AN ACCELERATED MANN-TYPE METHOD FOR SPLIT COMMON FIXED POINT PROBLEM IN HILBERT SPACES

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In this paper, we introduce a new method to solve split common fixed point problem of demicontractive operators in Hilbert spaces, which is based on the Wang method, inertial method and Mann method. Strong convergence result of the suggested algorithm is proved, and our results are utilized to study split feasibility problem and split variational inequality problem in Hilbert spaces. Finally, we compare the convergence speed of our algorithm with the Wang algorithm by a numerical example.

Keywords: Mann-type method, Inertial technique, split common fixed point problem
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1. Introduction

In 2009, Censor and Segal [1] first introduced the split common fixed point problem (SCFP), motivated by its applications to signal processing and image restoration. In time, this idea stimulated many authors to consider this direction (see [2–6]). More specific, let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two real Hilbert spaces. Then SCFP is to find a point $\omega^\ddagger$ satisfying:

$$\omega^\ddagger \in \mathrm{Fix}(\mathbb{S}) \text{ such that } \mathbb{A}\omega^\ddagger \in \mathrm{Fix}(\mathbb{T}). \tag{1}$$

where $\mathrm{Fix}(\mathbb{S})$ and $\mathrm{Fix}(\mathbb{T})$ denote the fixed point sets of $\mathbb{S}\colon \mathbb{H}_1 \to \mathbb{H}_1$ and $\mathbb{T}\colon \mathbb{H}_2 \to \mathbb{H}_2$, respectively, $\mathbb{A}\colon \mathbb{H}_1 \to \mathbb{H}_2$ be a linear bounded operator. We use Γ_1 to denote the solution set of problem (1) , i.e.,

$$\Gamma_1 := \{\omega^\ddagger \in \mathbb{H}_1 : \omega^\ddagger \in \mathrm{Fix}(\mathbb{S}) \text{ and } \mathbb{A}\omega^\ddagger \in \mathrm{Fix}(\mathbb{T})\}.$$

The problem (1) reduces to the split feasibility problem (SFP) if $\mathbb{S}$ and $\mathbb{T}$ are both metric projection operators. The SFP was introduced by Censor and Elfving [7] in 1994, and has been received a lot of attention (see [8–12]) because it has been used successfully in signal processing [13]. We underline that SFP can be expressed as: find a point $\omega^\ddagger$ such that

$$\omega^\ddagger \in \mathbb{C} \text{ and } \mathbb{A}\omega^\ddagger \in \mathbb{Q}. \tag{2}$$

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where $\mathbb{A}\colon \mathbb{C} \to \mathbb{Q}$ be a linear bounded operator, $\mathbb{C} \subset \mathbb{H}_1$ and $\mathbb{Q} \subset \mathbb{H}_2$ are two nonempty closed convex subsets. We use Γ_2 to denote the solution set of problem (2), i.e.,

$$\Gamma_2 := \{\omega^{\ddagger} \in \mathbb{C} \text{ and } \mathbb{A}\omega^{\ddagger} \in \mathbb{Q}\}.$$

Byrne [8] proposed the well-known CQ-algorithm to solve the problem (2), which is formulated as follows:

$$\mu^{k+1} = P_{\mathbb{C}}(\mu^k - \gamma\mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A}\mu^k. \tag{3}$$

where $P_{\mathbb{C}}$ and $P_{\mathbb{Q}}$ are the metric projection, $\gamma \in (0, \frac{2}{\lambda})$ and λ denote the spectral radius of the operator $\mathbb{A}^*\mathbb{A}$.

In order to solve problem (1), Censor and Segal [1] inspired by the algorithms (3) proposed the following iterative method:

$$\mu^{k+1} = \mathbb{S}(\mu^k + \gamma\mathbb{A}^*(\mathbb{T} - I)\mathbb{A}\mu^k, k \in N. \tag{4}$$

where γ is a correctly selected step size and $\mathbb{A}^*$ denote the adjoint of $\mathbb{A}$, $\mathbb{S}$ and $\mathbb{T}$ are two directed operators. It is well-known that a sequence generated by (4) weak convergences to the solution of problem (1) if γ is chosen in $(0, \frac{2}{\|A\|^2})$ and solution exists. This iterative method also can be generalized to the quasi-nonexpansive mapping [14], demicontractive operators [15] and finite many directed operators [16].

But in algorithm (4), it is difficult to calculate the norm of $\mathbb{A}$ and the step size γ depends on $\|\mathbb{A}\|$, that is why Cui and Wang [17] proposed the following variable step size in order to avoid calculating $\|\mathbb{A}\|$:

$$\tau^k = \frac{\|(I - \mathbb{S})\mathbb{A}\mu^k\|^2}{\|\mathbb{A}^*(I - \mathbb{S})\mathbb{A}\mu^k\|^2}.$$

It is easy to see that the above step selection do not need prior information about the $\|\mathbb{A}\|$. In 2017, Wang [18] proposed a new iterative algorithm to solve problem (1) of directed operators as follows:

$$\mu^{k+1} = \mu^k - \tau^k[(I - \mathbb{S})\mu^k + \mathbb{A}^*(I - \mathbb{T})\mathbb{A}\mu^k], \quad k \in \mathbb{N}. \tag{5}$$

where the step size is set as

$$\tau^k = \frac{\|(I - \mathbb{S})\mu^k\|^2 + \|(I - \mathbb{T})\mathbb{A}\mu^k\|^2}{\|(I - \mathbb{S})\mu^k + \mathbb{A}^*(I - \mathbb{T})\mathbb{A}\mu^k\|^2}. \tag{6}$$

where $\{\tau^k\}$ is an iterative sequence with self-adaptive step size. Further, this iteration scheme converges weakly.

The inertial method has been successfully used to solve various optimization problems from applied sciences [20, 21], which was first introduced in [19] and also be applied to solve the split feasibility problem [22, 23]. Extending the inertial method to the split common fixed point problem is natural because inertial method is really speed up the original algorithm's convergence. Following this research direction, Cui et al. [24] improved algorithm (4) as following form:

$$\begin{cases} \rho^k = \mu^k + \theta^k(\mu^k - \mu^{k-1}) \\ \mu^{k+1} = \mathbb{S}(\rho^k + \tau^k\mathbb{A}^*(\mathbb{T} - I)\mathbb{A}\rho^k \end{cases}$$

where $0 \le \theta^k < \theta < 1$, τ^k is defined as in (6). Note that θ^k and $\theta^k(\mu^k - \mu^{k-1})$ are called inertial parameter and inertial term, respectively.

On the basis of the above work, we introduce an accelerated iterative algorithm to solve problem (1) on demicontractive operators in Hilbert spaces. Our algorithm replaces self-adaptive step size $\{\tau^k\}$ with a positive real numbers sequence $\{\delta^k\}$ in algorithms (5), and then combining (5) with inertial iterative method and Mann type methods [25, 26, 27].

This article is organized as following: we review some basic knowledge for further use in Section 2. Section 3 introduces an accelerated iterative algorithm and obtained the strong convergence theorem, then we generalize our algorithm to solve split feasibility problem in Corollary 3.1. In Section 4, we utilize our results to study the split variational inequality problem in Hilbert spaces, and compare the convergence speed of our algorithm with the algorithm of Wang by a numerical example in section 5.

2. **Preliminaries**

We introduce some definitions and lemmas for further use in this section. In this article, let $\mathbb{H}$ is a Hilbert space, $\langle\cdot,\cdot\rangle$ denotes the inner product and $\|\cdot\|$ stands for the induced norm and $\mathbb{C}$ be a nonempty closed convex subset of $\mathbb{H}$, I denotes the identity operator on $\mathbb{H}$. The sequence μ^k weak convergence to $\omega^\ddagger$ is denoted by $\mu^k \rightharpoonup \omega^\ddagger$ and strong convergence to $\omega^\ddagger$ is denoted by $\mu^k \to \omega^\ddagger$, and denote the fixed point sets of $\mathbb{S}$ and $\mathbb{T}$ by $\mathrm{Fix}(\mathbb{S})$ and $\mathrm{Fix}(\mathbb{T})$, respectively.

Definition 2.1. Let $\mathbb{T}\colon \mathbb{H} \to \mathbb{H}$ be an operator and $\mathrm{Fix}(\mathbb{T}) \neq \emptyset$, then

1. The operator $\mathbb{T}$ is said nonexpansive if
$$\|\mathbb{T}\omega^\ddagger - \mathbb{T}\omega^\dagger\| \le \|\omega^\ddagger - \omega^\dagger\|,\ \forall \omega^\ddagger,\ \omega^\dagger \in \mathbb{H}.$$
2. The operator $\mathbb{T}$ is said quasi-nonexpansive if
$$\|\mathbb{T}\omega^\ddagger - \omega^\dagger\| \le \|\omega^\ddagger - \omega^\dagger\|,\ \forall \omega^\ddagger \in \mathbb{C},\ \omega^\dagger \in \mathrm{Fix}(\mathbb{T}).$$
3. The operator $\mathbb{T}$ is said strictly pseudo-contractive if $\exists \beta \in [0,1)$ satisfy
$$\|\mathbb{T}\omega^\ddagger - \mathbb{T}\omega^\dagger\|^2 \le \|\omega^\ddagger - \omega^\dagger\|^2 + \beta\|\omega^\ddagger - \mathbb{T}\omega^\ddagger - (\omega^\dagger - \mathbb{T}\omega^\dagger)\|^2,\ \forall \omega^\ddagger, \omega^\dagger \in \mathbb{H}.$$
4. The operator $\mathbb{T}$ is said demicontractive if $\exists \beta \in (0,1)$ satisfy
$$\|\mathbb{T}\omega^\ddagger - \omega^\dagger\|^2 \le \|\omega^\ddagger - \omega^\dagger\|^2 + \beta\|\omega^\ddagger - \mathbb{T}\omega^\ddagger\|^2,\ \forall \omega^\ddagger \in \mathbb{H},\ \omega^\dagger \in \mathrm{Fix}(\mathbb{T}).$$
or
$$\langle \omega^\ddagger - \mathbb{T}\omega^\ddagger, \omega^\ddagger - \omega^\dagger\rangle \ge \frac{1-\beta}{2}\|\omega^\ddagger - \mathbb{T}\omega^\ddagger\|^2. \tag{7}$$

Definition 2.2. Let inner product $\langle \phi_1, \phi_2\rangle = \phi_1\phi_2$, and norm $\|\cdot\|$, for all $\phi_1, \phi_2 \in \mathbb{H}$ and $\varsigma \in (0,1)$, we have

1. $\|\phi_1 + \phi_2\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + 2\langle \phi_1, \phi_2\rangle \le \|\phi_1\|^2 + 2\langle \phi_2, \phi_1 + \phi_2\rangle$;
2. $\|\varsigma\phi_1 + (1-\varsigma)\phi_2\|^2 = \varsigma\|\phi_1\|^2 + (1-\varsigma)\|\phi_2\|^2 - \varsigma(1-\varsigma)\|\phi_1 - \phi_2\|^2$.

Definition 2.3. For each point $\omega^\ddagger \in \mathbb{H}$, $P_{\mathbb{C}}\omega^\ddagger$ denotes a unique nearest point in $\mathbb{C}$, such that
$$\|\omega^\ddagger - P_{\mathbb{C}}\omega^\ddagger\| = \inf\{\|\omega^\ddagger - \omega^\dagger\| :\ \omega^\dagger \in \mathbb{C}\}.$$
$P_{\mathbb{C}}$ is a metric projection from $\mathbb{H}$ to $\mathbb{C}$ and $P_{\mathbb{C}}$ is nonexpansive.

Lemma 2.1 ([28])**.** For all $\omega^\ddagger \in \mathbb{H}$ and $\omega^\dagger \in \mathbb{C}$, we have
$$\omega^\dagger = P_{\mathbb{C}}\omega^\ddagger \Leftrightarrow \langle \omega^\ddagger - \omega^\dagger, \omega^\dagger - \nu^\dagger\rangle \ge 0,\ \forall \nu^\dagger \in \mathbb{C}$$

Lemma 2.2 ([28])**.** Let $\mathbb{T} : \mathbb{H} \to \mathbb{H}$ be a nonlinear operator. For every sequence $\{\mu^k\}$ in $\mathbb{H}$, $I - \mathbb{T}$ is demiclosed at zero if following expression holds:

$$\mu^k \rightharpoonup \omega^\ddagger \text{ and } (I - \mathbb{T})\mu^k \to 0 \Rightarrow \omega^\ddagger \in \mathrm{Fix}(\mathbb{T}).$$

Lemma 2.3 ([29])**.** Let $\{g^k\}$ be a sequence of nonnegative real numbers such that

$$g^{k+1} \leq (1 - \beta^k)g^k + \beta^k b^k.$$

for all $k \geq 0$, where $\{\beta^k\} \subset (0, 1)$ and $\{b^k\}$ is a sequence such that

1. $\sum_{k=0}^{\infty} \beta^k = \infty$;
2. $\limsup_{k\to\infty} b^k \leq 0$.

Then $\lim_{k\to\infty} g^k = 0$.

3. Main results

We design the following iterative scheme to solve the approximate solution of problem (1).

Algorithm 3.1 For $\mu^0 \in \mathbb{H}_1$, the sequence $\{\mu^k\}$ is defined as follows:

$$\begin{cases} \rho^k = \mu^k + \theta^k(\mu^k - \mu^{k-1}) \\ u^k = \rho^k - \delta^k[(I - \mathbb{S})\rho^k + \mathbb{A}^*(I - \mathbb{T})\mathbb{A}\rho^k] \\ \mu^{k+1} = (1 - \nu^k - \beta^k)\mu^k + \nu^k u^k, \end{cases} \tag{8}$$

where $\{\theta^k\} \subset (0, 1), \nu^k \subset (c, d) \subset (0, 1 - \beta^k)$ and $\{\beta^k\} \subset (0, 1)$, $\{\delta^k\}$ is a positive real numbers sequence, satisfying the following conditions:

$$\lim_{k\to\infty} \beta^k = 0, \sum_{k=1}^{\infty} \beta^k = \infty, \sum_{k=0}^{\infty} \delta^k = \infty,\ \sum_{k=0}^{\infty} (\delta^k)^2 < \infty.$$

Theorem 3.1. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two real Hilbert spaces with its inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, linear bounded operator is denoted by $\mathbb{A}\colon \mathbb{H}_1 \to \mathbb{H}_2$, $\mathbb{S}\colon \mathbb{H}_1 \to \mathbb{H}_1$ and $\mathbb{T}\colon \mathbb{H}_2 \to \mathbb{H}_2$ are two demicontractive mappings and demiclosed at zero, $\lim_{k\to\infty} \frac{\theta^k}{\beta^k}\|\mu^k - \mu^{k-1}\| = 0, \Gamma_1 \neq 0$.

Then the sequence $\{\mu^k\}$ converges strongly to a point $\omega^\ddagger \in \Gamma_1$, which is generated by Algorithm 3.1, where $\|\omega^\ddagger\| = \min\{\|\omega^\dagger\| : \omega^\dagger \in \Gamma_1\}$.

Proof. We will prove the statement in three steps.

Step 1. We prove that $\{\mu^k\},\{u^k\},\{\rho^k\}$ are bounded. Let $\omega^\ddagger\in\Gamma_1$. From (7) and (8), we have

$$\begin{aligned}\|u^k-\omega^\ddagger\|^2 &= \|\rho^k-\omega^\ddagger-\delta^k[(I-\mathbb{S})\rho^k+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k]\|^2\\ &= \|\rho^k-\omega^\ddagger\|^2+(\delta^k)^2\|(I-\mathbb{S})\rho^k+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k\|^2\\ &- 2\delta^k\langle (I-\mathbb{S})\rho^k+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k,\rho^k-\omega^\ddagger\rangle\\ &= \|\rho^k-\omega^\ddagger\|^2+(\delta^k)^2\|(I-\mathbb{S})\rho^k+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k\|^2\\ &- 2\delta^k\langle (I-\mathbb{S})\rho^k,\rho^k-\omega^\ddagger\rangle-2\delta^k\langle (I-\mathbb{T})\mathbb{A}\rho^k,\mathbb{A}\rho^k-\mathbb{A}\omega^\ddagger\rangle\\ &\le \|\rho^k-\omega^\ddagger\|^2+(\delta^k)^2\|(I-\mathbb{S})\rho^k+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k\|^2\\ &- 2\delta^k\frac{1-\alpha}{2}\|\rho^k-\mathbb{S}\rho^k\|^2-2\delta^k\frac{1-\mu}{2}\|(I-\mathbb{T})\mathbb{A}\rho^k\|^2\\ &\le \|\rho^k-\omega^\ddagger\|^2+2(\delta^k)^2[\|(I-\mathbb{S})\rho^k\|^2+\|\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k\|^2]\\ &- \delta^k[(1-\alpha)\|\rho^k-\mathbb{S}\rho^k\|^2+(1-\mu)\|(I-\mathbb{T})\mathbb{A}\rho^k\|^2].\end{aligned}$$

Since $\sum_{k=0}^{\infty}\delta^k=\infty$, $\sum_{k=0}^{\infty}(\delta^k)^2<\infty$, if $\omega^\ddagger$ solves problem (1), then $\omega^\ddagger=\mathbb{S}\omega^\ddagger,(I-\mathbb{T})\mathbb{A}\omega^\ddagger=0$, it is obvious that $\|\omega^\ddagger-\mathbb{S}\omega^\ddagger+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\omega^\ddagger\|=0$ and $\alpha,\mu\in[0,1)$. Hence, we have

$$2(\delta^k)^2[\|(I-\mathbb{S})\rho^k\|^2+\|\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k\|^2]\to 0$$

and

$$\delta^k[(1-\alpha)\|\rho^k-\mathbb{S}\rho^k\|^2+(1-\mu)\|(I-\mathbb{T})\mathbb{A}\rho^k\|^2]\ge 0.$$

Hence, we deduce that

$$\|u^k-\omega^\ddagger\|^2\le\|\rho^k-\omega^\ddagger\|^2.$$

Next, by applying the triangle inequality and the definition of ρ^k, we have

$$\begin{aligned}\|\rho^k-\omega^\ddagger\| &= \|\mu^k+\theta^k(\mu^k-\mu^{k-1})-\omega^\ddagger\|\\ &\le \|\mu^k-\omega^\ddagger\|+\beta^k\frac{\theta^k}{\beta^k}\|\mu^k-\mu^{k-1}\|.\end{aligned}$$

Since $\theta^k\in(0,1)$, for all $k\ge 1$, it follows from $\lim_{k\to\infty}\frac{\theta^k}{\beta^k}\|\mu^k-\mu^{k-1}\|=0$ that exists $M_1\ge 0$ such that $\frac{\theta^k}{\beta^k}\|\mu^k-\mu^{k-1}\|\le M_1$, and we have

$$\|u^k-\omega^\ddagger\|\le\|\rho^k-\omega^\ddagger\|\le\|\mu^k-\omega^\ddagger\|+\beta^k M_1. \tag{9}$$

Therefore, we have

$$\begin{aligned}\|\mu^{k+1}-\omega^\ddagger\| &= \|(1-\nu^k-\beta^k)\mu^k+\nu^k u^k-\omega^\ddagger\|\\ &= \|(1-\nu^k-\beta^k)(\mu^k-\omega^\ddagger)+\nu^k(u^k-\omega^\ddagger)-\beta^k\omega^\ddagger\|\\ &\le \|(1-\nu^k-\beta^k)(\mu^k-\omega^\ddagger)+\nu^k(u^k-\omega^\ddagger)\|+\beta^k\|\omega^\ddagger\|.\end{aligned} \tag{10}$$

Now, we analyse $(1-\nu^k-\beta^k)(\mu^k-\omega^\ddagger)+\nu^k(u^k-\omega^\ddagger)$. Note that

$$\begin{aligned}
&\|(1-\nu^k-\beta^k)(\mu^k-\omega^\ddagger)+\nu^k(u^k-\omega^\ddagger)\|^2\\
&=(1-\nu^k-\beta^k)^2\|\mu^k-\omega^\ddagger\|^2+(\nu^k)^2\|u^k-\omega^\ddagger\|^2\\
&+2(1-\nu^k-\beta^k)\nu^k\langle\mu^k-\omega^\ddagger,u^k-\omega^\ddagger\rangle\\
&\le(1-\nu^k-\beta^k)^2\|\mu^k-\omega^\ddagger\|^2+(\nu^k)^2\|u^k-\omega^\ddagger\|^2\\
&+2(1-\nu^k-\beta^k)\nu^k\|\mu^k-\omega^\ddagger\|\|u^k-\omega^\ddagger\|\\
&\le(1-\nu^k-\beta^k)^2\|\mu^k-\omega^\ddagger\|^2+(\nu^k)^2\|u^k-\omega^\ddagger\|^2\\
&+(1-\nu^k-\beta^k)\nu^k(\|\mu^k-\omega^\ddagger\|^2+\|u^k-\omega^\ddagger\|^2)\\
&\le(1-\nu^k-\beta^k)(1-\beta^k)\|\mu^k-\omega^\ddagger\|^2+(1-\beta^k)\nu^k\|u^k-\omega^\ddagger\|^2.
\end{aligned}\tag{11}$$

According to (9) and (11), we obtain

$$\begin{aligned}
&\|(1-\nu^k-\beta^k)(\mu^k-\omega^\ddagger)+\nu^k(u^k-\omega^\ddagger)\|^2\\
&\le(1-\nu^k-\beta^k)(1-\beta^k)\|\mu^k-\omega^\ddagger\|^2+(1-\beta^k)\nu^k(\|\mu^k-\omega^\ddagger\|+\beta^kM_1)^2\\
&\le(1-\nu^k-\beta^k)(1-\beta^k)\|\mu^k-\omega^\ddagger\|^2+(1-\beta^k)\nu^k\|\mu^k-\omega^\ddagger\|^2\\
&+2(1-\beta^k)\nu^k\beta^k\|\mu^k-\omega^\ddagger\|M_1+(\beta^k)^2M_1^2\\
&\le(1-\beta^k)^2\|\mu^k-\omega^\ddagger\|^2+2(1-\beta^k)\beta^k\|\mu^k-\omega^\ddagger\|M_1+(\beta^k)^2M_1^2\\
&=[(1-\beta^k)\|\mu^k-\omega^\ddagger\|+\beta^kM_1]^2.
\end{aligned}$$

Therefore, we have

$$\|(1-\nu^k-\beta^k)(\mu^k-\omega^\ddagger)+\nu^k(u^k-\omega^\ddagger)\|\le(1-\beta^k)\|\mu^k-\omega^\ddagger\|+\beta^kM_1.\tag{12}$$

Combining (10) and (12), we get

$$\begin{aligned}
\|\mu^{k+1}-\omega^\ddagger\|&\le(1-\beta^k)\|\mu^k-\omega^\ddagger\|+\beta^kM_1+\beta^k\|\omega^\ddagger\|\\
&=(1-\beta^k)\|\mu^k-\omega^\ddagger\|+\beta^k(M_1+\|\omega^\ddagger\|)\\
&\le\max\{\|\mu^k-\omega^\ddagger\|,M_1+\|\omega^\ddagger\|\}\\
&\le\cdots\\
&\le\max\{\|\mu^0-\omega^\ddagger\|,M_1+\|\omega^\ddagger\|\}
\end{aligned}$$

Thus $\{\mu^k\}$ is bounded, $\{\rho^k\}$ and $\{u^k\}$ are also bounded.

Step 2. Prove the following inequality holds:

$$g^{k+1}\le(1-\beta^k)g^k+\beta^kb^k,$$

where we define $g^k:=\|\mu^k-\omega^\ddagger\|^2$ and

$$\begin{aligned}
b^k:=&\frac{\theta^k}{\beta^k}\|\mu^k-\mu^{k-1}\|(1-\beta^k)M_2\\
&+2\nu^k\|\mu^k-u^k\|\|\omega^\ddagger-\mu^{k+1}\|+2\langle\omega^\ddagger,\omega^\ddagger-\mu^{k+1}\rangle.
\end{aligned}$$

Indeed, we have

$$\mu^{k+1}=(1-\nu^k)\mu^k+\nu^ku^k-\beta^k\mu^k.$$

Let $t^k = (1-\nu^k)\mu^k + \nu^k u^k$, then we have

$$\begin{aligned}
\|t^k - \omega^\ddagger\|^2 &= \|(1-\nu^k)\mu^k + \nu^k u^k - \omega^\ddagger\|^2 \\
&= \|(1-\nu^k)(\mu^k - \omega^\ddagger) + \nu^k(u^k - \omega^\ddagger)\|^2 \\
&= (1-\nu^k)^2\|\mu^k - \omega^\ddagger\|^2 + (\nu^k)^2\|u^k - \omega^\ddagger\|^2 \\
&+ 2\nu^k(1-\nu^k)\langle \mu^k - \omega^\ddagger, u^k - \omega^\ddagger\rangle \\
&\le (1-\nu^k)^2\|\mu^k - \omega^\ddagger\|^2 + (\nu^k)^2\|u^k - \omega^\ddagger\|^2 \\
&+ 2\nu^k(1-\nu^k)\|\mu^k - \omega^\ddagger\|\|u^k - \omega^\ddagger\| \\
&\le (1-\nu^k)^2\|\mu^k - \omega^\ddagger\|^2 + (\nu^k)^2\|u^k - \omega^\ddagger\|^2 \\
&+ \nu^k(1-\nu^k)(\|\mu^k - \omega^\ddagger\|^2 + \|u^k - \omega^\ddagger\|^2) \\
&= (1-\nu^k)\|\mu^k - \omega^\ddagger\|^2 + \nu^k\|u^k - \omega^\ddagger\|^2 \\
&\le (1-\nu^k)\|\mu^k - \omega^\ddagger\|^2 + \nu^k\|\rho^k - \omega^\ddagger\|^2
\end{aligned} \tag{13}$$

On the other hand, we have

$$\begin{aligned}
\|\rho^k - \omega^\ddagger\|^2 &= \|\mu^k + \theta^k(\mu^k - \mu^{k-1}) - \omega^\ddagger\|^2 \\
&= \|\mu^k - \omega^\ddagger\|^2 + (\theta^k)^2\|\mu^k - \mu^{k-1}\|^2 + 2\theta^k\langle \mu^k - \omega^\ddagger, \mu^k - \mu^{k-1}\rangle \\
&\le \|\mu^k - \omega^\ddagger\|^2 + (\theta^k)^2\|\mu^k - \mu^{k-1}\|^2 + 2\theta^k\|\mu^k - \omega^\ddagger\|\|\mu^k - \mu^{k-1}\| \\
&\le \|\mu^k - \omega^\ddagger\|^2 + \theta^k\|\mu^k - \mu^{k-1}\|[\theta^k\|\mu^k - \mu^{k-1}\| + 2\|\mu^k - \omega^\ddagger\|] \\
&\le \|\mu^k - \omega^\ddagger\|^2 + \theta^k\|\mu^k - \mu^{k-1}\|M_2
\end{aligned} \tag{14}$$

for some $M_2 > 0$, combining (13) and (14), we have

$$\begin{aligned}
\|t^k - \omega^\ddagger\|^2 &\le (1-\nu^k)\|\mu^k - \omega^\ddagger\|^2 + \nu^k\|\mu^k - \omega^\ddagger\|^2 + \nu^k\theta^k\|\mu^k - \mu^{k-1}\|M_2 \\
&\le \|\mu^k - \omega^\ddagger\|^2 + \theta^k\|\mu^k - \mu^{k-1}\|M_2
\end{aligned} \tag{15}$$

Since $t^k = (1-\nu^k)\mu^k + \nu^k u^k$, we have $\mu^k - t^k = \nu^k(\mu^k - u^k)$. Therefore, it follows that

$$\begin{aligned}
\mu^{k+1} &= t^k - \beta^k\mu^k \\
&= (1-\beta^k)t^k - \beta^k(\mu^k - t^k) \\
&= (1-\beta^k)t^k - \beta^k\nu^k(\mu^k - u^k)
\end{aligned}$$

This implies that

$$\begin{aligned}
\|\mu^{k+1} - \omega^\ddagger\|^2 &= \|(1-\beta^k)t^k - \beta^k\nu^k(\mu^k - u^k) - \omega^\ddagger\|^2 \\
&= \|(1-\beta^k)(t^k - \omega^\ddagger) - [\beta^k\nu^k(\mu^k - u^k) + \beta^k\omega^\ddagger]\|^2 \\
&\le (1-\beta^k)^2\|t^k - \omega^\ddagger\|^2 - 2\langle\beta^k\nu^k(\mu^k - u^k) + \beta^k\omega^\ddagger, \mu^{k+1} - \omega^\ddagger\rangle \\
&\le (1-\beta^k)\|t^k - \omega^\ddagger\|^2 \\
&+ 2\beta^k\nu^k\|\mu^k - u^k\|\|\omega^\ddagger - \mu^{k+1}\| + 2\beta^k\langle\omega^\ddagger, \omega^\ddagger - \mu^{k+1}\rangle
\end{aligned} \tag{16}$$

Thus, from (15) and (16), it follows that

$$\begin{aligned}\|\mu^{k+1}-\omega^\ddagger\|^2 &\le (1-\beta^k)\|\mu^k-\omega^\ddagger\|^2+(1-\beta^k)\theta^k\|\mu^k-\mu^{k-1}\|M_2\\ &+2\beta^k\nu^k\|\mu^k-u^k\|\|\omega^\ddagger-\mu^{k+1}\|+2\beta^k\langle\omega^\ddagger,\omega^\ddagger-\mu^{k+1}\rangle\\ &=(1-\beta^k)\|\mu^k-\omega^\ddagger\|^2+\beta^k[\frac{\theta^k}{\beta^k}\|\mu^k-\mu^{k-1}\|(1-\beta^k)M_2\\ &+2\nu^k\|\mu^k-u^k\|\|\omega^\ddagger-\mu^{k+1}\|+2\langle\omega^\ddagger,\omega^\ddagger-\mu^{k+1}\rangle]\end{aligned}$$

Step 3. Prove $\{\mu^k\}$ converges strongly to $\omega^\ddagger$.

Since $\{\mu^k\}$ and $\{\rho^k\}$ are bounded. According to Lemma 2.3, we next show that $\limsup_{k\to\infty} b^k \le 0$.

Let $z^k=\rho^k-\mathbb{S}\rho^k+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k$, $\forall\alpha,\mu\in[0,1)$, $\omega^\ddagger\in\Gamma_1$, from (7), we have

$$\begin{aligned}\langle z^k,\rho^k-\omega^\ddagger\rangle &= \langle\rho^k-\mathbb{S}\rho^k+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k,\rho^k-\omega^\ddagger\rangle\\ &=\langle\rho^k-\mathbb{S}\rho^k,\rho^k-\omega^\ddagger\rangle+\langle(I-\mathbb{T})\mathbb{A}\rho^k,\mathbb{A}\rho^k-\mathbb{A}\omega^\ddagger\rangle\\ &\ge\frac{1-\alpha}{2}\|\rho^k-\mathbb{S}\rho^k\|^2+\frac{1-\mu}{2}\|(I-\mathbb{T})\mathbb{A}\rho^k\|^2\\ &=\frac{1-\alpha}{2}\|\rho^k-\mathbb{S}\rho^k\|^2+\frac{(1-\mu)\|\mathbb{A}\|^2}{2\|\mathbb{A}\|^2}\|(I-\mathbb{T})\mathbb{A}\rho^k\|^2\\ &\ge\frac{1-\alpha}{2}\|\rho^k-\mathbb{S}\rho^k\|^2+\frac{(1-\mu)}{2\|\mathbb{A}\|^2}\|\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k\|^2\\ &\ge\frac{\min\{1-\alpha,1-\mu\}}{2\max\{1,\|\mathbb{A}\|^2\}}(\|\rho^k-\mathbb{S}\rho^k\|^2+\|\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k\|^2)\\ &\ge\frac{\min\{1-\alpha,1-\mu\}}{4\max\{1,\|\mathbb{A}\|^2\}}(\|\rho^k-\mathbb{S}\rho^k\|+\|\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k\|)^2\\ &\ge\frac{\min\{1-\alpha,1-\mu\}}{4\max\{1,\|\mathbb{A}\|^2\}}\|\rho^k-\mathbb{S}\rho^k+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k\|^2\\ &=\frac{\min\{1-\alpha,1-\mu\}}{4\max\{1,\|\mathbb{A}\|^2\}}\|z^k\|^2\\ &=\kappa\|z^k\|^2\end{aligned}\tag{17}$$

where $\kappa=\frac{\min\{1-\alpha,1-\mu\}}{4\max\{1,\|\mathbb{A}\|^2\}}$.

According to (8) and (17),we have

$$\begin{aligned}\|u^k-\omega^\ddagger\|^2 &= \|\rho^k-\omega^\ddagger-\delta^k z^k\|^2\\ &=\|\rho^k-\omega^\ddagger\|^2-2\delta^k\langle z^k,\rho^k-\omega^\ddagger\rangle+(\delta^k)^2\|z^k\|^2\\ &\le\|\rho^k-\omega^\ddagger\|^2-2\delta^k\kappa\|z^k\|^2+(\delta^k)^2\|z^k\|^2\end{aligned}\tag{18}$$

From (18) and (9), since $\{\mu^k\}$ is bounded, we obtain

$$\begin{aligned}2\delta^k\kappa\|z^k\|^2 &\le \|\rho^k-\omega^\ddagger\|^2-\|u^k-\omega^\ddagger\|^2+(\delta^k)^2\|z^k\|^2\\ &\le\|\mu^k-\omega^\ddagger\|^2-\|u^k-\omega^\ddagger\|^2+(\delta^k)^2M_3.\end{aligned}$$

where $M_3=\sup\{\|z^k\|^2\}$.

An induction induces that

$$2\kappa\sum_{k=0}^{\infty}\delta^k\|z^k\|^2\le\|\mu^0-\omega^\ddagger\|^2+M_3\sum_{k=0}^{\infty}(\delta^k)^2<\infty,$$

which implies that

$$\lim_{k\to\infty}\inf\|z^k\|=0$$

due to $\sum_{k=0}^{\infty}\delta^k=\infty$. So

$$\lim_{k\to\infty}\|z^k\|=\lim_{k\to\infty}\|\rho^k-\mathbb{S}\rho^k+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k\|=0. \tag{19}$$

Observe that

$$\begin{aligned}&\frac{1-\alpha}{2}\|\rho^k-\mathbb{S}\rho^k\|^2+\frac{1-\mu}{2}\|(I-\mathbb{T})\mathbb{A}\rho^k\|^2\\&\le\langle\rho^k-\mathbb{S}\rho^k,\rho^k-\omega^\ddagger\rangle+\langle\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k,\rho^k-\omega^\ddagger\rangle\\&=\langle\rho^k-\mathbb{S}\rho^k+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k,\rho^k-\omega^\ddagger\rangle\\&\le\|\rho^k-\mathbb{S}\rho^k+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k\|\|\rho^k-\omega^\ddagger\|\end{aligned} \tag{20}$$

According to (19) and (20),we have

$$\lim_{k\to\infty}\|\rho^k-\mathbb{S}\rho^k\|=0,\lim_{k\to\infty}\|(I-\mathbb{T})\mathbb{A}\rho^k\|=0 \tag{21}$$

From $\lim_{k\to\infty}\frac{\theta^k}{\beta^k}\|\mu^k-\mu^{k-1}\|=0$ and (8), we have

$$\|\rho^k-\mu^k\|=\theta^k\|\mu^k-\mu^{k-1}\|=\beta^k\frac{\theta^k}{\beta^k}\|\mu^k-\mu^{k-1}\|\to 0. \tag{22}$$

and according to (8), (21) and (22), we get

$$\begin{aligned}\|u^k-\mu^k\|&=\|\rho^k-\mu^k-\delta^k[(I-\mathbb{S})\rho^k+\mathbb{A}^*(I-\mathbb{T})\mathbb{A}\rho^k]\|.\\&\le\|\rho^k-\mu^k\|+\delta^k[\|(I-\mathbb{S})\rho^k\|+\|(I-\mathbb{T})\mathbb{A}\rho^k\|]\to 0\end{aligned} \tag{23}$$

Therefore,we have

$$\begin{aligned}\|\mu^{k+1}-\mu^k\|&=\|(1-\nu^k-\beta^k)\mu^k+\nu^ku^k-\mu^k\|\\&=\|\nu^k(u^k-\mu^k)-\beta^k\mu^k\|\le\nu^k\|u^k-\mu^k\|+\beta^k\|\mu^k\|\to 0\end{aligned} \tag{24}$$

There exists a subsequence $\{\mu^{kj}\}$ of $\{\mu^k\}$ such that $\mu^{kj}\rightharpoonup\omega^\dagger$ and

$$\lim_{k\to\infty}\sup\langle\omega^\ddagger,\omega^\ddagger-\mu^k\rangle=\lim_{j\to\infty}\langle\omega^\ddagger,\omega^\ddagger-\mu^{kj}\rangle=\langle\omega^\ddagger,\omega^\ddagger-\omega^\dagger\rangle$$

Further, from (22), we have $\{\rho^{kj}\}\rightharpoonup\omega^\dagger$. Since $\mathbb{A}$ be a linear bounded operator, we have $\mathbb{A}\rho^{kj}\rightharpoonup\mathbb{A}\omega^\dagger$, and $\mathbb{S}$ and $\mathbb{T}$ are demiclosed at zero. Combine (21), we have $\omega^\dagger\in\mathrm{Fix}(\mathbb{S})$ and $\mathbb{A}\omega^\dagger\in\mathrm{Fix}(\mathbb{T})$, which implies that $\omega^\dagger\in\Gamma_1$.

In addition, it follows from Theorem 3.1 that $\omega^\ddagger\in\Gamma_1$, where $\|\omega^\ddagger\|=\min\{\|\omega^\dagger\|:\omega^\dagger\in\Gamma_1\}$, i.e., $\|0-\omega^\ddagger\|=\min\{\|0-\omega^\dagger\|:\omega^\dagger\in\Gamma_1\}$. So according to Definition 2.3, we have $\omega^\ddagger=P_{\Gamma_1}0$, from characterization of P_{Γ_1} that

$$\lim_{k\to\infty}\sup\langle\omega^\ddagger,\omega^\ddagger-\mu^k\rangle=\langle\omega^\ddagger,\omega^\ddagger-\omega^\dagger\rangle\le 0$$

Since $\|\mu^{k+1}-\mu^k\|\to 0$, we get

$$\lim_{k\to\infty}\sup\langle\omega^\ddagger,\omega^\ddagger-\mu^{k+1}\rangle\le 0 \tag{25}$$

Therefore, combine (23) and (24) with (25), we have $\limsup_{k\to\infty}b^k\le 0$. According to Lemma 2.3, we get $\lim_{k\to\infty}\|\mu^{k+1}-\omega^\ddagger\|^2=0, i.e., \mu^k\to\omega^\ddagger$. This completes the proof. □

Now, we solve the problem (2) with the above result.

Algorithm 3.2 For $\mu^0 \in \mathbb{H}_1$, the sequence $\{\mu^k\}$ is defined as follows:

$$\begin{cases} \rho^k = \mu^k + \theta^k(\mu^k - \mu^{k-1}) \\ u^k = \rho^k - \delta^k[(I - P_{\mathbb{C}})\rho^k + \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A}\rho^k] \\ \mu^{k+1} = (1 - \nu^k - \beta^k)\mu^k + \nu^k u^k. \end{cases}$$

where $\{\theta^k\} \subset (0,1), \nu^k \subset (c,d) \subset (0, 1-\beta^k)$ and $\{\beta^k\} \subset (0,1)$, $\{\delta^k\}$ is a positive real numbers sequence, they satisfying the following conditions:

$$\lim_{k\to\infty} \beta^k = 0, \sum_{k=1}^{\infty} \beta^k = \infty, \sum_{k=0}^{\infty} \delta^k = \infty, \ \sum_{k=0}^{\infty} (\delta^k)^2 < \infty.$$

Corollary 3.1. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, linear bounded operator is denoted by $\mathbb{A}\colon \mathbb{H}_1 \to \mathbb{H}_2$, $\mathbb{C} \subset \mathbb{H}_1$ and $\mathbb{Q} \subset \mathbb{H}_2$ are two closed convex subsets, $\lim_{k\to\infty} \frac{\theta^k}{\beta^k} \|\mu^k - \mu^{k-1}\| = 0$, and $\Gamma_2 \neq \emptyset$. Then the sequence $\{\mu^k\}$ converges strongly to a point $\omega^\ddagger \in \Gamma_2$, which is generated by Algorithm 3.2, where $\|\omega^\ddagger\| = \min\{\|\omega^\dagger\| : \omega^\dagger \in \Gamma_2\}$.

4. Application to split variational inequality problem

We apply Algorithm 3.1 to solve split variational inequality problem in this section. Let $\mathbb{C} \subset \mathbb{H}_1$ and $\mathbb{Q} \subset \mathbb{H}_2$ be two nonempty closed convex subsets, $\mathbb{H}_1$ and $\mathbb{H}_2$ be two Hilbert spaces, $\mathbb{A} : \mathbb{H}_1 \to \mathbb{H}_2$ is a linear bounded operator.

Then the variational inequality problem (VIP) is to find a point $\omega^\ddagger$ such that

$$\langle \mathbb{A}\omega^\ddagger, \omega^\dagger - \omega^\ddagger \rangle \geq 0, \forall \omega^\dagger \in \mathbb{C}.$$

We use $VI(\mathbb{C}, \mathbb{A})$ to denote the solution set of VIP.

Definition 4.1. Given nonlinear operators $f : \mathbb{H}_1 \to \mathbb{H}_1$ and $g : \mathbb{H}_2 \to \mathbb{H}_2$. In 2012, Censor [30] studied the split variational inequality problem (SVIP), which is to find a point $\omega^\ddagger$ such that

$$\langle f\omega^\ddagger, \omega^\dagger - \omega^\ddagger \rangle \geq 0, \forall \omega^\dagger \in \mathbb{C}.$$

and

$$\langle g(\mathbb{A}\omega^\ddagger), \nu^\dagger - \mathbb{A}\omega^\ddagger \rangle \geq 0, \forall \nu^\dagger \in \mathbb{Q}.$$

The solution set of SVIP is denoted by Γ_3, i.e.,

$$\Gamma_3 = \{\omega^\ddagger \in VI(\mathbb{C}, f), \mathbb{A}\omega^\ddagger \in VI(\mathbb{Q}, g)\}$$

Next, we introduce following iterative algorithm to solve SVIP:

Algorithm 4.1 For $\mu^0 \in \mathbb{H}_1$, the sequence $\{\mu^k\}$ is defined as follows:

$$\begin{cases} \rho^k = \mu^k + \theta^k(\mu^k - \mu^{k-1}) \\ u^k = \rho^k - \delta^k[(I - \mathbb{S})\rho^k + \mathbb{A}^*(I - \mathbb{T})\mathbb{A}\rho^k] \\ \mu^{k+1} = (1 - \nu^k - \beta^k)\mu^k + \nu^k u^k. \end{cases}$$

where $\mathbb{S} = P_{\mathbb{C}}(I - \lambda f), \mathbb{T} = P_{\mathbb{Q}}(I - \lambda g), \{\theta^k\} \subset (0,1), \nu^k \subset (c,d) \subset (0, 1-\beta^k)$ and $\{\beta^k\} \subset (0,1), \{\delta^k\}$ is a positive real numbers sequence, they satisfying the following conditions:

$$\lim_{k\to\infty} \beta^k = 0, \sum_{k=1}^{\infty} \beta^k = \infty, \sum_{k=0}^{\infty} \delta^k = \infty, \ \sum_{k=0}^{\infty} (\delta^k)^2 < \infty.$$

Theorem 4.1. Let $\mathbb{S}\colon \mathbb{H}_1 \to \mathbb{H}_1$ and $\mathbb{T}\colon \mathbb{H}_2 \to \mathbb{H}_2$ be two demicontractive mappings, $\lim_{k\to\infty} \frac{\theta^k}{\beta^k}\|\mu^k - \mu^{k-1}\| = 0, \Gamma_3 \neq 0.$

Then the sequence $\{\mu^k\}$ converges strongly to a point $\omega^\ddagger \in \Gamma_3$, which is generated by Algorithm 4.1, where $\|\omega^\ddagger\| = \min\{\|\omega^\dagger\| : \omega^\dagger \in \Gamma_3\}$.

5. Numerical Example

Now, we compare the convergence speed of our Algorithm 3.1 with the algorithm of Wang [18] by a numerical example. All the codes were written in Matlab (R2021b) and run on PC-202310311101 with 11th Gen Intel(R) Core(TM) i5-11400 @ 2.60GHz 2.59 GHz, RAM 8.00GB.

Theorem 5.1. [18]If $\lim_{k\to\infty} \beta^k = 0$ *and* $\sum_{k=0}^{\infty} \beta^k = \infty$, the sequence $\{\mu^k\}$ is generated by following algorithm:

$$\mu^{k+1} = \beta^k u + (1-\beta^k)[\mu^k - \delta^k((\mu^k - \mathbb{S}\mu^k) + \mathbb{A}^*(I-\mathbb{T})\mathbb{A}\mu^k)],$$

where

$$\delta^k = \frac{\|(I-\mathbb{S})\mu^k\|^2 + \|(I-\mathbb{T})\mathbb{A}\mu^k\|^2}{\|(I-\mathbb{S})\mu^k + \mathbb{A}^*(I-\mathbb{T})\mathbb{A}\mu^k\|^2}.$$

Then $\{\mu^k\}$ converges strongly to an element $\omega^\ddagger \in \Gamma_1$,where $\omega^\ddagger = P_{\Gamma_1}(u)$.

Example 5.1. [31]Let $\mathbb{H} = \mathbb{R}$, inner product $\langle \mu, \rho\rangle = \mu\rho$, and norm $|\cdot|$. Let $\mu \in \mathbb{C}, \mathbb{C} = [0, +\infty)$ and $\mathbb{S}\mu = \mu + \frac{4}{\mu+1} - 1$. Then $Fix(\mathbb{S}) = 3$, and

$$\langle \mu - \rho, \mathbb{S}\mu - \mathbb{S}\rho\rangle = \langle \mu - \rho, \mu + \frac{4}{\mu+1} - \rho - \frac{4}{\rho+1}\rangle \leq \|\mu - \rho\|^2$$

for all $\mu, \rho \in \mathbb{C}, \mathbb{S}$ is a demicontractive operator, so $\mathbb{T}\mu = \mu + \frac{3}{\mu+2} - 1$.

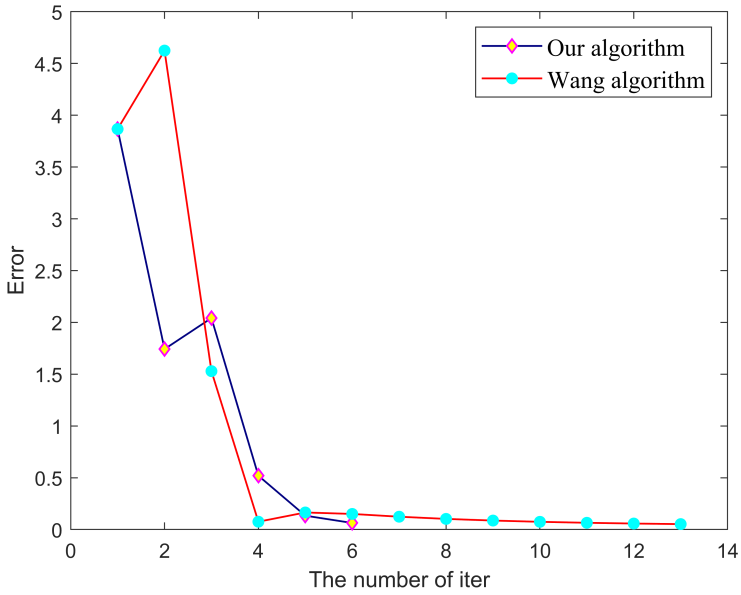
Let $\mu \in \mathbb{R}, \mathbb{A}\mu = \frac{1}{3}\mu, k \geq 1, \nu^k = \frac{1}{8}, \beta^k = \frac{1}{k}, \theta^k = \frac{1}{3}$, Obviously, $\mathbb{A}^* = \mathbb{A}$, Fix$(\mathbb{S}) = 3$ and Fix$(T) = 1$. Next, we rewrite our Algorithm 3.1 and Wang Algorithm as follows:

$$\begin{aligned} \mu^{k+1} &= (\frac{7}{8} - \frac{1}{k})\mu^k + \frac{1}{8}\left[\mu^k + \frac{1}{3}(\mu^k - \mu^{k-1})\right] \\ &\quad - \frac{1}{8}\delta^k\left(\frac{\mu^k + \frac{1}{3}(\mu^k - \mu^{k-1}) - 3}{\mu^k + \frac{1}{3}(\mu^k - \mu^{k-1}) + 1} + \frac{1}{3}\frac{\mu^k + \frac{1}{3}(\mu^k - \mu^{k-1}) - 3}{\mu^k + \frac{1}{3}(\mu^k - \mu^{k-1}) + 6}\right) \\ \mu^{k+1} &= \frac{1}{8}u + \frac{7}{8}\left(\mu^k - \frac{3(\mu^k+6)(\mu^k-3)}{(4\mu^k+19)(\mu^k+1)} - \frac{3(\mu^k+1)(\mu^k-3)}{(4\mu^k+19)(\mu^k+6)}\right) \end{aligned} \tag{26}$$

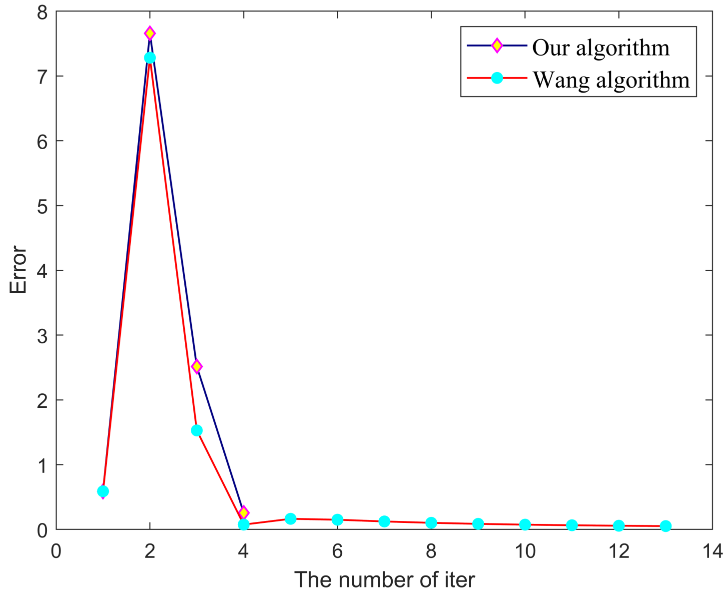
We show our calculation results in Table 1 and Figure 1.

TABLE 1. **The numerical results of Example 5.1**

Cases	Initial values	our Iter	CPU(s)	Wang Iter	CPU(s)
1	$x_0 = 0.1567, x_1 = 4.8530$	6	0.0004	13	0.0006
2	$x_0 = 4.5054, x_1 = 0.8382$	4	0.0001	13	0.0004



(A) Case1



(B) Case2

FIGURE 1. The calculation results of Example 5.1

We can see from Table 1 and Figures 1 that the convergence speed of our Algorithm 3.1 may be faster than the algorithm of Wang by comparing the iteration steps and CPU times.

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