] and [10].
For a finite-valued Σ-measurable weight function ω, the weighted composition operator on $L^2(\Sigma)$ induced by φ and ω is given by

$$(\omega C_\varphi)f = \omega \cdot f \circ \varphi, \quad f \in L^2(\Sigma),$$

where C_φ is the composition operator on $L^2(\Sigma)$, defined by $C_\varphi f = f \circ \varphi$. Note that the non-singularity of φ guarantees that ωC_φ is well-defined as a mapping of equivalence classes of functions on $\sigma(\omega)$. Boundedness of weighted composition operators $L^p(\Sigma)$ spaces has already being studied in [6]. Namely, ωC_φ is bounded on $L^p(\Sigma)$ for $1 \leq p < \infty$ if and only if $J := h_\varphi E(|\omega|^p) \circ \varphi^{-1} \in L^\infty(\Sigma)$. Throughout this paper, we assume that $J \in L^\infty(\Sigma)$. The major properties of these operators have been studied by Harrington and Whitly [5], Lambert [6], [8] and by many other mathematicians.
The theory of weighted translation pre-frame operators is the generalizations of the theory of c-frames and c-Bessel mappings. The properties of c-frames and c-Bessel mappings have been studied in [4].
The change of variable formula will be frequently used throughout this paper and we remind it here as follows:

$$\int_{\varphi^{-1}(B)} f \circ \varphi d\mu = \int_B h_\varphi f d\mu, \quad B \in \Sigma, f \in L^1(\Sigma).$$

Definition 1.1. *Let $L^2(X, \mathcal{H})$ be the class of all measurable mappings $f : X \to \mathcal{H}$ such that*

$$\|f\|_2^2 = \int_X \|f(x)\|^2 \, d\mu < \infty.$$

For any $f, g \in L^2(X, \mathcal{H})$, based on the polar identity, we may conclude that the mapping $x \mapsto \langle f(x), g(x)\rangle$ of X to $\mathbb{C}$, is measurable and it can be seen that $L^2(X, \mathcal{H})$ is a Hilbert space with the inner product defined by

$$\langle f, g\rangle_{L^2} = \int_X \langle f(x), g(x)\rangle \, d\mu.$$

We shall write $L^2(X)$ when $\mathcal{H} = \mathbb{C}$.

2. **Weighted translation pre-frame operator**

Definition 2.1. *Let $f : X \to \mathcal{H}$ be a mapping. We say that f is weakly measurable if for each $h \in \mathcal{H}$, the mapping $x \mapsto \langle h, f(x)\rangle$ of X to $\mathbb{C}$ is measurable.*

Definition 2.2. *Let $f : X \to \mathcal{H}$ be weakly measurable. We say that f is a* c*-frame for $\mathcal{H}$, if there exist $0 < A \leq B < \infty$ such that*

$$A\|h\|^2 \leq \int_X |\langle h, f(x)\rangle|^2 \, d\mu \leq B\|h\|^2, \quad h \in \mathcal{H}.$$

If only the right hand inequality is satisfied, then we say that f is a c*-Bessel mapping for $\mathcal{H}$. Let $f : X \to \mathcal{H}$ be a* c*-Bessel for $\mathcal{H}$. Let $T_{\omega,\varphi,f} : L^2(X) \to \mathcal{H}$ be defined by*

$$\langle T_{\omega,\varphi,f}(g), h\rangle = \int_X \omega(x)(g \circ \varphi)(x)\langle f(x), h\rangle \, d\mu(x), \quad h \in \mathcal{H}, g \in L^2(X).$$

It is evident that $T_{\omega,\varphi,f}$ is well-defined and linear. For each $g \in L^2(X)$ and $h \in \mathcal{H}$, we have

$$\begin{aligned}
\|T_{\omega,\varphi,f}(g)\| &= \sup_{h\in\mathcal{H}_1} |\langle T_{\omega,\varphi,f}(g), h\rangle| = \sup_{h\in\mathcal{H}_1} \Big| \int_X \omega g\circ\varphi(x)\langle f(x), h\rangle \, d\mu \Big| \\
&\leq \Big(\int_X |\omega g\circ\varphi|^2 \, d\mu\Big)^{1/2} \sup_{h\in\mathcal{H}_1} \Big(\int_X |\langle f(x), h\rangle|^2 \, d\mu\Big)^{1/2} \\
&= \Big(\int_X E|\omega g\circ\varphi|^2 \, d\mu\Big)^{1/2} \sup_{h\in\mathcal{H}_1} \Big(\int_X |\langle f(x), h\rangle|^2 \, d\mu\Big)^{1/2} \\
&= \Big(\int_X |\ g\circ\varphi|^2 E(|\omega|^2) \, d\mu\Big)^{1/2} \sup_{h\in\mathcal{H}_1} \Big(\int_X |\langle f(x), h\rangle|^2 \, d\mu\Big)^{1/2} \\
&= \Big(\int_X |\ g|^2 h_\varphi E(|\omega|^2)\circ\varphi^{-1} \, d\mu\Big)^{1/2} \sup_{h\in\mathcal{H}_1} \Big(\int_X |\langle f(x), h\rangle|^2 \, d\mu\Big)^{1/2} \\
&\leq\ B^{\frac{1}{2}} \|g\|_2 \|J\|_\infty^{\frac{1}{2}}.
\end{aligned}$$

Consequently, $T_{\omega,\varphi,f}$ is bounded. We shall denote $T_{\omega,\varphi,f} : L^2(X) \to \mathcal{H}$ by

$$T_{\omega,\varphi,f}(g) = \int_X \omega g\circ\varphi f \, d\mu, \quad g \in L^2(X),$$

and we call it the weighted translation pre-frame operator of f. For each $g \in L^2(X)$ and $h \in \mathcal{H}$, by an application of the conditional expectation properties and the change of variable formula, we have

$$\begin{aligned}
\langle g, T^*_{\omega,\varphi,f}(h)\rangle = \langle T_{\omega,\varphi,f}(g), h\rangle &= \int_X \omega(x)(g\circ\varphi)(x)\langle f(x), h\rangle \, d\mu \\
&= \int_X E(\omega(x)(g\circ\varphi)(x)\langle f(x), h\rangle) \, d\mu \\
&= \int_X (g\circ\varphi)(x) E(\omega(x)\langle f(x), h\rangle) \, d\mu \\
&= \int_X g(x) h_\varphi E(\omega\,\langle f, h\rangle)\circ\varphi^{-1} \, d\mu \\
&= \Big\langle g, h_\varphi E(\overline{\omega}\,\langle h, f\rangle)\circ\varphi^{-1}\Big\rangle.
\end{aligned}$$

Thus, $T^*_{\omega,\varphi,f}(h) = h_\varphi E(\overline{\omega}\,\langle h, f\rangle)\circ\varphi^{-1}$. Also, for each $h \in \mathcal{H}$, we have

$$\|T^*_{\omega,\varphi,f}(h)\|^2 = \langle T^*_{\omega,\varphi,f}(h), T^*_{\omega,\varphi,f}(h)\rangle = \int_X |h_\varphi E(\overline{\omega}\,\langle h, f\rangle)\circ\varphi^{-1}|^2 \, d\mu.$$

Therefore, $\|T_{\omega,\varphi,f}\| = \|T^*_{\omega,\varphi,f}\| = \Big(\sup_{h\in\mathcal{H}_1} \int_X |h_\varphi E(\overline{\omega}\,\langle h, f\rangle)\circ\varphi^{-1}|^2 \, d\mu\Big)^{\frac{1}{2}}$. The mapping $T^*_{\omega,\varphi,f} : \mathcal{H} \to L^2(X)$ is called the weighted translation analysis operator of f. We define, $S_{\omega,\varphi,f} : \mathcal{H} \to \mathcal{H}$ by

$$\begin{aligned}
S_{\omega,\varphi,f}(h) = T_{\omega,\varphi,f} T^*_{\omega,\varphi,f}(h) &= T_{\omega,\varphi,f}\Big(h_\varphi E(\overline{\omega}\,\langle h, f\rangle)\circ\varphi^{-1}\Big) \\
&= \int_X \omega h_\varphi\circ\varphi E(\overline{\omega}\langle h, f\rangle) f \, d\mu.
\end{aligned}$$

and we call it the weighted translation frame operator of f.

Note that each element of $L^2(X, \mathcal{H})$ is a c-Bessel mapping for $\mathcal{H}$. Moreover, for each c-Bessel mapping $f : X \to \mathcal{H}$, it is not so difficult to verify that $S_{\omega,\varphi,f}$ is invertible if and only if $T_{\omega,\varphi,f}$ is surjective, like as done in [3, Theorem 2.5].

Theorem 2.3. *Suppose that a weight function ω is bounded away from zero and for each $x \in X$, the map $x \mapsto \overline{\omega(x)}\langle h, f(x)\rangle$ is $\varphi^{-1}\Sigma$-measurable. Then $f : X \to \mathcal{H}$ is a* c*-frame for $\mathcal{H}$ if and only if the operator $T_{\omega,\varphi,f}$ is a bounded and onto operator.*

Proof. Let f be c-frame. By Definition 2.2, it is clear that $T_{\omega,\varphi,f}$ is bounded. We have to prove only that $T_{\omega,\varphi,f}$ is onto. Since $h_\varphi \circ \varphi > 0$ almost everywhere, we may assume that $h_\varphi \circ \varphi \geq \delta$ for some $\delta > 0$. Further, the fact that ω is assumed to be bounded away from zero, allows us to write $|\omega| \geq c$ for some $c > 0$. Then, by using the change of variable formula, we have

$$\begin{aligned}
\|T^*_{\omega,\varphi,f}(h)\|^2 &= \int_X |h_\varphi E(\overline{\omega}\,\langle h, f\rangle) \circ \varphi^{-1}|^2\, d\mu = \int_X h_\varphi^2 |E(\overline{\omega}\,\langle h, f\rangle) \circ \varphi^{-1}|^2\, d\mu \\
&= \int_X h_\varphi |E(\overline{\omega}\,\langle h, f\rangle) \circ \varphi^{-1}|^2\, d\mu \circ \varphi^{-1} = \int_X h_\varphi \circ \varphi |E(\overline{\omega}\,\langle h, f\rangle)|^2\, d\mu \\
&\geq \delta \int_X E|\overline{\omega}\,\langle h, f\rangle|^2\, d\mu = \delta \int_X |\omega|^2\, |\langle h, f\rangle|^2\, d\mu \geq \delta c \int_X |\langle h, f\rangle|^2\, d\mu \geq \delta c A\|h\|^2.
\end{aligned}$$

Therefore, by [1, Lemma 2.4.1], $T_{\omega,\varphi,f}$ is onto.
Conversely, let $T_{\omega,\varphi,f}$ be a bounded and onto operator. By [1, Lemma 2.4.1], there exits $c > 0$ such that for each $h \in \mathcal{H}$, $c\|h\|^2 \leq \|T^*_{\omega,\varphi,f}(h)\|^2$. On the other hand, by the change of variable formula, we have

$$\begin{aligned}
c\|h\|^2 &\leq \|T^*_{\omega,\varphi,f}(h)\|^2 = \int_X |h_\varphi E(\overline{\omega}\,\langle h, f\rangle) \circ \varphi^{-1}|^2 d\mu \\
&= \int_X h_\varphi |E(\overline{\omega}\,\langle h, f\rangle) \circ \varphi^{-1}|^2 d\mu \circ \varphi^{-1} = \int_X h_\varphi \circ \varphi |E(\overline{\omega}\,\langle h, f\rangle)|^2 d\mu \\
&= \int_X h_\varphi \circ \varphi E(|\overline{\omega}\,\langle h, f\rangle|^2) d\mu \leqslant \|h_\varphi \circ \varphi\|_\infty \int_X |\overline{\omega}\,\langle h, f\rangle|^2 d\mu \\
&\leqslant \|h_\varphi \circ \varphi\|_\infty \|\omega\|^2_\infty \int_X |\langle h, f\rangle|^2 d\mu.
\end{aligned}$$

Since $\|h_\varphi \circ \varphi\|_\infty \|\omega\|^2_\infty > 0$, we get that $A\|h\|^2 \leq \int_X |\langle h, f\rangle|^2 d\mu$ for some constant $A > 0$. The only thing to be proved is that f is c-Bessel. For this, the change of variable formula and the properties of the conditional expectation are essentially used to obtain that

$$\begin{aligned}
\delta c \int_X |\langle h, f\rangle|^2\, d\mu &\leq \delta \int_X |\omega|^2\, |\langle h, f\rangle|^2\, d\mu = \delta \int_X E|\overline{\omega}\,\langle h, f\rangle|^2\, d\mu \\
&\leq \int_X h_\varphi \circ \varphi |E(\overline{\omega}\,\langle h, f\rangle)|^2\, d\mu = \int_X h_\varphi |E(\overline{\omega}\,\langle h, f\rangle) \circ \varphi^{-1}|^2 d\mu \circ \varphi^{-1} \\
&= \int_X h_\varphi |E(\overline{\omega}\,\langle h, f\rangle) \circ \varphi^{-1}|^2\, h_\varphi d\mu = \int_X |h_\varphi E(\overline{\omega}\,\langle h, f\rangle) \circ \varphi^{-1}|^2 d\mu \\
&= \|T^*_{\omega,\varphi,f}(h)\|^2 \leqslant \|T^*_{\omega,\varphi,f}\|^2 \|h\|^2.
\end{aligned}$$

Hence, $\int_X |\langle h, f\rangle|^2\, d\mu \leq B\|h\|^2$ for some $B > 0$, since $\|T^*_{\omega,\varphi,f}\| > 0$. □

Let $f : X \to \mathcal{H}$ be a c-Bessel mapping for $\mathcal{H}$ and let F be a measurable subset of X. Then, it is clear that $f\chi_F : X \to \mathcal{H}$ and $f\,|_F\colon F \to \mathcal{H}$ are c-Bessel mappings for $\mathcal{H}$. So, we can embed $L^2(F)$ in $L^2(X)$ as a closed subspace. Now, for each $h \in \mathcal{H}$ and each $g \in L^2(X)$, consider the following:

$$\int_X \omega g \circ \varphi(x)\langle f(x)\chi_F(x), h\rangle d\mu = \int_F \omega g \circ \varphi(x)\langle f(x), h\rangle\, d\mu.$$

According to this equation, the operators $T_{\omega,\varphi,f|_F}$ and $T_{\omega,\varphi,f\chi_F}$ can be uniquely determined in such a way that for any disjoint measurable subsets $F, G \subseteq X$, one has

$$T_{\omega,\varphi,f|_F} + T_{\omega,\varphi,f|_G} = T_{\omega,\varphi,f\chi_F} + T_{\omega,\varphi,f\chi_G} = T_{\omega,\varphi,f\chi_{(F\bigcup G)}} = T_{\omega,\varphi,f|(F\bigcup G)}.$$

Similar to [9, Theorem 3.1], for a c-Bessel mapping $f : X \to \mathcal{H}$ and a sequence of measurable subsets of X, namely $\{F_i\}_{i=1}^{\infty}$, we have $\lim_{n\to\infty} \|T_{\omega,\varphi,f|_{\bigcup_{i=1}^{n} F_i}}\| = \|T_{\omega,\varphi,f|_{\bigcup_{i=1}^{\infty} F_i}}\|$. Furthermore, for a sequence $\{F_i\}_{i=1}^{\infty}$ of pairwise disjoint and measurable subsets of X, $\sum_{i=1}^{\infty} T_{\omega,\varphi,f|_{F_i}} = T_{\omega,\varphi,f|_{\bigcup_{i=1}^{\infty} F_i}}$.

Theorem 2.4. *Let K be a Hilbert space, $f : X \to \mathcal{H}$ be a* c*-Bessel mapping for $\mathcal{H}$, and $u : \mathcal{H} \to K$ be a bounded linear mapping. Then:*

(i) *The mapping $uf : X \to K$ is a* c*-Bessel mapping for K , and*

$$uT_{\omega,\varphi,f} = T_{\omega,\varphi,uf}$$

(ii) *Suppose that a weight function ω is bounded away from zero and for each $x \in X$, the map $x \mapsto \overline{\omega(x)}\langle h, f(x)\rangle$ is $\varphi^{-1}\Sigma$-measurable. Let f be a* c*-frame for $\mathcal{H}$. Then, uf is a* c*-frame for K if and only if u is surjective.*

Proof. (i) Since

$$\sup_{h\in\mathcal{H}_1} \int_X |\langle h, u(f(x))\rangle|^2 d\mu \le \|u\|^2 \sup_{h\in\mathcal{H}_1} \int_X |\langle h, f(x)\rangle|^2 \, d\mu,$$

uf is a c-Bessel mapping for K. For each $g \in L^2(X)$, we have

$$\begin{aligned}\langle T_{\omega,\varphi,uf}(g), k\rangle &= \int_X \omega g \circ \varphi(x)\langle u(f(x)), k\rangle \, d\mu \\ &= \int_X \omega g \circ \varphi(x)\langle f(x), u^*(k)\rangle \, d\mu = \langle T_{\omega,\varphi,f}(g), u^*(k)\rangle = \langle uT_{\omega,\varphi,f}(g), k\rangle.\end{aligned}$$

Hence, $T_{\omega,\varphi,uf} = uT_{\omega,\varphi,f}$.
(ii) Suppose that u is surjective. By (i), it is clear that $T_{\omega,\varphi,uf}$ is also surjective. Hence, by Theorem 2.3, uf is a c-frame for K. Conversely, suppose that uf is a c-frame for K. Then by Theorem 2.3, $T_{\omega,\varphi,uf}$ is surjective and again by (i), u is clearly surjective. □

Proposition 2.5. *Let $f : X \to \mathcal{H}$ be a* c*-frame for $\mathcal{H}$. Then :*

(i) $\sup_{h\in\mathcal{H}_1} \|T^*_{\omega,\varphi,f}(h)\|^2 = \|S_{\omega,\varphi,f}\|$.

(ii) $\inf_{h\in\mathcal{H}_1} \|T^*_{\omega,\varphi,f}(h)\|^2 = \|S^{-1}_{\omega,\varphi,f}\|^{-1}$.

(iii) $\|S^{-1}_{\omega,\varphi,f}\|$ *and* $\|S_{\omega,\varphi,f}\|$ *are the optimal values satisfying*

$$\|S^{-1}_{\omega,\varphi,f}\|^{-1} \le S_{\omega,\varphi,f} \le \|S_{\omega,\varphi,f}\|.$$

Proof. By scrutinizing the proof of [9, Theorem 3.3], the desired results can be similarly proved. □

3. dual of *c*-Bessel mapping

Definition 3.1. *Let f, g be* c*-Bessel mappings for $h \in \mathcal{H}$. We say that f equals weakly to g whenever $T^*_{\omega,\varphi,f} = T^*_{\omega,\varphi,g}$, which is equivalent with $\langle h, f\rangle = \langle h, g\rangle$ a.e., for all $h \in \mathcal{H}$.*

Theorem 3.2. *Let f, g be* c*-Bessel mappings for $\mathcal{H}$. Then, the following assertions are equivalent:*

1) *For each $h \in \mathcal{H}$, $h = T_{\omega,\varphi,f}\left(\langle h, g \circ \varphi^{-1}\rangle\right)$.*
2) *For each $k \in \mathcal{H}$, $k = T_{\overline{\omega},\varphi,g}\left(\langle k, f \circ \varphi^{-1}\rangle\right)$.*
3) *For each $h, k \in \mathcal{H}$, $\langle h, k\rangle = \int_X \omega\langle h, g(x)\rangle\langle f(x), k\rangle d\mu$.*
4) *For each $h \in \mathcal{H}$, $\|h\|^2 = \int_X \omega\langle h, g(x)\rangle\langle f(x), h\rangle d\mu$.*
5) *For each orthonormal bases $\{\gamma_j\}_{j\in J}$ and $\{e_i\}_{i\in I}$ for $\mathcal{H}$,*

$$\langle e_i, \gamma_j\rangle = \int_X \omega\langle e_i, g(x)\rangle\langle f(x), \gamma_j\rangle d\mu \quad i \in I, j \in J.$$

6) *For each orthonormal bases* $\{e_i\}_{i\in I}$ *for* $\mathcal{H}$,

$$\int_X \omega\langle e_i, g(x)\rangle\langle f(x), e_j\rangle d\mu = \langle e_i, e_j\rangle \quad i,j\in I.$$

Proof. (1) → (2). Choose $h, k \in \mathcal{H}$ arbitrarily. Then,

$$\begin{aligned}\langle h,k\rangle &= \Big\langle T_{\omega,\varphi,f}(\langle h, g\circ\varphi^{-1}\rangle), k\Big\rangle = \int_X \omega\langle h, g(x)\rangle\langle f(x), k\rangle d\mu\\ &= \overline{\int_X \overline{\omega}\langle k, f(x)\rangle\langle g(x), h\rangle d\mu} = \overline{\Big\langle T_{\overline{\omega},\varphi,g}(\langle k, f\circ\varphi^{-1}\rangle), h\Big\rangle}\\ &= \Big\langle h, T_{\overline{\omega},\varphi,g}(\langle k, f\circ\varphi^{-1}\rangle)\Big\rangle.\end{aligned}$$

Hence, $k = T_{\overline{\omega},\varphi,g}(\langle k, f\circ\varphi^{-1}\rangle)$. The implication (2) → (3) is proved in a similar way. The proofs of the other implications can be approached like [9, Theorem 3.4] and we omit them. □

Definition 3.3. *Let* f, g *be* c*-Bessel mappings for* $\mathcal{H}$. *We say that* f, g *is a dual pair, if one of the assertions of Theorem 3.2 is satisfied.*

Remark 3.4. *Let* f, g *be a dual pair. Since every function in* $L^2(X)$ *is* c*-Bessel mapping, thus, for each* $h\in\mathcal{H}$, *we have*

$$\begin{aligned}\|h\|^2 &= \int_X \omega\langle h, g(x)\rangle\langle f(x), h\rangle\, d\mu \le \int_X |\omega\langle h, g(x)\rangle\langle f(x), h\rangle|\, d\mu\\ &\le \left(\int_X |\langle h, g(x)\rangle|^2\, d\mu\right)^{1/2}\left(\int_X |\omega\langle h, f(x)\rangle|^2\, d\mu\right)^{1/2}\\ &\le \left(\int_X |\langle h, g(x)\rangle|^2\, d\mu\right)^{1/2}\|h\|B^{\frac{1}{2}}\|\omega\|_\infty.\end{aligned}$$

Hence, g *is a* c*-frame for* $\mathcal{H}$.

Theorem 3.5. *Suppose that a weight function* ω *is bounded away from zero and for each* $x\in X$ *and* $h\in\mathcal{H}$, *the map* $x\mapsto \overline{\omega(x)}\langle h, f(x)\rangle$ *is* $\varphi^{-1}\Sigma$*-measurable. Let* f *be a* c*-frame for* $\mathcal{H}$. *Then the following arguments hold.*

1) *For each* $h\in\mathcal{H}$, *we find the following formulas*

$$h = T_{\omega,\varphi,S^{-1}_{\omega,\varphi,f}f}\Big(h_\varphi E(\overline{\omega}\,\langle h, f\rangle)\circ\varphi^{-1}\Big),$$

and

$$h = T_{\omega,\varphi,f}\Big(h_\varphi E(\overline{\omega}\,\langle S^{-1}_{\omega,\varphi,f}(h), f\rangle)\circ\varphi^{-1}\Big).$$

2) *In the formula* $h = T_{\omega,\varphi,f}\Big(h_\varphi E(\overline{\omega}\,\langle S^{-1}_{\omega,\varphi,f}(h), f\rangle)\circ\varphi^{-1}\Big)$, $h_\varphi E(\overline{\omega}\,\langle h, S^{-1}_{\omega,\varphi,f}f\rangle)\circ\varphi^{-1}$ *has the least norm among all of the retrieval formulas.*
3) *For each* $h\in\mathcal{H}$, $h = T_{\omega,\varphi,f}\langle h, g\circ\varphi^{-1}\rangle$ *if and only if there exists a* c*-Bessel mapping* l *for* $\mathcal{H}$ *such that* $g\circ\varphi^{-1} = S^{-1}_{\omega,\varphi,f}f + l$, *where for each* $k\in\mathcal{H}$, $\langle k, l\rangle \in ker(T_{\omega,\varphi,f})$.
4) *The map* f *has just one dual if and only if* $\mathcal{R}(T^*_{\omega,\varphi,f}) = L^2(X)$.

Proof. (1) Since f is c-frame, then by Theorem 2.3, $T_{\omega,\varphi,f}$ is onto and hence $S_{\omega,\varphi,f}$ is an invertible operator as argued earlier. Consequently, for each $h \in \mathcal{H}$, we obtain that

$$\begin{aligned} h &= S^{-1}_{\omega,\varphi,f} S_{\omega,\varphi,f}(h) = S^{-1}_{\omega,\varphi,f} T_{\omega,\varphi,f} T^*_{\omega,\varphi,f}(h) \\ &= T_{\omega,\varphi,S^{-1}_{\omega,\varphi,f}f}\Big(h_\varphi E(\overline{\omega}\,\langle h, f\rangle) \circ \varphi^{-1}\Big). \end{aligned}$$

Moreover, we have

$$\begin{aligned} h &= S_{\omega,\varphi,f} S^{-1}_{\omega,\varphi,f}(h) = T_{\omega,\varphi,f} T^*_{\omega,\varphi,f}\Big(S^{-1}_{\omega,\varphi,f}(h)\Big) \\ &= T_{\omega,\varphi,f}\Big(h_\varphi E(\overline{\omega}\,\langle S^{-1}_{\omega,\varphi,f}(h), f\rangle) \circ \varphi^{-1}\Big). \end{aligned}$$

(2) Choose $\phi \in L^2(X)$ and let $h = T_{\omega,\varphi,f}(\phi)$. Then, for each $g \in \mathcal{H}$, we have

$$\begin{aligned} \langle h, g\rangle &= \Big\langle T_{\omega,\varphi,f}\Big(h_\varphi E(\overline{\omega}\,\langle S^{-1}_{\omega,\varphi,f}(h), f\rangle) \circ \varphi^{-1}\Big), g\Big\rangle \\ &= \int_X \omega\Big(h_\varphi E(\overline{\omega}\,\langle S^{-1}_{\omega,\varphi,f}(h), f\rangle) \circ \varphi^{-1}\Big) \circ \varphi \langle f, g\rangle \, d\mu \\ &= \int_X \omega(x)(h_\varphi \circ \varphi)(x) E(\overline{\omega}\,\langle S^{-1}_{\omega,\varphi,f}(h), f(x)\rangle)\langle f(x), g\rangle \, d\mu. \end{aligned}$$

In the same way, we get that

$$\langle h, g\rangle = \langle T_{\omega,\varphi,f}(\phi), g\rangle = \int_X \omega(x)(\phi \circ \varphi)(x)\langle f(x), g\rangle \, d\mu.$$

Therefore,

$$\begin{aligned} \langle h, g\rangle - \langle h, g\rangle &= \Big\langle T_{\omega,\varphi,f}\Big(h_\varphi E(\overline{\omega}\langle S^{-1}_{\omega,\varphi,f}(h), f\rangle) \circ \varphi^{-1} - \phi\Big), g\Big\rangle \\ = \int_X \omega\Big(h_\varphi E(\overline{\omega}\langle S^{-1}_{\omega,\varphi,f}(h), f\rangle) &\circ \varphi^{-1} - \phi(x)\Big) \circ \varphi\langle f(x), g\rangle \, d\mu = 0. \end{aligned}$$

So, $T_{\omega,\varphi,f}\Big(h_\varphi E(\overline{\omega}\,\langle S^{-1}_{\omega,\varphi,f}(h), f\rangle) \circ \varphi^{-1} - \phi\Big) = 0$.
Eventually, $h_\varphi E(\overline{\omega}\,\langle S^{-1}_{\omega,\varphi,f}(h), f\rangle) \circ \varphi^{-1} - \phi \in ker(T_{\omega,\varphi,f})$. Since f is a c-Bessel mapping for $\mathcal{H}$, we obtain that $h_\varphi E(\overline{\omega}\,\langle S^{-1}_{\omega,\varphi,f}(h), f\rangle) \circ \varphi^{-1} \in \mathcal{R}(T^*_{\omega,\varphi,f})$. But, $L^2(X) = ker(T_{\omega,\varphi,f}) \oplus \mathcal{R}(T^*_{\omega,\varphi,f})$. Consequently,

$$\|\phi\|^2 = \|h_\varphi E(\overline{\omega}\,\langle S^{-1}_{\omega,\varphi,f}(h), f\rangle) \circ \varphi^{-1} - \phi\|^2 + \|h_\varphi E(\overline{\omega}\,\langle S^{-1}_{\omega,\varphi,f}(h), f\rangle) \circ \varphi^{-1}\|^2$$

and (2) is proved.
(3) Let g be a c-Bessel mapping for $\mathcal{H}$. For each $h \in \mathcal{H}$, assume that $h = T_{\omega,\varphi,f}\langle h, g \circ \varphi^{-1}\rangle$. Let $g \circ \varphi^{-1} - S^{-1}_{\omega,\varphi,f} f = l$. By Theorem 3.2, for each $h, k \in \mathcal{H}$, we have

$$\begin{aligned} \Big\langle T_{\omega,\varphi,f}\langle k, l\rangle, h\Big\rangle &= \Big\langle T_{\omega,\varphi,f}\langle k, g \circ \varphi^{-1}\rangle, h\Big\rangle - \Big\langle T_{\omega,\varphi,f}\langle k, S^{-1}_{\omega,\varphi,f} f\rangle, h\Big\rangle \\ = \int_X \omega\langle k, g\rangle\langle f, h\rangle - &\int_X \omega\langle k, S^{-1}_{\omega,\varphi,f} f \circ \varphi\rangle\langle f, h\rangle = \langle k, h\rangle - \langle k, h\rangle = 0. \end{aligned}$$

Hence, for each $k \in \mathcal{H}$, $\langle k, l\rangle \in \mathcal{R}(T^*_{\omega,\varphi,f})^\perp = ker(T_{\omega,\varphi,f})$. Now, let $g \circ \varphi^{-1} = S^{-1}_{\omega,\varphi,f} f + l$. Then, for each $h \in \mathcal{H}$, we have

$$\begin{aligned} \int_X \omega\langle f, h\rangle\langle k, g\rangle \, d\mu &= \int_X \omega\langle f, h\rangle, \langle k, (S^{-1}_{\omega,\varphi,f} f + l) \circ \varphi\rangle \, d\mu \\ = \int_X \omega(x)\langle f(x), h\rangle\langle k, S^{-1}_{\omega,\varphi,f} f \circ \varphi\rangle \, d\mu &+ \int_X \omega(x)\langle f(x), h\rangle\langle k, l \circ \varphi\rangle \, d\mu \\ = \langle k, h\rangle + \Big\langle T_{\omega,\varphi,f}\langle k, l\rangle, h\Big\rangle &= \langle k, h\rangle. \end{aligned}$$

Thus, by Theorem 3.2, $h = T_{\omega,\varphi,f}\langle h, g\circ\varphi^{-1}\rangle$.
(4) Let $\mathcal{R}(T^*_{\omega,\varphi,f}) \neq L^2(X)$. Pick $l \in \mathcal{R}(T^*_{\omega,\varphi,f})^\perp$ with $\|l\| = 1$. Consider the map $k : X \to L^2(X)$ defined by $k(x) = l\circ\varphi(x)l$. For each $t \in L^2(X)$, the map $X \to \mathbb{C}$, defined by the assignment $x \mapsto \langle t, k(x)\rangle$ is Σ-measurable and $\int_X |\langle t, k(x)\rangle|^2\, d\mu = \int_X |\langle t, l\circ\varphi(x)l\rangle|^2\, d\mu = \int_X |\langle t, l\rangle|^2 |l\circ\varphi(x)|^2\, d\mu = |\langle t, l\rangle|^2 \leq \|t\|^2$. Thus, k is a c-Bessel mapping for $L^2(X)$. Let $\upsilon : L^2(X) \to \mathcal{H}$ be a mapping such that $\upsilon(l) \neq 0$. Then υk is c-Bessel mapping for $\mathcal{H}$, and so $S^{-1}_{\omega,\varphi,f} f + \upsilon k$ is a c-Bessel mapping for $\mathcal{H}$. Let $h \in \mathcal{H}$. Since

$$\begin{aligned}
&\int_X \omega\langle h, S^{-1}_{\omega,\varphi,f} f(x) + \upsilon k(x)\rangle\langle f(x), h\rangle\, d\mu \\
&\qquad = \int_X \omega\langle h, S^{-1}_{\omega,\varphi,f} f(x)\rangle\langle f(x), h\rangle\, d\mu + \int_X \omega\langle h, \upsilon k(x)\rangle\langle f(x), h\rangle\, d\mu \\
&\qquad = \|h\|^2 + \langle \upsilon^*(h), l\rangle \int_X \omega\overline{l\circ\varphi(x)}\langle f(x), h\rangle\, d\mu \\
&\qquad = \|h\|^2 + \langle \upsilon^*(h), l\rangle\Big\langle T_{\omega,\varphi,f}(l), h\Big\rangle = \|h\|^2,
\end{aligned}$$

it is inferred that $S^{-1}_{\omega,\varphi,f} f + \upsilon k$ is the dual of f. Moreover, the equation

$$\langle \upsilon(l), \upsilon k(x)\rangle = \langle \upsilon(l), l\circ\varphi(x)\upsilon(l)\rangle = \overline{l\circ\varphi(x)}\langle \upsilon(l), \upsilon(l)\rangle$$

implies that $S^{-1}_{\omega,\varphi,f} f + \upsilon k$ is not weakly equal to $S^{-1}_{\omega,\varphi,f} f$. Conversely, suppose that $L^2(X) = \mathcal{R}(T^*_{\omega,\varphi,f})$. Now, put $g\circ\varphi^{-1} = S^{-1}_{\omega,\varphi,f} f + l$, where for each $k \in \mathcal{H}$, $\langle k, l\rangle \in ker(T_{\omega,\varphi,f}) = \mathcal{R}(T^*_{\omega,\varphi,f})^\perp = \{0\}$. Therefore, $l = 0$ weakly, so f has just one dual. $\square$

Example 3.6. *Let $X = [0, 2]$, Σ be the Lebesgue measurable subsets of X and μ be the Lebesgue measure on X. Also let $\varphi : X \to X$ be a non-singular measurable transformation with $h_\varphi \in L^\infty$ and let $f : X \to L^2(X)$ be a c-Bessel mapping for $L^2(X)$. For $A \subseteq X$, put $h = \chi_A$. Then, for $T_{\omega,\varphi,f} : L^2(X) \to L^2(X)$ the following is obtained*

$$\langle T_{\omega,\varphi,f}(g), \chi_A\rangle = \int_0^2 \int_A \omega f(x) d\nu\, g\circ\varphi d\mu,$$

where ν is the Lebesgue measure on A. Since $g\circ\varphi \in L^1(\mu)$ and $f(x) \in L^1(\nu)$, then by the Fubini's theorem, $f(x)g\circ\varphi \in L^1(\mu\times\nu)$ and

$$\int_A T_{\omega,\varphi,f}(g) d\nu = \int_A \int_0^2 \omega f(x) g\circ\varphi d\mu d\nu.$$

It follows that the formula of the bounded operator $T_{\omega,\varphi,f}$ on $L^2(X)$ is $T_{\omega,\varphi,f}(g) = \int_0^2 \omega f(x) g\circ\varphi d\mu$.

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