φ - CONTRACTIVE ORBITAL AFFINE ITERATED FUNCTION SYSTEMS

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In this paper, we introduce the notion of φ-contractive orbital affine iterated function system (oAIFS for short), which is based on the notions of affine iterated function system and φ-contractive orbital iterated function system. We present two results which give a description of the functions of an oAIFS and establish sufficient conditions to exist a norm with specific properties on the linear spaces where the functions are defined. Two examples are provided.

Keywords: attractor, affine iterated function systems, comparison function, orbital iterated function systems.

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1. Introduction

The concept of iterated function system (IFS for short) was introduced by J. Hutchinson in [9] and in the last decades, many generalizations of this concept have been considered. For example, there have been studied IFSs consisting of φ-contractions (see [10]), convex contractions (see [7]), systems with an infinite number of functions (see [6], [8], [11]), graph-directed Markov systems (see [13]) and others. Also, important contibutions in the study of IFSs are presented in [12] and [22]. For certain types of IFSs, the fractal operator associated is Picard (see [15]), but there have been introduced and studied IFSs for which the fractal operator is weakly Picard. For the last case, let us mention the orbital iterated function systems, which have been studied largely in [16], [17] and [20]. An orbital iterated function system is a finite family of continuous functions defined on a metric space (X, d) having the property that on the orbit of every element $x \in X$, the functions are contractions with the same contractivity constant. It was proved (see [16]) that the fractal operator associated to such a system is weakly Picard.

An important type of IFS is represented by the affine iterated function systems. For example, in [1], the authors studied the hyperbolic affine iterated function systems and in [14], R. Miculescu and A. Mihail gave an alternative characterization of hyperbolic affine infinite iterated function systems defined on an arbitrary normed space. Moreover, in [4], the authors studied an application of affine iterated function systems, namely equilibrium states of generalized singular value potentials. Affine iterated function systems have been also studied in [2], [3], [5], [18], [19] and [23]).

In this paper, we use the notions of affine iterated function system and φ-contractive orbital iterated function system in order to introduce the notion of φ-contractive orbital affine iterated function system (oAIFS for short). This is an affine iterated function system defined on the normed space $(\mathbb{R}^n, \|\cdot\|)$, consisting of a finite family of functions $(f_i)_{i\in I}$ having

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the property that for every $i \in I$, there exist $\widetilde{A}_i \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $\widetilde{a}_i \in \mathbb{R}^n$ such that $f_i(x) = \widetilde{A}_i x + \widetilde{a}_i$ for all $x \in \mathbb{R}^n$. Moreover, there exists a comparison function $\varphi\colon [0,\infty) \to [0,\infty)$ such that $d(f_i(y), f_i(z)) \le \varphi(d(y,z))$ for every $x \in \mathbb{R}^n$, $y, z \in \mathcal{O}(x)$ and $i \in I$. We denote such a system by $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$. We present a result (see Theorem 3.1) which states that there exist two linear subspaces $Y, Z \subset \mathbb{R}^n$ such that $Y + Z = \mathbb{R}^n$, $Y \cap Z = \{0_{\mathbb{R}^n}\}$ and for every $i \in I$, there exist $B_i \in L(Z,Z)$, $C_i \in L(Y,Z)$ and $b_i \in Z$ such that $\widetilde{A}_i = \begin{bmatrix} I_Y & O_{Z,Y} \\ C_i & B_i \end{bmatrix}$ and $\widetilde{a}_i = \begin{bmatrix} 0_Y \\ b_i \end{bmatrix}$. We also prove that there exists a norm $\|\cdot\|_Z$ on Z such that $\|B_i\|_Z < c$ for all $i \in I$. Moreover, we prove that we can find a norm $\|\cdot\|_\theta\colon \mathbb{R}^n \to [0,\infty)$, such that $\left\|\widetilde{A}_i\right\|_\theta \le 1$ and $\|B_i\|_\theta < c$ for all $i \in I$ (see Theorem 3.2). Two examples are provided.

2. Preliminaries

Notations and terminology

Given a set X, a function $f\colon X \to X$ and $n \in \mathbb{N}^*$, by f^n we mean $f \circ f \circ \cdots \circ f$ by n times. By f^0 we mean the identity function.

Given a metric space (X,d), by:

- diam (A) we mean the diameter of the subset A of X;
- $P_{cp}(X)$ we mean the set of non-empty compact subsets of X;
- the Hausdorff-Pompeiu metric we mean $h\colon P_{cp}(X) \times P_{cp}(X) \to [0,\infty)$ given by $h(A,B) = \max\{d(A,B), d(B,A)\}$ for all $A, B \in P_{cp}(X)$, where $d(A,B) = \sup\limits_{x\in A}\inf\limits_{y\in B} d(x,y)$;
- a weakly Picard operator we mean a function $f\colon X \to X$ having the property that for every $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent to a fixed point of f.

Results regarding the Hausdorff-Pompeiu metric

Proposition 2.1 (see [21]). *For a metric space (X,d), we have*

$$h\left(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i\right) \le \sup_{i\in I} h(A_i, B_i) \tag{1}$$

for every $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ families of elements from $P_{cp}(X)$.

Proposition 2.2 (see [21]). *For a metric space (X,d), we have*

$$h(A,B) \le \operatorname{diam}(A \cup B)$$

for every $A, B \in P_{cp}(X)$.

Proposition 2.3 (see [21]). *If the metric space (X,d) is complete, then the metric space $(P_{cp}(X), h)$ is complete.*

Let Y, $Z \subset \mathbb{R}^n$ be two real linear spaces. By $Y + Z$ we mean $\{y + z \mid y \in Y,\ z \in Z\}$ and by $L(Y,Z)$ we denote the space of linear applications from Y to Z.

By 0_Y we mean the null vector from Y and by I_Y we mean the identity function from $L(Y,Y)$. By $\dim Y$ we mean the dimension of the space Y.

By $O_{Y,Z}$ we denote the linear application from Y to Z which, applied to every element from Y, is equal to 0_Z. If $Y = Z$, $O_{Y,Y}$ will be denoted by O_Y.

If $\|\cdot\|_Y$ is a norm on Y and $A \in L(Y,Y)$, by $\|A\|_Y$ we mean $\sup\limits_{y\in Y, y\neq 0_Y} \frac{\|Ay\|_Y}{\|y\|_Y}$.

Let Y, $Z \subset \mathbb{R}^n$ be two real linear spaces such that $Y + Z = \mathbb{R}^n$ and $Y \cap Z = \{0_{\mathbb{R}^n}\}$. If $x \in \mathbb{R}^n$, then there exist a unique $y \in Y$ and a unique $z \in Z$ such that $y + z = x$. In this case, we make the notation $x = \begin{bmatrix} y \\ z \end{bmatrix}$. Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $x = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^n$. We make the notation $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where $A_{11} \in L(Y,Y)$, $A_{12} \in L(Z,Y)$, $A_{21} \in L(Y,Z)$ and $A_{22} \in L(Z,Z)$ are defined by $Ay = \begin{bmatrix} A_{11}y \\ A_{21}y \end{bmatrix}$ and $Az = \begin{bmatrix} A_{12}z \\ A_{22}z \end{bmatrix}$ for every $y \in Y$ and $z \in Z$. Let us note that $Ax = \begin{bmatrix} A_{11}y \\ A_{21}y \end{bmatrix} + \begin{bmatrix} A_{12}z \\ A_{22}z \end{bmatrix}$ for every $x = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^n$.

The shift (code) space

Given two sets A and B, by B^A we mean the set of all functions from A to B.

For a set I and $n \in \mathbb{N}^*$, we use the notation $I^{\{1,2,\cdots,n\}} \overset{not}{=} \Lambda_n(I)$. If $n = 0$, then $\Lambda_0(I)$ consists of a single element, namely the empty word, denoted by λ.

For $n \in \mathbb{N}^*$, the elements of $\Lambda_n(I)$ are finite words with n letters from I, namely $\alpha = \alpha_1\alpha_2\cdots\alpha_n$. In this case, n is called the length of α and it is denoted by $|\alpha|$. For $\alpha \in \Lambda_n(I)$ and $m \in \mathbb{N}^*$, by $[\alpha]_m$ we mean the word formed with the first m letters from α if $m \le n$, or the word α if $n < m$. By $[\alpha]_0$ we mean the word λ.

Let $n \in \mathbb{N}^*$. For a family of functions $(f_i)_{i\in I}$, where $f_i\colon X \to X$ for all $i \in I$ and $\alpha = \alpha_1\alpha_2\cdots\alpha_n \in \Lambda_n(I)$, we use the notation $f_\alpha = f_{\alpha_1} \circ \cdots \circ f_{\alpha_n}$. By f_λ we mean the identity function.

Let Y, $Z \subset \mathbb{R}^n$ be two real linear spaces, $(B_i)_{i\in I} \subset L(Y,Z)$, $m \in \mathbb{N}^*$ and $\alpha = i_1i_2\cdots i_m \in \Lambda_m(I)$. We use the notation $B_\alpha = B_{i_1} \circ B_{i_2} \circ \cdots \circ B_{i_m}$.

By $\Lambda^*(I)$ we mean the set of all finite words with letters from I, namely $\Lambda^*(I) = \bigcup_{n\in\mathbb{N}} \Lambda_n(I)$.

By $\Lambda(I)$ we mean the set $I^{\mathbb{N}^*}$. The elements of $\Lambda(I)$ can be written as infinite words, namely $\alpha = \alpha_1\alpha_2\cdots\alpha_n\cdots$. For $\alpha \in \Lambda(I)$ and $n \in \mathbb{N}^*$, by α_n we mean the letter on position n in α. By $[\alpha]_n$ we mean the word formed with the first n letters from α. By $[\alpha]_0$ we mean λ.

Orbital iterated function systems

Definition 2.1. *Let (X,d) be a complete metric space and $(f_i)_{i\in I}$ a finite family of continuous functions, with $f_i\colon X \to X$ for all $i \in I$. Let $B \in P_{cp}(X)$. By the orbit of B we mean the set $\mathcal{O}(B) = \bigcup_{n\in\mathbb{N}} \bigcup_{\alpha\in\Lambda(I)} f_{[\alpha]_n}(B)$. If $B = \{x\}$, for the orbit of $\{x\}$ we make the notation $\mathcal{O}(x)$.*

Definition 2.2. *A function $\varphi\colon [0,\infty) \to [0,\infty)$ is called comparison function if $\varphi(r) < r$ for all $r > 0$ and φ is increasing and right continuous on $[0,\infty)$.*

We note that if $\varphi\colon [0,\infty) \to [0,\infty)$ is a comparison function, then $\lim_{n\to\infty} \varphi^n(r) = 0$ for all $r > 0$.

Definition 2.3. *By an iterated function system (IFS) we mean a pair denoted by $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$, where (X,d) is a complete metric space and $(f_i)_{i\in I}$ is a finite family of continuous functions, with $f_i\colon X \to X$ for all $i \in I$. If there exists $\varphi\colon [0,\infty) \to [0,\infty)$ a comparison function such that*

$$d(f_i(y), f_i(z)) \le \varphi(d(y,z))$$

for every $i \in I$ and $y,z \in X$, $\mathcal{S}$ is called φ-contractive iterated function system. If there exists a constant $C \in [0,1)$ such that $\varphi(t) = C \cdot t$ for all $t \in [0,\infty)$, $\mathcal{S}$ is called C-contractive iterated function system.

Definition 2.4. *By a φ-contractive orbital iterated function system (oIFS for short) we mean a pair denoted by $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$, where (X,d) is a complete metric space and $(f_i)_{i\in I}$ is a finite family of continuous functions, with $f_i\colon X \to X$ for all $i \in I$, having the property that there exists $\varphi\colon [0,\infty) \to [0,\infty)$ a comparison function such that*

$$d\left(f_i(y), f_i(z)\right) \le \varphi\left(d(y,z)\right)$$

for every $x \in X$, $i \in I$ and $y, z \in \mathcal{O}(x)$. If there exists $C \in [0,1)$ such that $\varphi(t) = C \cdot t$ for all $t \in [0,\infty)$, $\mathcal{S}$ is called C-contractive orbital iterated function system.

Definition 2.5. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIFS. The fractal operator associated to $\mathcal{S}$ is the function $F_{\mathcal{S}}\colon P_{cp}(X) \to P_{cp}(X)$ defined by $F_{\mathcal{S}}(K) = \underset{i\in I}{\cup} f_i(K)$, for every $K \in P_{cp}(X)$.*

Definition 2.6. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an IFS. By an attractor of $\mathcal{S}$ we mean a fixed point of the fractal operator associated to $\mathcal{S}$. We say that $\mathcal{S}$ has a unique attractor if there exists a set denoted by A such that $\lim\limits_{n\to\infty} h\left(F_S^n(K), A\right) = 0$ and $F_S(A) = A$ for every $K \in P_{cp}(X)$.*

Theorem 2.1 (see [17])**.** *Every oIFS has at least one attractor. More precisely, the associated fractal operator is weakly Picard.*

Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIFS. Then, for every $K \in P_{cp}(X)$, there exists a set (an attractor) corresponding to K, denoted by $A_K \in P_{cp}(X)$, such that $\lim\limits_{n\to\infty} h\left(F_S^n(K), A_K\right) = 0$ and $F_S(A_K) = A_K$. If $K = \{x\}$, we will denote its corresponding attractor by A_x.

Remark 2.1. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIFS. As for every $x \in X$ the sequence $\left(F_{\mathcal{S}}^n(\{x\})\right)_{n\in\mathbb{N}}$ is convergent to A_x, we have $\overline{\mathcal{O}(x)} = \mathcal{O}(x) \cup A_x$. We note that $\mathcal{O}(x)$ is bounded for every $x \in X$.*

Using a technique similar with the one used in [17] and [20], one can prove the following:

Proposition 2.4. *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIFS, $(K_n)_{n\in\mathbb{N}}$ a sequence with $K_n \in P_{cp}(X)$ for all $n \in \mathbb{N}$ and $K \in P_{cp}(X)$ such that $\lim\limits_{n\to\infty} h(K_n, K) = 0$.*

Then, $\lim\limits_{n\to\infty} h\left(A_{K_n}, A_K\right) = 0$.

Proposition 2.5 (see [17])**.** *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIFS. Then, $A_B = A_x$ for every $x \in X$ and $B \in P_{cp}\left(\overline{\mathcal{O}(x)}\right)$.*

Proposition 2.6 (see [17])**.** *Let $\mathcal{S} = \left((X,d), (f_i)_{i\in I}\right)$ be an oIFS. Then,*

$$\underset{n\ge 1}{\cap} f_{[\alpha]_n}(A_x) = \{a_\alpha(x)\}$$

and

$$d\left(f_{[\alpha]_n}(x), a_\alpha(x)\right) \le \varphi^n\left(\operatorname{diam}\left(\mathcal{O}(x)\right)\right)$$

for all $x \in X$, $n \in \mathbb{N}$ and $\alpha \in \Lambda(I)$.

Definition 2.7. *Let $x, y \in X$. We say that x is equivalent with y and we use the notation $x \backsim y$ if $A_x = A_y$.*

Remark 2.2. *The above relation is an equivalence relation. For an element $x \in X$, we denote its class by $\hat{x}$.*

Remark 2.3. *If $x, y \in X$ such that $x \backsim y$, then using Proposition 2.6, we obtain $a_\alpha(x) = a_\alpha(y)$.*

Remark 2.4. *If $x, y \in X$ such that $A_x \cap A_y \neq \varnothing$, then $A_x = A_y$.*

Affine iterated function systems

Definition 2.8. *On $\mathbb{R}^n$ we consider a fixed norm denoted by $\|\cdot\|$. By an affine iterated function system we mean a pair denoted by $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$, where $(f_i)_{i\in I}$ is a finite family of continuous functions, with $f_i\colon \mathbb{R}^n \to \mathbb{R}^n$ for all $i \in I$, having the property that for every $i \in I$, there exist $\widetilde{A}_i \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $\widetilde{a}_i \in \mathbb{R}^n$ such that $f_i(x) = \widetilde{A}_i x + \widetilde{a}_i$ for all $x \in \mathbb{R}^n$.*

Definition 2.9. *Let $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$ be an affine iterated function system. $\mathcal{S}$ is called hyperbolic if there exists a norm $|||\cdot|||$ on $\mathbb{R}^n$ and $C \in [0,1)$ such that the system $\left((\mathbb{R}^n, |||\cdot|||), (f_i)_{i\in I}\right)$ is a C-contractive iterated function system.*

Definition 2.10. *Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a comparison function and $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$ an affine iterated function system. S is called φ-hyperbolic if there exists a norm $|||\cdot|||$ on $\mathbb{R}^n$ such that $\left((\mathbb{R}^n, |||\cdot|||), (f_i)_{i\in I}\right)$ is a φ-contractive iterated function system.*

Theorem 2.2 (see [14])**.** *Let $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$ be an affine iterated function system. Then, the following statements are equivalent:*

1) $\mathcal{S}$ is hyperbolic;

2) there exists a comparison function φ_0 such that $\mathcal{S}$ is φ_0-hyperbolic;

3) $\mathcal{S}$ has a unique attractor.

Definition 2.11. *By a φ-contractive orbital affine iterated function system (oAIFS for short) we mean a pair denoted by $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$ which is an affine iterated function system and has the property that there exists a comparison function $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\mathcal{S}$ is a φ-contractive orbital iterated function system.*

3. Main results

Proposition 3.1. *Let $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$ be an oAIFS, $m \in \mathbb{N}$, $m \geq 2$ and $\alpha \in \Lambda_m(I)$. Then,*

$$f_\alpha(x) = \widetilde{A}_\alpha x + \widetilde{a}_{\alpha_1} + \sum_{k=2}^{m} \widetilde{A}_{[\alpha]_{k-1}} \widetilde{a}_{\alpha_k}$$

for all $x \in \mathbb{R}^n$.

Proof. By mathematical induction. □

Remark 3.1. *Using Proposition 3.1, we have that*

$$f_\alpha(x_1) - f_\alpha(x_2) = \widetilde{A}_\alpha (x_1 - x_2) \tag{2}$$

for all $x_1, x_2 \in \mathbb{R}^n$, $m \in \mathbb{N}^*$ and $\alpha \in \Lambda_m(I)$.

Theorem 3.1. *Let $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i\in I}\right)$ be an oAIFS. Then, there exist two linear subspaces $Y, Z \subset \mathbb{R}^n$ such that*

1) $Y + Z = \mathbb{R}^n$, $Y \cap Z = \{0_{\mathbb{R}^n}\}$;

2) for every $i \in I$, there exist $B_i \in L(Z,Z)$, $C_i \in L(Y,Z)$ and $b_i \in Z$ such that $\widetilde{A}_i = \begin{bmatrix} I_Y & O_{Z,Y} \\ C_i & B_i \end{bmatrix}$ and $\widetilde{a}_i = \begin{bmatrix} 0_Y \\ b_i \end{bmatrix}$;

3) there exist $c \in (0,1)$ and a norm $\|\cdot\|_Z$ on Z such that $\|B_i\|_Z < c$ for all $i \in I$.

Proof. Let us consider

$$Z = \left\{ z \in \mathbb{R}^n \mid \lim_{m\to\infty} \widetilde{A}_{[\alpha]_m} z = 0_{\mathbb{R}^n} \text{ for all } \alpha \in \Lambda(I) \right\}.$$

One can easily prove that Z is a linear subspace of $\mathbb{R}^n$. It results that there exists a subspace of $\mathbb{R}^n$ denoted by Y such that $Y + Z = \mathbb{R}^n$ and $Y \cap Z = \{0_{\mathbb{R}^n}\}$.

This implies that there exist $B_i \in L(Z,Z)$, $C_i \in L(Y,Z)$, $D_i \in L(Y,Y)$, $E_i \in L(Z,Y)$, $b_i \in Z$ and $c_i \in Y$ such that

$$\widetilde{A}_i = \begin{bmatrix} D_i & E_i \\ C_i & B_i \end{bmatrix} \text{ and } \widetilde{a}_i = \begin{bmatrix} c_i \\ b_i \end{bmatrix}.$$

Claim 1. $\lim_{m\to\infty} \widetilde{A}_{[\alpha]_m}(x_1 - x_2) = 0$ for all $x_1, x_2 \in \mathbb{R}^n$ with $x_1 \backsim x_2$.

Justification: Let $x_1, x_2 \in \mathbb{R}^n$ with $x_1 \backsim x_2$. From Remark 3.1, Proposition 2.6 and Remark 2.3, we have

$$\lim_{m\to\infty} \widetilde{A}_{[\alpha]_m}(x_1 - x_2) = \lim_{m\to\infty} \left(f_{[\alpha]_m}(x_1) - f_{[\alpha]_m}(x_2)\right) = a_\alpha(x_1) - a_\alpha(x_2) = 0.$$

Claim 2. $\hat{x} = x + Z$ for all $x \in \mathbb{R}^n$.

Justification: Let $x \in \mathbb{R}^n$. Applying Proposition 2.6, we have

$$\lim_{m\to\infty} \sup_{\alpha\in\Lambda(I)} \left\| f_{[\alpha]_m}(x) - a_\alpha(x) \right\| \le \lim_{m\to\infty} \varphi^m(\operatorname{diam}(\mathcal{O}(x))) = 0,$$

so

$$\lim_{m\to\infty} \sup_{\alpha\in\Lambda(I)} \left\| f_{[\alpha]_m}(x) - a_\alpha(x) \right\| = 0 \tag{3}$$

for every $x \in \mathbb{R}^n$.

Let $x_1, x_2 \in \mathbb{R}^n$ with $x_1 \backsim x_2$. Using Remark 2.3, we have $a_\alpha(x_1) = a_\alpha(x_2)$ for all $\alpha \in \Lambda(I)$. From Remark 3.1 and the triangle inequality, we obtain

$$\lim_{m\to\infty} \sup_{\alpha\in\Lambda(I)} \left\| \widetilde{A}_{[\alpha]_m}(x_1 - x_2) \right\| = \lim_{m\to\infty} \sup_{\alpha\in\Lambda(I)} \left\| f_{[\alpha]_m}(x_1) - f_{[\alpha]_m}(x_2) \right\|$$

$$= \lim_{m\to\infty} \sup_{\alpha\in\Lambda(I)} \left\| f_{[\alpha]_m}(x_1) - a_\alpha(x_1) + a_\alpha(x_2) - f_{[\alpha]_m}(x_2) \right\|$$

$$\le \lim_{m\to\infty} \sup_{\alpha\in\Lambda(I)} \left\| f_{[\alpha]_m}(x_1) - a_\alpha(x_1) \right\| + \lim_{m\to\infty} \sup_{\alpha\in\Lambda(I)} \left\| a_\alpha(x_2) - f_{[\alpha]_m}(x_2) \right\|.$$

Using relation (3) and the above inequality, it results

$$\lim_{m\to\infty} \sup_{\alpha\in\Lambda(I)} \left\| \widetilde{A}_{[\alpha]_m}(x_1 - x_2) \right\| = 0.$$

We deduce that $x_1 - x_2 \in Z$, so $x_1 \in x_2 + Z$. Hence, we proved that $\hat{x} \subset x + Z$ for all $x \in \mathbb{R}^n$.

Let us consider $x_1 \in x + Z$. Then,

$$f_{[\alpha]_m}(x_1) = f_{[\alpha]_m}(x) + f_{[\alpha]_m}(x_1) - f_{[\alpha]_m}(x)$$

$$= f_{[\alpha]_m}(x) + \widetilde{A}_{[\alpha]_m}(x_1 - x)$$

for all $m \in \mathbb{N}^*$ and $\alpha \in \Lambda(I)$. By passing to limit as $m \to \infty$ in the above relation and using the fact that $x_1 - x \in Z$, we obtain $a_\alpha(x_1) = a_\alpha(x)$. Applying Proposition 2.6 and Remark 2.4, we infer that $x_1 \backsim x$, so $x_1 \in \hat{x}$. Therefore, $x + Z \subset \hat{x}$, for all $x \in \mathbb{R}^n$.

From the both inclusions we deduce the conclusion of the claim.

For all $p \in \mathbb{N}^*$ and $\beta \in \Lambda_p(I)$, we have $f_\beta(x) \in \mathcal{O}(x)$ and using Proposition 2.5, we deduce that $A_{f_\beta(x)} = A_x$. Hence, $f_\beta(x) \in \hat{x}$ and it results that $\mathcal{O}(x) \subset \hat{x}$ for all $x \in \mathbb{R}^n$.

Let $x \in \mathbb{R}^n$ and $i \in I$. As $f_i(x) \in \mathcal{O}(x)$, we have $f_i(x) \in \hat{x}$. Applying Claim 2, we obtain that $f_i(x) \in x + Z$. Thus, there exists $t_x \in Z$ such that $\widetilde{A}_i x + \widetilde{a}_i = x + t_x$. So,

$$\left(\widetilde{A}_i - I_{\mathbb{R}^n}\right) x + \widetilde{a}_i = t_x. \tag{4}$$

As $Y+Z=\mathbb{R}^n$ and $Y\cap Z=\{0_{\mathbb{R}^n}\}$, there exist $y\in Y$ and $z\in Z$ such that $x=\begin{bmatrix} y\\ z\end{bmatrix}$. Therefore, relation (4) is equivalent with

$$\begin{bmatrix} D_i-I_Y & E_i\\ C_i & B_i-I_Z\end{bmatrix}\begin{bmatrix} y\\ z\end{bmatrix}+\begin{bmatrix} c_i\\ b_i\end{bmatrix}=\begin{bmatrix} 0\\ t_x\end{bmatrix}.$$

As $x\in\mathbb{R}^n$ was arbitrary chosen, we obtain

$$\begin{cases} (D_i-I_Y)\,y+E_iz+c_i=0\\ C_iy+(B_i-I_Z)\,z+b_i=t_x\end{cases}$$

for every $y\in Y$ and $z\in Z$. Taking $y=0_Y$ and $z=0_Z$ in the first equation of the system, it results $c_i=0_Y$ for all $i\in I$. Taking $y=0_Y$ and $z\neq 0_Z$ in first equation, we have $E_i=O_{Z,Y}$ for all $i\in I$. In the same equation, for $z=0_Z$ and $y\neq 0_Y$, it results that $D_i=I_y$ for all $i\in I$. Thus,

$$f_i\left(\begin{bmatrix} y\\ z\end{bmatrix}\right)=\begin{bmatrix} I_Y & O_{Z,Y}\\ C_i & B_i\end{bmatrix}\begin{bmatrix} y\\ z\end{bmatrix}+\begin{bmatrix} 0_Y\\ b_i\end{bmatrix}=\begin{bmatrix} y\\ C_iy+B_iz+b_i\end{bmatrix}$$

for all $y\in Y$, $z\in Z$ and $i\in I$.

Using the mathematical induction, one can prove that

$$\widetilde{A}_{[\alpha]_m}\begin{bmatrix} 0_Y\\ z\end{bmatrix}=\begin{bmatrix} 0_Y\\ B_{[\alpha]_m}z\end{bmatrix} \tag{5}$$

for all $m\in\mathbb{N}^*$, $\alpha\in\Lambda(I)$ and $z\in Z$.

Let us consider the system $\mathcal{S}_Z=\left((Z,\|\cdot\|),(g_i)_{i\in I}\right)$ with $g_i\colon Z\to Z$ given by $g_i(z)=B_iz$ for all $z\in Z$ and $i\in I$. As $g_i(0_Z)=0_Z$ for every $i\in I$, we deduce that $F_{\widetilde{\mathcal{S}}_Z}(\{0_Z\})=\{0_Z\}$.

Using the fact that $\hat{x}=x+Z$ for all $x\in\mathbb{R}^n$, by taking $x=0_Z$, we obtain that $\hat{0}_Z=Z$.

Let us consider $B\in P_{cp}(Z)$ and $z\in B\cup\{0_Z\}\subset Z$. As $z\in\hat{0}_Z$, we infer that $a_\alpha(z)=a_\alpha(0_Z)$.

For $\alpha\in\Lambda(I)$, we have that

$$B_{[\alpha]_p}z\overset{(5)}{=}\widetilde{A}_{[\alpha]_p}z=\widetilde{A}_{[\alpha]_p}(z-0_Z)\overset{(2)}{=}f_{[\alpha]_p}(z)-f_{[\alpha]_p}(0_Z) \tag{6}$$

for every $p\in\mathbb{N}^*$. Thus,

$$B_{[\alpha]_p}z=f_{[\alpha]_p}(z)-f_{[\alpha]_p}(0_Z)$$

for every $p\in\mathbb{N}^*$. Moreover,

$$\begin{aligned}\sup_{|\alpha|=p}\left\|B_{[\alpha]_p}z\right\| &\overset{(6)}{=}\sup_{|\alpha|=p}\left\|f_{[\alpha]_p}(z)-f_{[\alpha]_p}(0_Z)\right\|\\ &=\sup_{|\alpha|=p}\left\|f_{[\alpha]_p}(z)-f_{[\alpha]_p}(0_Z)-a_\alpha(z)+a_\alpha(0_Z)\right\|\\ &\le\sup_{|\alpha|=p}\left\|f_{[\alpha]_p}(z)-a_\alpha(z)\right\|+\sup_{|\alpha|=p}\left\|f_{[\alpha]_p}(0_Z)-a_\alpha(0_Z)\right\|\\ &\le 2\sup_{|\alpha|=p}\ \sup_{x\in B\cup\{0_Z\}}\left\|f_{[\alpha]_p}(x)-a_\alpha(x)\right\|\le 2\varphi^p\left(\operatorname{diam}\left(\mathcal{O}\left(B\cup\{0_Z\}\right)\right)\right)\end{aligned}$$

for every $p\in\mathbb{N}^*$. Hence,

$$\sup_{x\in B}\sup_{|\alpha|=p}\left\|B_{[\alpha]_p}x\right\|\le 2\varphi^p\left(\operatorname{diam}\left(\mathcal{O}\left(B\cup\{0_Z\}\right)\right)\right)$$

for every $p\in\mathbb{N}^*$. We have

$$h\left(F^p_{\widetilde{\mathcal{S}}_Z}(B),\{0_Z\}\right)=h\left(\underset{|\alpha|=p}{\cup}g_\alpha(B),\{0_Z\}\right)$$

$$\leq \sup_{|\alpha|=p} h\left(g_\alpha\left(B\right), \{0_Z\}\right) \leq \sup_{x\in B}\sup_{|\alpha|=p} \left\|g_\alpha\left(x\right)\right\|$$
$$= \sup_{x\in B}\sup_{|\alpha|=p} \left\|B_\alpha x\right\| \leq 2\varphi^p\left(\operatorname{diam}\left(\mathcal{O}\left(B\cup\{0_Z\}\right)\right)\right)$$

for every $p \in \mathbb{N}^*$.

By taking into consideration the above relation, we obtain that $\lim_{p\to\infty} h\left(F^p_{\widetilde{\mathcal{S}}_Z}\left(B\right), \{0_Z\}\right) = 0$ for every $B \in P_{cp}\left(Z\right)$. As $F_{\widetilde{\mathcal{S}}_Z}\left(\{0_Z\}\right) = \{0_Z\}$, we infer that $\{0_Z\}$ is an attractor of $\mathcal{S}_Z$. Using Theorem 2.2, we deduce that there exists a norm $\left\|\cdot\right\|_Z$ on Z such that $\max_{i\in I}\left\|B_i\right\|_Z < 1$. Hence, $\left(\left(Z, \left\|\cdot\right\|_Z\right), \left(g_i\right)_{i\in I}\right)$ is a C-contractive IFS.

□

Theorem 3.2. *Let $\mathcal{S} = \left(\left(\mathbb{R}^n, \left\|\cdot\right\|\right), \left(f_i\right)_{i\in I}\right)$ be an oAIFS. Let Y and Z be the linear subspaces of $\mathbb{R}^n$ which result from Theorem 3.1 and let $\left\|\cdot\right\|_Y$ be a norm defined on Y. Let $\mu = \max_{i\in I}\left\|B_i\right\|_Z$, $\beta = \max_{i\in I}\left\|C_i\right\|_{Y,Z}$ and $\theta \in \left(0, \frac{1-\mu}{\beta}\right)$. We consider the norm $\left\|\cdot\right\|_\theta : \mathbb{R}^n \to [0,\infty)$ defined by*

$$\left\|\begin{bmatrix} y \\ z \end{bmatrix}\right\|_\theta = \max\left\{\left\|y\right\|_Y, \theta\left\|z\right\|_Z\right\}$$

for all $y \in Y$ and $z \in Z$ and the norm $|||\cdot||| : Z \to [0,\infty)$ given by $|||z||| = \theta||z||_Z$ for all $z \in Z$. Then, $\left\|\widetilde{A}_i\right\|_\theta \leq 1$ and $|||B_i||| = \left\|B_i\right\|_Z < 1$ for all $i \in I$.

Proof. Let $i \in I$, $\theta \in \left(0, \frac{1-\mu}{\beta}\right)$, $y \in Y$ and $z \in Z$.

Case 1.

$$\theta\left\|z\right\|_Z \leq \left\|y\right\|_Y. \tag{7}$$

We have

$$\left\|\widetilde{A}_i\begin{bmatrix} y \\ z \end{bmatrix}\right\|_\theta = \left\|\begin{bmatrix} y \\ C_iy + B_iz \end{bmatrix}\right\|_\theta = \max\left\{\left\|y\right\|_Y,\ \theta\left\|C_iy + B_iz\right\|_Z\right\}$$
$$\leq \max\left\{\left\|y\right\|_Y,\ \theta\left(\left\|C_i\right\|_{Y,Z}\left\|y\right\|_Y + \left\|B_i\right\|_Z\left\|z\right\|_Z\right)\right\}$$
$$\leq \max\left\{\left\|y\right\|_Y,\ \theta\left(\beta\left\|y\right\|_Y + \mu\left\|z\right\|_Z\right)\right\}.$$

Applying (7), we have

$$\theta\left(\beta\left\|y\right\|_Y + \mu\left\|z\right\|_Z\right) \leq \beta\theta\left\|y\right\|_Y + \mu\left\|y\right\|_Y = \left(\beta\theta + \mu\right)\left\|y\right\|_Y.$$

Using the fact that $\theta < \frac{1-\mu}{\beta}$, we deduce that $\theta\beta + \mu < 1$, so

$$\theta\left(\beta\left\|y\right\|_Y + \mu\left\|z\right\|_Z\right) \leq \left\|y\right\|_Y.$$

Thus,

$$\left\|\widetilde{A}_i\begin{bmatrix} y \\ z \end{bmatrix}\right\|_\theta \leq \left\|y\right\|_Y = \left\|\begin{bmatrix} y \\ z \end{bmatrix}\right\|_\theta$$

and we obtain the conclusion.

Case 2.

$$\left\|y\right\|_Y \leq \theta\left\|z\right\|_Z. \tag{8}$$

Similarly with the first case, we have

$$\left\|\widetilde{A}_i\begin{bmatrix} y \\ z \end{bmatrix}\right\|_\theta = \left\|\begin{bmatrix} y \\ C_iy + B_iz \end{bmatrix}\right\|_\theta \leq \max\left\{\left\|y\right\|_Y,\ \theta\left(\beta\left\|y\right\|_Y + \mu\left\|z\right\|_Z\right)\right\}.$$

Applying (8), we have

$$\left\|\widetilde{A}_i\begin{bmatrix} y \\ z \end{bmatrix}\right\|_\theta \leq \max\left\{\theta\left\|z\right\|_Z,\ \theta\left(\beta\theta\left\|z\right\|_Z + \mu\left\|z\right\|_Z\right)\right\}$$

$$= \theta \max \left\{ \|z\|_Z , \ (\beta\theta + \mu) \|z\|_Z \right\}.$$

As $\theta < \frac{1-\mu}{\beta}$, it results

$$(\beta\theta + \mu) \|z\|_Z \le \|z\|_Z ,$$

so

$$\left\| \widetilde{A}_i \begin{bmatrix} y \\ z \end{bmatrix} \right\|_\theta \le \theta \|z\|_Z = \left\| \begin{bmatrix} y \\ z \end{bmatrix} \right\|_\theta .$$

Again, we obtained the conclusion. □

Remark 3.2. *Let $\mathcal{S}$ be an oAIFS as in Theorem 3.2. Then, for any norm on $\mathbb{R}^n$, it doesn't exist, in general, a constant $\gamma < 1$ such that $\left\| \widetilde{A}_i \right\| < \gamma$ for all $i \in I$. For example, we consider $\mathcal{S} = ((\mathbb{R}, |\cdot|), f)$ with $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ for all $x \in \mathbb{R}$. It can be seen that $\mathcal{S}$ is an oAIFS with $\widetilde{A} = 1$ and $\widetilde{a} = 0$. In this case, $|\widetilde{A}| = 1$.*

4. **Remarks and examples**

Let $\mathcal{S} = \left((\mathbb{R}^n, \|\cdot\|), (f_i)_{i \in I} \right)$ be an oAIFS. Applying Theorem 3.1, we obtain that there exist two linear subspaces $Y, Z \subset \mathbb{R}^n$ such that

1) $Y + Z = \mathbb{R}^n$ and $Y \cap Z = \{0_{\mathbb{R}^n}\}$;

2) for every $i \in I$, there exist $B_i \in L(Z, Z)$, $C_i \in L(Y, Z)$ and $b_i \in Z$ such that $\widetilde{A}_i = \begin{bmatrix} I_Y & O_{Z,Y} \\ C_i & B_i \end{bmatrix}$ and $\widetilde{a}_i = \begin{bmatrix} 0_Y \\ b_i \end{bmatrix}$;

3) there exist $c \in (0, 1)$ and a norm $\|\cdot\|_Z$ on Z such that $\|B_i\|_Z < c$ for all $i \in I$.

By mathematical induction, one can prove

$$\widetilde{A}_{i_1 \cdots i_m} = \begin{bmatrix} I_Y & O_{Z,Y} \\ C^{i_1 \cdots i_m} & B_{i_1} \cdots B_{i_m} \end{bmatrix},$$

where

$$C^{i_1 \cdots i_m} = C_{i_1} + \sum_{k=2}^{m} B_{i_1 \cdots i_{k-1}} C_{i_k}$$

for all $m \in \mathbb{N}$, $m \ge 2$ and $i_j \in I$, with $j \in \{1, \cdots, m\}$.

Let $m \in \mathbb{N}^*$ and $\alpha \in \Lambda(I)$. Let $y \in Y$ and $z \in Z$ such that $x = \begin{bmatrix} y \\ z \end{bmatrix}$. It results

$$\widetilde{A}_{[\alpha]_m} x = \begin{bmatrix} y \\ C^{\alpha_1 \cdots \alpha_m} y + B_{\alpha_1 \cdots \alpha_m} z \end{bmatrix}$$

and

$$\widetilde{A}_{[\alpha]_{k-1}} \widetilde{a}_{\alpha_k} = \begin{bmatrix} I_Y & O_{Z,Y} \\ C^{\alpha_1 \cdots \alpha_{k-1}} & B_{\alpha_1 \cdots \alpha_{k-1}} \end{bmatrix} \begin{bmatrix} 0_Y \\ b_{\alpha_k} \end{bmatrix} = \begin{bmatrix} 0_Y \\ B_{\alpha_1 \cdots \alpha_{k-1}} b_{\alpha_k} \end{bmatrix}$$

for all $\alpha \in \Lambda(I)$, $k \in \{2, \cdots, m\}$ and $m \in \mathbb{N}$, $m \ge 2$. Using Proposition 3.1, we have

$$f_{[\alpha]_m}(x) = \widetilde{A}_{[\alpha]_m} x + \widetilde{a}_{\alpha_1} + \sum_{k=2}^{m} \widetilde{A}_{[\alpha]_{k-1}} \widetilde{a}_{\alpha_k}$$

for all $x \in \mathbb{R}^n$.

As $\|B_i\|_Z < c < 1$ for all $i \in I$, we deduce

$$a_\alpha(x) = \lim_{m \to \infty} f_{[\alpha]_m}(x)$$

$$= \lim_{m \to \infty} \left(\begin{bmatrix} y \\ \left(C_{i_1} + \sum_{k=2}^{m} B_{[\alpha]_{k-1}} C_{\alpha_k} \right) y + B_{[\alpha]_m} z \end{bmatrix} + \begin{bmatrix} 0_Y \\ \widetilde{a}_{\alpha_1} \end{bmatrix} + \sum_{k=2}^{m} \begin{bmatrix} 0_Y \\ B_{[\alpha]_{k-1}} b_{\alpha_k} \end{bmatrix} \right)$$

$$= \begin{bmatrix} y \\ \left(C_{i_1} + \sum_{k\geq 2} B_{[\alpha]_{k-1}} C_{\alpha_k} \right) y + \widetilde{a}_{\alpha_1} + \sum_{k\geq 2} B_{[\alpha]_{k-1}} b_{\alpha_k} \end{bmatrix}$$

for all $\alpha \in \Lambda(I)$. Therefore,

$$a_\alpha \left(\begin{bmatrix} y \\ z \end{bmatrix} \right) = \begin{bmatrix} y \\ \widetilde{a}_{\alpha_1} + C_{i_1} + \sum_{k\geq 2} B_{[\alpha]_{k-1}} \left(C_{\alpha_k} y + b_{\alpha_k} \right) \end{bmatrix} \tag{9}$$

for all $\alpha \in \Lambda(I)$, $y \in Y$ and $z \in Z$.

Example A. Let us consider the normed space $\left(\mathbb{R}^4, \|\cdot\|\right)$, where $\|\cdot\|$ is the Euclidean norm. Let $(e_i)_{i\in\overline{1,3}}$ be the canonical basis in $\mathbb{R}^3$. We consider $I = \{1,2,3\}$ and the family of functions $(f_i)_{i\in I}$ where $f_i : \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$f_i(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \end{pmatrix} x + \begin{pmatrix} 0 \\ e_i \end{pmatrix}$$

for all $x \in \mathbb{R}^4$ and $i \in I$. Thus, we obtained the system $\mathcal{S} = \left(\left(\mathbb{R}^4, \|\cdot\|\right), (f_i)_{i\in I}\right)$. One can easily prove that $\mathcal{S}$ is an oAIFS. From the proof of the Theorem 3.1, one can easily see that $Z = \left\{ \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix} \mid a,b,c \in \mathbb{R} \right\}$. We consider $Y = \left\{ \begin{pmatrix} y_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mid y_0 \in \mathbb{R} \right\}$. Let $x \in \mathbb{R}^4$. Then, there exist $y \in Y$ and $z \in Z$ such that $x = \begin{bmatrix} y \\ z \end{bmatrix}$. Since $y \in Y$, there exists $y_0 \in Y$ such that $y = \begin{pmatrix} y_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Using relation (9), we deduce that

$$a_\alpha \left(\begin{bmatrix} y \\ z \end{bmatrix} \right) = \begin{bmatrix} y \\ \sum_{k\geq 1} \left(\frac{1}{2}\right)^{k-1} \left(\frac{y_0}{3} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ e_{\alpha_k} \end{pmatrix} \right) \end{bmatrix}$$

for all $y \in \mathbb{R}$, $z \in \mathbb{R}^3$ and $\alpha \in \Lambda(I)$.

If $y_0 = 0$, we obtain that A_y is the Sierpinski triangle (denoted by T) with vertices in $\begin{pmatrix} 0 \\ e_1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ e_2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e_3 \end{pmatrix}$. For $y_0 \neq 0$, we obtain that A_y is T translated by

$$\begin{bmatrix} y \\ \sum_{k\geq 1} \left(\frac{1}{2}\right)^{k-1} \frac{y_0}{3} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} y \\ \frac{2y_0}{3} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \end{bmatrix} = \begin{pmatrix} y_0 \\ \frac{2y_0}{3} \\ \frac{2y_0}{3} \\ \frac{2y_0}{3} \end{pmatrix}.$$

For a set $K \in P_{cp}(X)$, if we want to find A_K, we use the fact that $A_K = \underset{x\in K}{\cup} A_x$.

Example B. Let us consider the normed space $\left(\mathbb{R}^3, \|\cdot\|\right)$, where $\|\cdot\|$ is the Euclidean norm. Let $(e_i)_{i\in\overline{1,2}}$ be the canonical basis in $\mathbb{R}^2$. We consider $I = \{1,2\}$ and the family of functions $(g_i)_{i\in I}$ where $g_i : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $g_i(x) = Bx + b_i$ for all $x \in \mathbb{R}^3$ and $i \in I$, where $B = \begin{pmatrix} \frac{23}{6} & \frac{7}{4} & \frac{7}{4} \\ -14 & -10 & -7 \\ \frac{25}{3} & \frac{25}{6} & \frac{9}{2} \end{pmatrix}$, $b_1 = \begin{pmatrix} -2 \\ 9 \\ -5 \end{pmatrix}$ and $b_2 = \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$. Thus, we obtained the system $\mathcal{S} = \left(\left(\mathbb{R}^3, \|\cdot\|\right), (g_i)_{i\in I}\right)$. One can easily prove that $\mathcal{S}$ is an oAIFS. In order to find the spaces Y and Z from the Theorem 3.1, we change the basis in $\mathbb{R}^3$, by considering the matrix $D = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ -3 & 1 & 3 \end{pmatrix}$. Its inverse is $D^{-1} = \begin{pmatrix} 3 & -2 & 1 \\ -12 & 9 & -5 \\ 7 & -5 & 3 \end{pmatrix}$. In this case, we obtain the functions

$$f_i(x) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{3} & 0 \\ \frac{1}{4} & 0 & \frac{1}{3} \end{pmatrix} x + \begin{pmatrix} 0 \\ e_i \end{pmatrix}$$

for all $x \in \mathbb{R}^3$ and $i \in I$. If we apply Theorem 3.1 for the system $\left(\left(\mathbb{R}^3, \|\cdot\|\right), (f_i)_{i\in I}\right)$, we have that Y is the space generated by the vector $\begin{pmatrix} 3 \\ -12 \\ 7 \end{pmatrix}$ and Z is the space generated by the vectors $\begin{pmatrix} -2 \\ 9 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$.

5. **Conclusions**

In this paper we introduce the notion of φ-contractive orbital affine iterated function system (oAIFS), which represents a type of IFS for which the component functions are affine and they are endowed with weaker contractivity conditions. We present two results which give a description of the functions of an oAIFS and establish sufficient conditions to exist a norm with specific properties on the linear spaces where the functions are defined. Also, we provide two examples for such type of systems.

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