PROJECTIVE J-REPRESENTATIONS ASSOCIATED WITH PROJECTIVE u-COVARIANT (α)-COMPLETELY POSITIVE LINEAR MAPS AND THEIR CORRESPONDING ρ-MAPS

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In this paper we construct a Krein space, a J-representation and a projective J-unitary representation associated with a unital projective J-covariant completely positive linear map. We also find a Krein space representation for a unital projective (θ, u)-covariant α-completely positive linear map. For a given projective unital (θ, u)-covariant α-completely positive linear map ρ and a projective (τ, σ, u)-covariant ρ-map we construct a projective covariant J-representation of a C^*-dynamical system. We form projective J-unitary representations associated to a projective (u, u')-covariant completely positive ρ-maps. Also, we prove that there is a projective covariant representation associated with a projective covariant α-completely positive linear map on an S-module.

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1. Introduction

Krein spaces as indefinite generalization of Hilbert spaces were first used in the quantum field theory by Dirac [6] and Pauli [17] and then formally defined by Ginzburg [7]. Krein spaces arise naturally in situations where the indefinite inner product has an analitically useful property (such as Lorentz invariance) which the Hilbert inner product lacks. Motivated by the physical fact that in massless quantum field theory the state space may be a space with an indefinite metric, many authors extended the GNS construction to Krein spaces. Heo, Hong, Ji [9] provided a KSGNS type representation on a Krein C^*-module for a C^*-algebra and a $*$-algebra introducing the notion of α-completely positive map as a generalization of a completely positive map. Moreover, Heo and Ji [10] constructed a Stinespring type covariant representation for a pair of a covariant completely positive map ρ and a covariant ρ-map. In [13], Heo introduced the notion of a covariant α-completely positive map of a topological group into a (locally) C^*-algebra, which is a counterpart of a covariant α-completely positive linear map between (locally) C^*-algebra [9], [11] and constructed a covariant KSGNS type representation of a group on a

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Krein module over a (locally) C^*-algebra, which is associated to a covariant α-completely positive map of a group system. In [16], M. S. Moslehian, M. Joiţa and U. C. Ji proved a KSGNS type theorem for α-completely positive maps on Hilbert C^*-modules and showed that the minimal KSGNS construction is unique up to unitary equivalence and studied a covariant version of the KSGNS type theorem for a covariant α-completely positive map. S. Dey and H. Trivedi introduced in [5] S-modules, generalizing the notion of Krein C^*-modules, where a fixed unitary replaces the symmetry of Krein C^*-modules and proved a KSGNS construction for α-completely positive maps in the context of S-modules.

In this paper we construct a Krein space, a J-representation and a projective J-unitary representation associated with a unital projective J-covariant completely positive linear map. We also find a Krein space representation for a unital projective (θ, u)-covariant α-completely positive linear map. For a given projective unital (θ, u)-covariant α-completely positive linear map ρ and a projective (τ, σ, u)-covariant ρ-map we construct a projective covariant J-representation of a C^*-dynamical system. We form projective J-unitary representations associated to a projective (u, u')-covariant completely positive ρ-maps. Also, we prove that there is a projective covariant representation associated with a projective covariant α-completely positive linear map on an S-module.

First we remind and introduce some notions and definitions that we'll use in what follows.

Definition 1.1. [9] *Let H be a Hilbert space with the inner product $(\cdot|\cdot)$ which is linear in the second variable and conjugate linear in the first variable. A fundamental symmetry J on H (i.e. $J = J^*$, $J^2 = I$ sau $J = J^* = J^{-1}$) induces an indefinite inner product $[x, y]_J := (Jx|y)$ $(x, y \in H)$ and the pair (H, J) is called **Krein space** or **J-space**.*

For each $T \in \mathcal{L}(H)$, there is an operator $T^J \in \mathcal{L}(H)$ such that $[T\xi, \eta]_J = [\xi, T^J\eta]_J$, $\xi, \eta \in H$ and then T^J is called the **J-adjoint** of T. It can be easily seen that $T^J = JT^*J$.

Let B be a C^*-algebra and let (H, J) be a Krein space. We denote by $\mathcal{U}_J(H)$ the set of all J-unitary operators in $\mathcal{L}(H)$, i.e for each $s \in G$, $u_s^J u_s = u_s u_s^J = I$, which is equivalent to $u_s^* = Ju_{s^{-1}}J$ or $u_s^J = u_{s^{-1}}$. $\mathcal{U}_J(H)$ is called **the J-unitary group**.

Definition 1.2. *A **projective J-unitary representation** of a locally compact group G into the J-unitary group $\mathcal{U}_J(H)$ is a map $u\colon G \to \mathcal{U}_J(H)$ that satisfies the following properties:*

(i) $u_s^* = Ju_{s^{-1}}J$ *or* $u_s^J = u_{s^{-1}}$;
(ii) $u_{st} = \omega(s, t)u_s u_t$ *for all* $s, t \in G$

Definition 1.3. *Let (G, A, θ) be a C^*-dynamical system and let u be a projective J-unitary representation of G on a Krein space (H, J). We say that a completely*

positive linear map ρ from A to $\mathcal{L}(H)$ is ***projective J-covariant*** *with respect to the C^*-dynamical system (G, A, θ) if $\rho(\theta_s(a)) = u_s\rho(a)u_s^J$ for all $a \in A$ and $s \in G$.*

Definition 1.4. *([8]) Let A be a C^*-algebra and let (H, J) be a Krein space. A representation $\pi\colon A \to \mathcal{L}(H)$ of A on the Hilbert space H is called a* ***J-representation*** *on the Krein space (H, J) if π is a representation of A on the Hilbert space H and $\pi(a^*) = \pi(a)^J = J\pi(a)^*J,\ a \in A$.*

Definition 1.5. *A* ***projective covariant J-representation*** *of a C^*- dynamical system (G, A, θ) on a Krein space (H, J) is a triple $(\pi, u, (H, J))$, where π is a J-representation of A on (H, J) and u is a projective J-unitary representation of G into $\mathcal{U}_J(H)$ such that the (θ, u)-covariance property holds: $\pi(\theta_s(a)) = u_s\pi(a)u_s^J$ for all $a \in A$ and $s \in G$.*

2. Main results

Following the results in [4] , we rewrite and prove them for Krein spaces.

Theorem 2.1. *Let (G, A, θ) be a unital C^*-dynamical system such that $\theta_s^2 = I_A$(=the identity map on A), for all $s \in G$, (H, J) a Krein space and u a projective J-unitary representation of G on H with the normalized multiplier ω. If $\rho\colon A \to \mathcal{L}(H)$ is a unital projective J-covariant completely positive linear map, then there are a Krein space (K, J), a J-representation π of A on the Krein space (K, J), a projective J-unitary representation v of G into $\mathcal{U}_J(K)$ with multiplier ω and an isometry $V\colon H \to K$ such that:*

i) $\rho(a) = V^*\pi(a)V$ *for all* $a \in A$;
ii) $u_s = V^*v_sV$ *for all* $s \in G$;
iii) $\pi(\theta_s(a)) = v_s\pi(a)v_s^J,$ *for all* $a \in A$ *and* $s \in G$.

Proof. Following the proof of Stinespring's Theorem (Th.1.1.1, [1]), we form the algebraic tensor product $A \otimes_{alg} H$ and endow it with a pre-inner product by setting $\langle a \otimes \xi, b \otimes \eta\rangle_{A\otimes_{alg}H} = (\xi|\rho(\theta_s(a^*)b)\eta)_H$. To obtain K we divide $A \otimes_{alg} H$ by the kernel $N = \{z \in A\otimes_{alg}H | \langle z, z\rangle_{A\otimes_{alg}H} = 0\}$ of $\langle\cdot,\cdot\rangle_{A\otimes_{alg}H}$ and complete. K becomes a Hilbert space with respect to the inner product given by $\langle x_1 + N, x_2 + N\rangle_K = \langle x_1, x_2\rangle_{A\otimes_{alg}H}$, $x_1, x_2 \in A \otimes_{alg} H$.

The map θ induces a linear involution J on the quotient space $A \otimes_{alg} H/N$ by $J(a \otimes \xi + N) = \theta_s(a) \otimes \xi + N$.

We define an indefinite inner product $[\cdot,\cdot]_J$ on the quotient space $A \otimes_{alg} H/N$ by $[a \otimes \xi + N, b \otimes \eta + N]_J = (\xi|\rho(a^*b)\eta)_H$.

We have $\langle J(a \otimes \xi + N), b \otimes \eta + N\rangle_K = \langle \theta_s(a) \otimes \xi + N, b \otimes \eta + N\rangle_K = \langle \theta_s(a) \otimes \xi, b \otimes \eta\rangle_{A\otimes_{alg}H} = (\xi|\rho(\theta_s(\theta_s(a)^*)b)\eta)_H = (\xi|\rho(\theta_s(\theta_s(a^*))b)\eta)_H = (\xi|\rho(\theta_s^2(a^*)b)\eta)_H = (\xi|\rho(a^*b)\eta)_H = [a \otimes \xi + N, b \otimes \eta + N]_J$, for all $a, b \in A, \xi, \eta \in H, s \in G$. So (K, J) becomes a Krein space.

For all $a \in A$ we define a linear map $\pi(a)\colon K \to K$ by

$$\pi(a)(b \otimes \xi + N) = (ab) \otimes \xi + N,$$

for all $\xi \in H, b \in A$.

For $a_1, a_2, b \in A, \xi, \eta \in H$, we have $\langle \pi(b)(a_1 \otimes \xi + N), a_2 \otimes \eta + N\rangle_K =$
$= \langle ba_1 \otimes \xi + N, a_2 \otimes \eta + N\rangle_K = (\xi|\rho(\theta_s((ba_1)^* a_2)\eta)_H = (\xi|\rho(\theta_s(a_1^* b^*)a_2)\eta)_H$

On the other hand, $\langle a_1 \otimes \xi + N, J\pi(b^*)J(a_2 \otimes \eta + N)\rangle_K =$
$= \langle a_1 \otimes \xi + N, J\pi(b^*)(\theta_s(a_2) \otimes \eta + N)\rangle_K = \langle a_1 \otimes \xi + N, J(b^*\theta_s(a_2) \otimes \eta + N)\rangle_K =$
$= \langle a_1 \otimes \xi + N, \theta_s(b^*\theta_s(a_2)) \otimes \eta + N)\rangle_K = \langle a_1 \otimes \xi + N, \theta_s(b^*)\theta_s^2(a_2) \otimes \eta + N)\rangle_K =$
$= \langle a_1 \otimes \xi + N, \theta_s(b^*)a_2 \otimes \eta + N)\rangle_K = \langle a_1 \otimes \xi, \theta_s(b^*)a_2 \otimes \eta\rangle_{A \otimes_{alg} H} =$
$= (\xi|\rho(\theta_s(a_1^*)\theta_s(b^*)a_2)\eta)_H = (\xi|\rho(\theta_s(a_1^* b^*)a_2)\eta)_H$

Hence, $\pi(b)^* = J\pi(b^*)J$ and π is a J-representation.

We define a linear map $V\colon H \to K$ by $V\xi = 1_A \otimes \xi + N$.

For $\xi, \eta \in H$ and $a \in A$, we have $\langle V\xi, a \otimes \eta + N\rangle_K =$
$= \langle 1_A \otimes \xi + N, a \otimes \eta + N\rangle_K = (\xi|\rho(\theta_s(1_A^*)a)\eta)_H = (\xi|\rho(\theta_s(1_A)a)\eta)_H = (\xi|\rho(a)\eta)_H$
which implies that $V^*(a \otimes \eta + N) = \rho(a)\eta$, so V is an isometry.

For $\xi, \eta \in H$ and $a \in A$, we have

$(\xi|V^*\pi(a)V\eta)_H = (\xi|V^*\pi(a)(1_A \otimes \eta + N))_H = (\xi|V^*(a \otimes \eta + N))_H =$
$= (\xi|\rho(a)\eta)_H$, so $V^*\pi(a)V = \rho(a)$ and i) is verified.

We define $v\colon G \to \mathcal{U}_J(K)$ by setting

$$v_s(a \otimes \xi + N) = \theta_s(a) \otimes u_s\xi + N \text{ for all } a \in A, s \in G, \xi \in H.$$

For $a, b \in A$ and $\xi, \eta \in H$, we have $\langle v_s(a \otimes \xi + N), b \otimes \eta + N\rangle_K =$
$= \langle \theta_s(a) \otimes u_s\xi + N, b \otimes \eta + N\rangle_K = \langle \theta_s(a) \otimes u_s\xi, b \otimes \eta\rangle_{A \otimes_{alg} H} =$
$= (u_s\xi|\rho(\theta_s(\theta_s(a)^*)b)\eta)_H = (u_s\xi|\rho(\theta_s(\theta_s(a^*))b)\eta)_H = (u_s\xi|\rho(\theta_s^2(a^*)b)\eta)_H =$
$= (u_s\xi|\rho(a^*b)\eta)_H$

On the other hand, we have $\langle a \otimes \xi + N, Jv_{s^{-1}}J(b \otimes \eta + N)\rangle_K =$
$= \langle a \otimes \xi + N, Jv_{s^{-1}}(\theta_s(b) \otimes \eta + N)\rangle_K = \langle a \otimes \xi + N, J(\theta_{s^{-1}}(\theta_s(b)) \otimes u_{s^{-1}}\eta + N)\rangle_K =$
$= \langle a \otimes \xi + N, J(b \otimes u_{s^{-1}}\eta + N)\rangle_K = \langle a \otimes \xi + N, \theta_s(b) \otimes u_{s^{-1}}\eta + N\rangle_K =$
$= (\xi|\rho(\theta_s(a^*)\theta_s(b))u_{s^{-1}}\eta)_H = (\xi|\rho(\theta_s(a^*b))u_{s^{-1}}\eta)_H = (\xi|u_s^*\rho(a^*b)\eta)_H = (u_s\xi|\rho(a^*b)\eta)_H$

Therefore, $v_s^* = Jv_{s^{-1}}J$, which means that v_s is a J-unitary representation.

We show now that v is a projective representation with multiplier ω. Let $a \in A, s, t \in G, \xi \in H$. Since θ is a group homomorphism and u is a projective representation with the multiplier ω, we have $v_{st}(a \otimes \xi + N) =$
$= \theta_{st}(a) \otimes u_{st}\xi + N = \theta_s(a)\theta_t(a) \otimes \omega(s,t)u_su_t\xi + N =$
$= \omega(s,t)\theta_s(\theta_t(a)) \otimes u_s(u_t\xi) + N = \omega(s,t)v_s(\theta_t(a) \otimes u_t\xi + N) =$
$= \omega(s,t)v_sv_t(a \otimes \xi + N)$.

We verify now condition ii). Let $s \in G$ and $\xi \in H$. We have $V^*v_sV\xi =$
$= V^*v_s(1_A \otimes \xi + N) = V^*(\theta_s(1_A) \otimes u_s\xi + N) = V^*(1_A \otimes u_s\xi + N) = \rho(1_A)u_s\xi =$
$= I_Hu_s\xi = u_s\xi$, because ρ is unital.

We prove condition iii). Let $a, b \in A, s \in G, \xi \in H$.

Then $v_s\pi(a)v_s^J(b \otimes \xi + N) = v_s\pi(a)v_{s^{-1}}(b \otimes \xi + N) =$
$= v_s\pi(a)(\theta_{s^{-1}}(b) \otimes u_{s^{-1}}\xi + N) = v_s(a\theta_{s^{-1}}(b) \otimes u_{s^{-1}}\xi + N) =$
$= \theta_s(a\theta_{s^{-1}}(b)) \otimes u_su_{s^{-1}}\xi + N) = \theta_s(a)\theta_s(\theta_{s^{-1}}(b)) \otimes \overline{\omega(s,s^{-1})}u_{ss^{-1}}\xi + N =$
$= \theta_s(a)b \otimes I_H\xi + N = \theta_s(a)b \otimes \xi + N = \pi(\theta_s(a))(b \otimes \xi + N)$. □

Definition 2.1. *Let (G, A, θ) be a C^*-dynamical system and let u be a projective unitary representation of G on a Hilbert space H. We say that an α-completely positive linear map φ from A to $\mathcal{L}(H)$ is* ***projective*** *(θ, u)-**covariant** with respect to the C^*-dynamical system (G, A, θ) if $\varphi(\alpha_s(a)) = u_s\varphi(a)u_s^*$ for all $a \in A$ and $s \in G$.*

Theorem 2.2. *Let (G, A, θ) be a unital C^*-dynamical system such that $\theta_s^2 = I_A$, let u be a projective unitary representation of G on a Hilbert space H with the normalized multiplier ω. If $\rho\colon A \to \mathcal{L}(H)$ is a unital projective (θ, u)-covariant α-completely positive linear map, where θ and α are equivariant, then there are a Krein space (K, J), a J-representation π of A on the Krein space (K, J), a projective J-unitary representation v of G into $\mathcal{U}_J(K)$ with multiplier ω and an isometry $V\colon H \to K$ such that:*

i) $\rho(a) = V^*\pi(a)V$ *for all* $a \in A$;
ii) $u_s = V^*v_sV$ *for all* $s \in G$;
iii) $\pi(\theta_s(a)) = v_s\pi(a)v_s^J$, *for all* $a \in A$ *and* $s \in G$.

Proof. Following the proof of Theorem 2.1 and endow the algebraic tensor product $A \otimes_{alg} H$ with a pre-inner product by setting

$$\langle a \otimes \xi, b \otimes \eta \rangle_{A\otimes_{alg}H} = (\xi|\rho(\alpha(a^*)b)\eta)_H.$$

We divide $A\otimes_{alg}H$ by the kernel $N = \{z \in A\otimes_{alg}H | \langle z, z\rangle_{A\otimes_{alg}H} = 0\}$ of $\langle\cdot,\cdot\rangle_{A\otimes_{alg}H}$ and complete and thus we obtain K, which becomes a Hilbert space with respect to the inner product given by $\langle x_1 + N, x_2 + N\rangle_K = \langle x_1, x_2\rangle_{A\otimes_{alg}H}$, $x_1, x_2 \in A \otimes_{alg} H$.

The map α induces a linear involution J on the quotient space $A \otimes_{alg} H/N$ by $J(a \otimes \xi + N) = \alpha(a) \otimes \xi + N$.

We define an indefinite inner product $[\cdot,\cdot]_J$ on the quotient space $A \otimes_{alg} H/N$ by $[a \otimes \xi + N, b \otimes \eta + N]_J = (\xi|\rho(a^*b)\eta)_H$.

We have $\langle J(a \otimes \xi + N), b \otimes \eta + N\rangle_K = \langle \alpha(a) \otimes \xi + N, b \otimes \eta + N\rangle_K =$
$= \langle \alpha(a) \otimes \xi, b \otimes \eta\rangle_{A\otimes_{alg}H} = (\xi|\rho(\alpha(\alpha(a)^*)b)\eta)_H = (\xi|\rho(\alpha^2(a^*)b)\eta)_H =$
$= (\xi|\rho(a^*b)\eta)_H = [a\otimes\xi+N, b\otimes\eta+N]_J$, for all $a, b \in A, \xi, \eta \in H$. So (K, J) becomes a Krein space.

We define a linear map $\pi(a)\colon A\otimes_{alg} H \to A\otimes_{alg} H$ as in the proof of Theorem 2.1: $\pi(a)(b \otimes \xi) = (ab) \otimes \xi$ for all $a, b \in A$ and $\xi \in H$.

We have $\langle\pi(a)(b \otimes \xi), \pi(a)(b \otimes \xi)\rangle_{A\otimes_{alg}H} = \langle(ab) \otimes \xi, (ab) \otimes \xi\rangle_{A\otimes_{alg}H} =$
$= (\xi|\rho(\alpha((ab)^*)ab)\xi)_H = (\xi|\rho(\alpha(ab)^*ab)\xi)_H \leq c(a)(\xi|\rho(\alpha(b)^*b)\xi)_H =$
$= c(a)\langle b \otimes \xi, b \otimes \xi\rangle_{A\otimes_{alg}H}$

Therefore $\pi(a)$ leaves N-invariant and naturally define a linear transformation on $A \otimes_{alg} H/N$. Since $\pi(a)$ is bounded, $\pi(a)$ extends to a bounded linear operator on K and it is denoted also by $\pi(a)$.

We prove that π is a J-representation. Let $a_1, a_2, b \in A$ and $\xi, \eta \in H$. We have $\langle\pi(b)(a_1 \otimes \xi + N), a_2 \otimes \eta + N\rangle_K = \langle ba_1 \otimes \xi + N, a_2 \otimes \eta + N\rangle_K =$
$= (\xi|\rho(\alpha((ba_1)^*)a_2)\eta)_H = (\xi|\rho(\alpha(a_1^*b^*)a_2)\eta)_H$.

On the other hand, $\langle a_1 \otimes \xi + N, J\pi(b^*)J(a_2 \otimes \eta + N)\rangle_K =$
$= \langle a_1 \otimes \xi + N, J\pi(b^*)(\alpha(a_2) \otimes \eta + N)\rangle_K = \langle a_1 \otimes \xi + N, J(b^*\alpha(a_2) \otimes \eta + N)\rangle_K =$
$= \langle a_1 \otimes \xi + N, \alpha(b^*\alpha(a_2)) \otimes \eta + N)\rangle_K = \big\langle a_1 \otimes \xi + N, \alpha(b^*)\alpha^2(a_2) \otimes \eta + N)\big\rangle_K =$
$= \langle a_1 \otimes \xi + N, \alpha(b^*)a_2 \otimes \eta + N)\rangle_K = (\xi|\rho(\alpha(a_1^*)\alpha(b^*)a_2)\eta)_H = (\xi|\rho(\alpha(a_1^*b^*)a_2)\eta)_H.$
This means that $\pi(b)^* = J\pi(b^*)J$.

We define V as in Theorem 2.1 and prove i) as in the proof of Theorem 2.1.

We define $v\colon G \to \mathcal{U}_J(K)$ by setting

$$v_s(a \otimes \xi + N) = \theta_s(a) \otimes u_s\xi + N \text{ for all } a \in A, s \in G, \xi \in H.$$

Let $a, b \in A$, $s \in G$ and $\xi, \eta \in H$. We have $\langle v_s(a \otimes \xi + N), b \otimes \eta + N)\rangle_K =$
$= \langle \theta_s(a) \otimes u_s\xi + N, b \otimes \eta + N)\rangle_K = (u_s\xi|\rho(\alpha(\theta_s(a)^*)b)\eta)_H =$
$= (u_s\xi|\rho(\alpha(\theta_s(a^*)b)\eta)_H = (u_s\xi|\rho(\alpha(\theta_s(a^*))\alpha(\alpha(b)))\eta)_H =$
$= (u_s\xi|\rho(\alpha(\theta_s(a^*)\alpha(b)))\eta)_H = (u_s\xi|\rho(\theta_s(a^*)\alpha(b))\eta)_H =$
$= (\xi|u_{s^{-1}}\rho(\theta_s(a^*)\alpha(b))\eta)_H = (\xi|u_{s^{-1}}\rho(\alpha(\theta_s(a^*)\alpha(b)))\eta)_H =$
$= (\xi|u_{s^{-1}}\rho(\alpha(\theta_s(a^*))\alpha(\alpha(b))))\eta)_H = (\xi|u_{s^{-1}}\rho(\theta_s(\alpha(a^*))b)\eta)_H =$
$= (\xi|u_{s^{-1}}\rho(\theta_s(\alpha(a^*)\theta_{s^{-1}}(b))\eta)_H$

On the other hand, $\langle a \otimes \xi + N, Jv_{s^{-1}}J(b \otimes \eta + N)\rangle_K =$
$= \langle a \otimes \xi + N, Jv_{s^{-1}}(\alpha(b) \otimes \eta + N)\rangle_K = \langle a \otimes \xi + N, J(\theta_{s^{-1}}(\alpha(b)) \otimes u_{s^{-1}}\eta + N)\rangle_K =$
$= \langle a \otimes \xi + N, \alpha(\theta_{s^{-1}}(\alpha(b))) \otimes u_{s^{-1}}\eta + N)\rangle_K = (\xi|\rho(\alpha(a^*)\alpha(\theta_{s^{-1}}(\alpha(b))))u_{s^{-1}}\eta)_H =$
$= (\xi|\rho(\alpha(a^*\theta_{s^{-1}}(\alpha(b))))u_{s^{-1}}\eta)_H = (\xi|\rho(a^*\theta_{s^{-1}}(\alpha(b)))u_{s^{-1}}\eta)_H =$
$= (\xi|\rho(a^*\alpha(\theta_{s^{-1}}(b)))u_{s^{-1}}\eta)_H = (\xi|\rho(\alpha(\alpha(a^*))\alpha(\theta_{s^{-1}}(b)))u_{s^{-1}}\eta)_H =$
$= (\xi|\rho(\alpha(\alpha(a^*)\theta_{s^{-1}}(b)))u_{s^{-1}}\eta)_H = (\xi|\rho(\alpha(a^*)\theta_{s^{-1}}(b))u_{s^{-1}}\eta)_H$

By the covariance of ρ, we get that

$$\langle v_s(a \otimes \xi + N), b \otimes \eta + N)\rangle_K = \langle a \otimes \xi + N, Jv_{s^{-1}}J(b \otimes \eta + N)\rangle_K,$$

so v is a J-representation.

As in the proof of Theorem 2.1 we can prove that v is a projective representation and that ii) and iii) hold.

□

We rewrite the definition in [12] for projective unitary representations and Krein spaces.

Definition 2.2. *Let (G, A, α) be a C^*-dynamical system and let (H, J_H) be a Krein space. A group homomorphism τ from G into $\mathcal{U}_{J_H}(H)$ such that for any $s \in G, a \in A$ and $\xi, \xi' \in H$,*

i) $\tau_s(\xi a) = \tau_s(\xi)\theta_s(a)$
ii) $[\tau_s(\xi), \tau_s(\xi')]_{J_H} = \theta_s([\xi, \xi']_{J_H})$

*is called a θ-**compatible action** of G on (H, J_H). Let (K, J_K) and (L, J_L) be Krein spaces. For a θ-compatible action τ of G on (H, J_H) and a map $\Phi\colon H \to L(K, L)$, if there are a projective J_K- unitary representation $v\colon G \to \mathcal{U}_{J_K}(K)$ and a projective J_L-unitary representation $\sigma\colon G \to \mathcal{U}_{J_L}(L)$ such that $\Phi(\tau_s(\xi)) = \sigma_s\Phi(\xi)v_s^{J_K}$ for any $\xi \in H$ and $s \in G$, then Φ is called **projective** (τ, σ, v)-**covariant**.*

In the following theorem we construct a projective covariant representation associated to a pair of two projective covariant maps as in [12].

Theorem 2.3. *Let H, K, L be Hilbert spaces, (G, A, θ) a C^*-dynamical system such that $\theta_s^2 = I_A$ and $u\colon G \to \mathcal{U}(H)$ a projective unitary representation with the normalized multiplier ω. If $\rho\colon A \to \mathcal{L}(K)$ is a unital projective (θ, u)-covariant α-completely positive map, where θ and α are equivariant and if $\Phi\colon H \to \mathcal{L}(K, L)$ is a projective (τ, σ, v)-covariant ρ-map, then there is a pair $((\pi, V, (K_1, J)), (\Pi, W, L_1))$ such that*

i) *(K_1, J) is a Krein space and L_1 is a Hilbert space*
ii) *$\pi\colon A \to \mathcal{L}(K_1)$ is a J-representation*
iii) *$\Pi\colon H \to \mathcal{L}(K_1, L_1)$ is a $J \circ \pi$-map*
iv) *$V \in \mathcal{L}(K, K_1)$ is an isometry and $W \in \mathcal{L}(L, L_1)$ is a projection satisfying conditions (i)-(iii) in Theorem 4.4, [9] and $\Phi(\xi) = W^*\Pi(\xi)V$ for all $\xi \in H$.*

Moreover, there is a projective J-unitary representation v and a map $\sigma'\colon G \to \mathcal{U}(L_1)$ such that

(1) *$(\pi, v, (K_1, J))$ is a projective covariant J-representation of (G, A, θ)*
(2) *Π is projective (τ, σ', v)-covariant.*

Proof. By Theorem 2.2, there are a Krein space K_1, a J-representation $\pi\colon A \to \mathcal{L}(K_1)$, a projective J-unitary representation $v\colon G \to \mathcal{U}_J(K)$ and an operator $V \in \mathcal{L}(K, K_1)$ such that conditions (i)-(iii) in Theorem 2.2 hold.

We prove the existence of L_1, Π, W, σ' as in the proof of Theorem 3.2, [12]. □

We remind some notions and remarks in [2], [18] and [16].

A **morphism of Hilbert C^*-modules** [2] or a generalized isometry [18] is a map $\psi\colon E \to F$ from a Hilbert A-module E to a Hilbert B-module F with the property that there is a C^*-morphism $\varphi\colon A \to B$ such that $\langle \psi(x), \psi(y)\rangle = \varphi(\langle x, y\rangle)$ for all $x, y \in E$. A map $\psi\colon E \to F$ is an **isomorphism of Hilbert C^*-modules** if it is invertible, ψ and ψ^{-1} are morphisms of Hilbert C^*-modules.

Let G be a locally compact group. A continuous action of G on a full Hilbert A-module E is a group morphism $\eta\colon G \to \mathrm{Aut}(E)$, where $\mathrm{Aut}(E)$ is the group of all isomorphisms of Hilbert C^*-modules from E to E, such that the map $(t, x) \longmapsto \eta_t(x)$ from $G \times E$ to E is continuous. The triple (G, E, η) is called a **dynamical system on Hilbert C^*-modules**. Any C^*-dynamical system (G, A, θ) can be regarded as a dynamical system on Hilbert C^*-modules. Any continuous action η of G on E induces a unique continuous action θ^η of G on A such that $\theta_s^\eta(\langle x, y\rangle) = \langle \eta_s(x), \eta_s(y)\rangle$ for all $x, y \in E, s \in G$ [16].

Definition 2.3. *Let u and u' be two projective unitary representations of G on the Hilbert spaces H and K, E a Hilbert module and (G, E, η) a dynamical system. A ρ-map $\Phi\colon E \to \mathcal{L}(H, K)$ is **projective (u', u)-covariant** with respect to (G, E, η) if $\Phi(\eta_s(x)) = u'_s\Phi(x)u_s^*$ for all $x \in E, s \in G$ and $\theta_s^\eta \circ \alpha =$*
$= \alpha \circ \theta_s^\eta$ for all $s \in G$.

Remark 2.1. *Clearly, if $\rho\colon E \to \mathcal{L}(H)$ is a completely positive map projective u-covariant with respect to the C^*-dynamical system (G, A, θ), then it is projective (u, u)-covariant with respect to the dynamical system on Hilbert C^*-modules (G, A, θ).*

Remark 2.2. *Let $\Phi\colon E \to \mathcal{L}(H, K)$ be a ρ-map. If Φ is projective (u', u)-covariant with respect to (G, E, η), then ρ is projective u-covariant with respect to θ^η, which means that $\rho(\theta^\eta_s(a)) = u_s\rho(a)u_s^*$ for all $a \in A, s \in G$.*

Definition 2.4. *Let (G, E, η) be a dynamical system on Hilbert C^*-modules. A **projective covariant representation** of (G, E, η) is a quadruple (π, v, w, H, K) consists of two Hilbert spaces H and K, a representation $\pi\colon E \to \mathcal{L}(H, K)$, a projective unitary representation v of G on H, a projective unitary representation w of G on K such that $\pi(\eta_s(x)) = w_s\pi(x)v_s^*$ for all $x \in E, s \in G$.*

Clearly, any projective covariant representation of a C^*-dynamical system (G, A, θ) is a projective covariant representation of (G, A, θ) regarded as dynamical system on Hilbert C^*-modules.

We prove now the projective version of Theorem 3.4 in [15] in terms of Krein spaces.

Theorem 2.4. *Let E be a Hilbert C^*-module over a C^*-algebra A, (G, E, η) be a dynamical system (H, J_1) and (K, J_2) two Krein spaces, $u\colon G \to \mathcal{L}(H)$ a projective J_1-unitary representation, $u'\colon G \to \mathcal{L}(K)$ a projective J_2-unitary representation with the normalized multiplier ω, $\rho\colon A \to \mathcal{L}(H)$ a unital projective J_1-covariant completely positive map with respect to (G, A, α^η) and $\Phi\colon E \to \mathcal{L}(H, K)$ a unital completely positive map projective (u, u')-covariant ρ-map with respect to (G, E, η). Then there are two Krein spaces (H_Φ, J_1) and (K_Φ, J_2), a J_1-representation $\pi\colon A \to \mathcal{L}(H_\Phi)$, v^Φ a projective J_1-unitary representation of G on H_Φ, w^Φ a projective J_2-unitary representation of G on K_Φ, a $J_1 \circ \pi$-map $\pi_\Phi\colon E \to \mathcal{L}(H_\Phi, K_\Phi)$, an isometry $V_\Phi\colon H \to H_\Phi$ and a coisometry $W_\Phi\colon K \to K_\Phi$ such that*

(a) $\Phi(\xi) = W_\Phi^*\pi_\Phi(\xi)V_\Phi$ *for all* $\xi \in E$
(b) $v_s^\Phi V_\Phi = V_\Phi u_s$ *for all* $s \in G$
(c) $w_s^\Phi W_\Phi = W_\Phi u'_s$ *for all* $s \in G$
(d) $[\pi_\Phi(E)V_\Phi H] = K_\Phi$
(e) $[\pi_\Phi(E)^*W_\Phi K] = H_\Phi$

Proof. By Theorem 2.1, there are a Krein space (H_Φ, J_1), a J_1-representation $\pi\colon A \to \mathcal{L}(H_\Phi)$, a projective J_1-unitary representation v^Φ of G on H_Φ and an isometry $V_\Phi\colon H \to H_\Phi$ such that

i) $\rho(a) = V_\Phi^*\pi(a)V_\Phi$ for all $a \in A$;
ii) $u_s^\Phi = V_\Phi^* v_s^\Phi V_\Phi$ for all $s \in G$;
iii) $\pi(\alpha_s^\eta(a)) = v_s^\Phi\pi(a)(v_s^\Phi)^J$, for all $a \in A$ and $s \in G$.

By Theorem 2.1 and Theorem 2.3, following the proof of Theorem 3.4, [15], we complete the proof. □

Definition 2.5. *([5]) Let $(E, \langle\cdot,\cdot\rangle)$ be a Hilbert C^*-module over a C^*-algebra A and let U be a unitary on E, i.e. U is an invertible adjointable map on E such that $U^* = U^{-1}$. We define an A-valued sesquilinear form by $[x, y] = \langle x, Uy\rangle$ for all $x, y \in E$. We say that (E, A, U) is an S-module. If $U = I$, then $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ coincide for the S-module (E, A, U). When $U = U^*$, the S-module (E, A, U) forms a Krein A-module.*

Let (E_1, B, U_1) and (E_2, B, U_2) be two S-modules, where E_1 and E_2 are Hilbert C^*-modules over a C^*-algebra B. For each $T \in \mathcal{L}(E_1, E_2)$ there is an operator $T^\sharp \in \mathcal{L}(E_2, E_1)$ such that $\langle T(x), U_2 y\rangle = \langle x, U_1 T^\sharp(y)\rangle$ for all $x \in E_1, y \in E_2$. In fact $T^\sharp = U_1^* T^* U_2$.

Suppose A is a C^*-algebra and (E, B, U) is an S-module. A homomorphism $\pi\colon A \to \mathcal{L}(E)$ is called an U-**representation** of A on (E, B, U) if $\pi(a^*) = U^*\pi(a)^*U = \pi(a)^\sharp$, i.e. $[\pi(a)x, y] = [x, \pi(a^*)y]$ for all $x, y \in E$.

Definition 2.6. *A **projective U-unitary representation** with the multiplier ω of a locally compact group G into $\mathcal{U}(E)$ is a map $v\colon G \to \mathcal{U}(E)$ that satisfies the following properties:*

i) $v_s^* = U v_{s^{-1}} U$ or $v_s^U = v_{s^{-1}}$

ii) $v_{st} = \omega(s, t) v_s v_t$ for all $s, t \in G$.

Definition 2.7. *A **projective covariant U-representation** of a C^*- dynamical system (G, A, θ) on an S-module (E, B, U) is a triple $(\pi, v, (E, B, U))$, where π is an U-representation of A on (E, B, U) and v is a projective U-unitary representation of G into $\mathcal{U}(E)$ such that $\pi(\theta_s(a)) = v_s \pi(a) v_s^U$ for all $a \in A$ and $s \in G$.*

Definition 2.8. *Let (G, A, θ) be a C^*-dynamical system, B a C^*-algebra, (E, B, U) an S-module and v a projective U-unitary representation of G into $\mathcal{U}(E)$. We say that a completely positive linear map ρ from A to $\mathcal{L}(E)$ is **projective U-covariant** with respect to the C^*-dynamical system (G, A, θ) if $\rho(\theta_s(a)) = v_s \rho(a) v_s^U$ for all $a \in A$ and $s \in G$.*

Theorem 2.5. *Let A and B two unital C^*-algebras and $\alpha\colon A \to A$ an automorphism. Let (E, B, U) be an S-module, (G, A, θ) be a C^*-dynamical system and v a projective U-unitary representation of G on E with the normalized multiplier ω. If $\rho\colon A \to \mathcal{L}(E)$ is a unital projective U-covariant α-completely positive map, then there are*

(i) *a Hilbert B-module E_0 and a unitary U_0 such that (E_0, B, U_0) is an S-module;*

(ii) *a map $V \in \mathcal{L}(E, E_0)$ such that $V^\sharp = V^*$, an U_0-representation π_0 of A on (E_0, B, U_0) satisfying $V^*\pi_0(a)^*\pi_0(b)V = V^*\pi_0(\alpha(a)^*b)V$ for all $a, b \in A$ and $\rho(a) = V^*\pi_0(a)V$ for all $a \in A$;*

(iii) *a projective U_0-unitary representation w of G into $\mathcal{U}(E_0)$ with the multiplier ω;*

(iv) *$\pi(\theta_s(a)) = w_s \pi(a) w_s^{U_0}$ for all $a \in A$ and $s \in G$.*

Proof. (i) Following the proof of Lemma 4.1, [9], we form the algebraic tensor product $A \otimes_{alg} E$ and endow it with a pre-inner product by setting

$$\left\langle \sum_{i=1}^{n} a_i \otimes x_i, \sum_{j=1}^{m} a'_j \otimes y_j \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \langle x_i, \rho(\alpha(a_i^*)a'_j)y_j \rangle$$

for all $a_1, a_2, \ldots, a_n, a'_1, a'_2, \ldots, a'_m \in A, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m \in E$.

By Cauchy-Schwarz inequality for positive-definite sesquilinear forms we observe that $N = \{\sum_{i=1}^{n} a_i \otimes x_i \in A \otimes_{alg} E / \sum_{i,j=1}^{n} \langle x_i, \rho(\alpha(a_i^*)a_j)x_j \rangle = 0\}$ is a submodule of $A \otimes_{alg} E$. Therefore $\langle \cdot, \cdot \rangle$ induces naturally on the quotient module $A \otimes_{alg} E/N$ as a B-valued inner product. We denote this inner product also by $\langle \cdot, \cdot \rangle$. To obtain E_0 we divide $A \otimes_{alg} E$ by the kernel N and complete and E_0 becomes a Hilbert B-module.

We define U_0 by $U_0(\sum_{i=1}^{n} a_i \otimes x_i + N) = \sum_{i=1}^{n} \alpha(a_i) \otimes Ux_i + N$ for all $a_i \in A, x_i \in E, i = \overline{1,n}$.

For all $a_i, a'_j \in A, x_i, y_j \in E, i = \overline{1,n}, j = \overline{1,m}$ we have

$$\left\langle U_0(\sum_{i=1}^{n} a_i \otimes x_i + N), U_0(\sum_{j=1}^{m} a'_j \otimes y_j + N) \right\rangle =$$

$$= \left\langle \sum_{i=1}^{n} \alpha(a_i) \otimes Ux_i + N, \sum_{j=1}^{m} \alpha(a'_j) \otimes Uy_j + N \right\rangle =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle Ux_i, \rho(\alpha(\alpha(a_i)^*)\alpha(a'_j))Uy_j \rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \langle x_i, \rho(\alpha(a_i)^*)a'_j)y_j \rangle =$$

$$= \left\langle \sum_{i=1}^{n} a_i \otimes x_i + N, \sum_{j=1}^{m} a'_j \otimes y_j + N \right\rangle, \text{ so } U_0 \text{ is unitary.}$$

Since $\left\langle U_0(\sum_{i=1}^{n} a_i \otimes x_i + N), \sum_{j=1}^{m} a'_j \otimes y_j + N \right\rangle =$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle \alpha(a_i) \otimes Ux_i + N, a'_j \otimes y_j + N \rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \langle Ux_i, \rho(\alpha(\alpha(a_i)^* a'_j))y_j \rangle =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle x_i, \rho(\alpha(a_i^*)\alpha^{-1}(a'_j))U^*y_j \rangle = \left\langle \sum_{i=1}^{n} a_i \otimes x_i + N, \sum_{j=1}^{m} \alpha^{-1}(a'_j) \otimes U^*y_j + N \right\rangle,$$

we get $U_0^*(\sum_{j=1}^{m} a'_j \otimes y_j + N) = \sum_{j=1}^{m} \alpha^{-1}(a'_j) \otimes U^*y_j + N$

(ii) We define $V\colon E \to E_0$ by $Vx = 1_A \otimes Ux + N, \ \ x \in E$.

$\|Vx\|^2 = \| \langle Vx, Vx \rangle \| = \| \langle 1_A \otimes Ux + N, 1_A \otimes Ux + N \rangle \| =$

$= \| \langle Ux, \rho(1_A)Ux \rangle \| \leq \|\rho(1_A)\| \cdot \|x\|^2$, then V is bounded.

For each $a_1, a_2, \ldots, a_n \in A$ and $x, y_1, y_2, \ldots, y_n \in E$, we have

$$\left\langle Vx, \sum_{i=1}^{n} a_i \otimes y_i + N \right\rangle = \left\langle 1_A \otimes Ux + N, \sum_{i=1}^{n} a_i \otimes y_i + N \right\rangle =$$

$$= \left\langle Ux, \sum_{i=1}^{n} \rho(\alpha(1_A)a_i)y_i \right\rangle = \left\langle x, \sum_{i=1}^{n} U^*\rho(a_i)y_i \right\rangle = \left\langle x, \sum_{i=1}^{n} \rho(a_i)U^*y_i \right\rangle \quad (1)$$

From Lemma 2.8, [9], there is $M > 0$ such that $\rho(a_i^*)\rho(a_j) \leq M\rho(\alpha(a_i^*)a_j)$. Hence, for $a_1, a_2, \ldots, a_n \in A$, $y_1, y_2, \ldots, y_n \in E$, we have

$$\|\sum_{i=1}^{n} \rho(a_i)U^*y_i\|^2 = \| \left\langle \sum_{i=1}^{n} \rho(a_i)U^*y_i, \sum_{j=1}^{n} \rho(a_j)U^*y_j \right\rangle \| =$$

$$= \| \sum_{i,j=1}^{n} \langle U^*y_i, \rho(a_i^*)\rho(a_j)U^*y_j \rangle \| \leq M\| \sum_{i,j=1}^{n} \langle U^*y_i, \rho(\alpha(a_i^*)a_j)Uy_j \rangle \| =$$

$$= M\|\sum_{i=1}^{n} a_i \otimes y_i + N\|^2 \quad (2)$$

By (1) and (2) it results that V is an adjointable map with the adjoint $V^*(\sum_{i=1}^{n} a_i \otimes x_i + N) = \sum_{i=1}^{n} \rho(a_i)U^*x_i$, $a_i \in A, x_i \in E, i = \overline{1,n}$.

Let $a_1, a_2, \ldots, a_n \in A$, $x_1, x_2, \ldots, x_n \in E$. We have

$$V^\sharp(\sum_{i=1}^{n} a_i \otimes x_i + N) = U^*V^*U_0(\sum_{i=1}^{n} a_i \otimes x_i + N) =$$

$$= U^*V^*(\sum_{i=1}^{n} \alpha(a_i) \otimes Ux_i + N) = U^*(\sum_{i=1}^{n} \rho(\alpha(a_i))U^*Ux_i) = U^*(\sum_{i=1}^{n} \rho(\alpha(a_i))x_i) =$$

$$= V^*(\sum_{i=1}^{n} a_i \otimes x_i + N).$$ This implies that $V^\sharp = V^*$.

We define $\pi_0' \colon A \to \mathcal{L}(E_0)$ by $\pi_0'(a)(\sum_{i=1}^{n} a_i \otimes x_i + N) = \sum_{i=1}^{n} aa_i \otimes x_i + N$ for all $a, a_1, a_2, \ldots, a_n \in A, x_1, x_2, \ldots, x_n \in E$.

We have $\|\pi_0'(a)(\sum_{i=1}^{n} a_i \otimes x_i + N)\|^2 = \|\sum_{i=1}^{n} aa_i \otimes x_i + N\|^2 =$

$$= \| \left\langle \sum_{i=1}^{n} aa_i \otimes x_i + N, \sum_{j=1}^{n} aa_j \otimes x_j + N \right\rangle \| = \| \sum_{i,j=1}^{n} x_i, \rho(\alpha(a_i^*a^*)aa_j)x_j \| \leq$$

$$\leq M(a)\| \sum_{i,j=1}^{n} \langle x_i, \rho(\alpha(a_i^*)a_j)x_j \rangle \| = M(a)\|\sum_{i=1}^{n} a_i \otimes x_i + N\|^2$$ for all $a, a_1, a_2, \ldots, a_n \in A, x_1, x_2, \ldots, x_n \in E$.

Thus for each $a \in A$, $\pi_0'(a)$ is a well defined bounded linear operator from E_0 to E_0.

Using $\left\langle \pi_0'(a)(\sum_{i=1}^{n} a_i \otimes x_i + N), \sum_{j=1}^{m} a_j' \otimes x_j' + N \right\rangle =$

$= \left\langle \sum_{i=1}^{n} aa_i \otimes x_i + N, \sum_{j=1}^{m} a_j' \otimes x_j' + N \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \langle x_i, \rho(\alpha(a_i^* a^*)a_j')x_j' \rangle =$

$= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle x_i, \rho(\alpha(a_i^*)\alpha(a^*)a_j')x_j' \rangle = \left\langle \sum_{i=1}^{n} a_i \otimes x_i + N, \sum_{j=1}^{m} \alpha(a^*)a_j' \otimes x_j' + N \right\rangle$

$U_0\pi_0'(a^*)U_0^*(\sum_{j=1}^{m} a_j' \otimes x_j' + N) = U_0\pi_0'(a^*)(\sum_{j=1}^{m} \alpha^{-1}(a_j') \otimes U^* x_j' + N) =$

$= U_0(\sum_{j=1}^{m} (a^*\alpha^{-1}(a_j')) \otimes U^* x_j' + N) = \sum_{j=1}^{m} \alpha(a^*)a_j' \otimes x_j' + N$ for all $a_1, a_2, \ldots, a_n$, $a_1', a_2', \ldots, a_m' \in A$ and $x_1, x_2, \ldots, x_n, x_1', x_2', \ldots, x_m' \in E$ and it follows that $\pi_0' \colon A \to \mathcal{L}(E_0)$ is a well defined map and is an U_0-representation.

We define an U_0-representation $\pi_0 \colon A \to \mathcal{L}(E_0)$ by $\pi_0(a) = \pi_0'(\alpha(a))$ for all $a \in A$.

We have $V^\sharp \pi_0'(a)Vx = V^*(a \otimes Ux + N) = U^*\rho(a)Ux = \rho(a)x$. Therefore $\rho(a) = V^\sharp \pi_0'(a)V$ for all $a \in A$.

For each $x \in E$ and $a, b \in A$, we get $V^*\pi_0'(a)^*\pi_0'(b)Vx =$

$= V^*U_0\pi_0'(a^*)U_0^*\pi_0'(b)Vx = V^*U_0\pi_0'(a^*)U_0^*\pi_0'(b)(1_A \otimes Ux + N) =$

$= V^*U_0\pi_0'(a^*)U_0^*(b \otimes Ux + N) = V^*U_0\pi_0'(a^*)(\alpha^{-1}(b) \otimes U^*Ux + N) =$

$= V^*U_0\pi_0'(a^*)(\alpha^{-1}(b) \otimes x + N) = V^*U_0(a^*\alpha^{-1}(b) \otimes x + N) =$

$= V^*(\alpha(a^*\alpha^{-1}(b)) \otimes Ux + N) = \rho(\alpha(a^*\alpha^{-1}(b)))U^*Ux =$

$= \rho(\alpha(a)^*b)x = V^*\pi_0'(\alpha(a)^*b)Vx$

We have $V^*\pi_0(a)^*\pi_0(b)Vx = V^*\pi_0'(\alpha(a))^*\pi_0'(\alpha(b))Vx =$

$= V^*\pi_0'(\alpha(\alpha(a))^*\alpha(b))Vx = V^*\pi_0'(\alpha(\alpha(a)^*b))Vx = V^*\pi_0(\alpha(a)^*b)Vx$ for all $a, b \in A$ and $x \in E$.

(iii) We define $w \colon G \to \mathcal{U}(E_0)$ by $w_s(\sum_{i=1}^{n} a_i \otimes x_i + N) = \sum_{i=1}^{n} \theta_s^\eta(a_i) \otimes v_s x_i + N$ for all $a_i \in A, x_i \in E, i = \overline{1,n}, s \in G$.

We have $\left\langle w_s(\sum_{i=1}^{n} a_i \otimes x_i + N), \sum_{j=1}^{m} b_j \otimes y_j + N \right\rangle =$

$= \left\langle \sum_{i=1}^{n} \theta_s^\eta(a_i) \otimes v_s x_i + N, \sum_{j=1}^{m} b_j \otimes y_j + N \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \langle v_s x_i, \rho(\alpha(\theta_s^\eta(a)^*)b_j)y_j \rangle =$

$= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle v_s x_i, \rho(\alpha(\theta_s^\eta(a^*))b_j)y_j \rangle$

On the other hand, $\left\langle \sum_{i=1}^{n} a_i \otimes x_i + N, U_0 w_{s^{-1}} U_0(\sum_{j=1}^{m} b_j \otimes y_j + N) \right\rangle =$

$= \left\langle \sum_{i=1}^{n} a_i \otimes x_i + N, U_0 w_{s^{-1}}(\sum_{j=1}^{m} \alpha(b_j) \otimes U y_j + N) \right\rangle =$

$$= \left\langle \sum_{i=1}^{n} a_i \otimes x_i + N, U_0(\sum_{j=1}^{m} \theta^{\eta}_{s^{-1}}(\alpha(b_j)) \otimes v_{s^{-1}} U y_j + N) \right\rangle =$$

$$= \left\langle \sum_{i=1}^{n} a_i \otimes x_i + N, \sum_{j=1}^{m} \alpha(\theta^{\eta}_{s^{-1}}(\alpha(b_j))) \otimes U v_{s^{-1}} U y_j + N \right\rangle =$$

$$= \left\langle \sum_{i=1}^{n} a_i \otimes x_i + N, \sum_{j=1}^{m} \alpha(\alpha(\theta^{\eta}_{s^{-1}}(b_j))) \otimes U v_{s^{-1}} U y_j + N \right\rangle =$$

$$= \left\langle \sum_{i=1}^{n} a_i \otimes x_i + N, \sum_{j=1}^{m} \theta^{\eta}_{s^{-1}}(b_j) \otimes U v_{s^{-1}} U y_j + N \right\rangle =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} \left\langle a_i \otimes x_i + N, \theta^{\eta}_{s^{-1}}(b_j) \otimes v_s^* y_j + N \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \left\langle x_i, \rho(\alpha(a_i^*)\theta^{\eta}_{s^{-1}}(b_j)) v_s^* y_j \right\rangle =$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \left\langle x_i, v_s^* \rho(\theta^{\eta}_s(\alpha(a_i^*)\theta^{\eta}_{s^{-1}}(b_j))) v_{s^{-1}}^* v_s^* y_j \right\rangle =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} \left\langle x_i, v_s^* \rho(\theta^{\eta}_s(\alpha(a_i^*))b_j)\omega(s^{-1}, s) v_{ss^{-1}}^* y_j \right\rangle =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle x_i, v_s^* \rho(\theta^{\eta}_s(\alpha(a_i^*))b_j) y_j \rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} \langle x_i, v_s^* \rho(\alpha(\theta^{\eta}_s(a_i^*))b_j) y_j \rangle$$

For all $a_i \in A, x_i \in E, i = \overline{1,n}, s, t \in G$, we have

$$w_{st}(\sum_{i=1}^{n} a_i \otimes x_i + N) = \sum_{i=1}^{n} \theta^{\eta}_{st}(a_i) \otimes v_{st} x_i + N =$$

$$= \sum_{i=1}^{n} \theta^{\eta}_s \theta^{\eta}_t(a_i) \otimes \omega(s,t) v_s v_t x_i + N = \omega(s,t) w_s(\sum_{i=1}^{n} \theta^{\eta}_t(a_i) \otimes v_t x_i + N) =$$

$$= \omega(s,t) w_s w_t(\sum_{i=1}^{n} a_i \otimes x_i + N)$$

Thus, we proved that w is a projective U_0-unitary representation.

(iv) For all $a, a_i \in A, x_i \in E, i = \overline{1,n}, s \in G$, we have

$$\pi_0(\theta^{\eta}_s(a))(\sum_{i=1}^{n} a_i \otimes x_i + N) = \pi_0'(\alpha(\theta^{\eta}_s(a)))(\sum_{i=1}^{n} a_i \otimes x_i + N) = \sum_{i=1}^{n} \alpha(\theta^{\eta}_s(a)) a_i \otimes x_i + N$$

On the other hand, $w_s \pi_0(a) w_s^{U_0}(\sum_{i=1}^{n} a_i \otimes x_i + N) =$

$$= w_s \pi_0(a) w_{s^{-1}}(\sum_{i=1}^{n} a_i \otimes x_i + N) = w_s \pi_0'(\alpha(a))(\sum_{i=1}^{n} \theta^{\eta}_{s^{-1}}(a_i) \otimes v_{s^{-1}} x_i + N) =$$

$$= w_s(\sum_{i=1}^{n} \alpha(a)\theta^{\eta}_{s^{-1}}(a_i) \otimes v_{s^{-1}} x_i + N) = \sum_{i=1}^{n} \theta^{\eta}_s(\alpha(a)\theta^{\eta}_{s^{-1}}(a_i)) \otimes v_s v_{s^{-1}} x_i + N =$$

$$= \sum_{i=1}^{n} \theta^{\eta}_s(\alpha(a)) a_i \otimes x_i + N = \sum_{i=1}^{n} \alpha(\theta^{\eta}_s(a)) a_i \otimes x_i + N$$

Therefore, condition (iv) is verified. □

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