ENERGY LEVELS OF WEAKLY BOUND NUCLEI WITH RELATIVISTIC EFFECTS

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Relativistic effects are employed to describe the weakly bound nuclei, e.g. He, Be, C and O near closed shells. In order to calculate the energy levels of the ground states and first excited states for these nuclei, we present a description based on the assumption of a two-body system formed by a neutron added to an inert core. We adopt the appropriate interaction between neutron and inert core which is a modified of Yukawa potential. Then we solve the Dirac equation with pseudospin symmetry in the shell model by using the basic concept of the supersymmetric shape-invariance method. The results obtained from this approach are compared with experiments and do not show any inversion in agreement with experiments.

Keywords: relativistic effects, weakly-bound nuclei, supersymmetry method.

1. Introduction

A unique feature of nuclear systems along the neutron drip line is the concentration of strength at excitation energies just above the continuum threshold. This concentration of strength is directly measured in breakup reactions. Howevr, it has also strong effects on other processes, such as elastic scattering or sub-barrier fusion reactions. It was proved that this peculiar feature was associated with the weakly bound nature of most nuclei at the dripline [1]. Neutron drip-line nuclei and, in particular, those near the neutron driplines and closed shells play an important role in nuclear astrophysics [2].

In last few decades, the relativistic mean field theory has been successful in describing the nuclear phenomena associated with unstable nuclei as much as stable nuclei [3-5]. Thus it is very helpful to use relativistic theories to study properties of the weakly bound nuclei near the neutron drip line. Compared to the non-relativistic mean field theory, the relativistic mean field theory can explain real nuclear saturation features in the nuclear matter and also presents the spin-orbit coupled potential [6]. The starting point of the relativistic mean field theory is the Lagrangian which describes the nucleons as Dirac spinors moving in the

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mean field. It includes the interaction between nucleons (protons and neutrons), mesons (σ, ω, ρ), and also the coulomb field. The main feature of the relativistic nuclear dynamics is the appearance of the attractive scalar field **S**, and the repulsive vector field **V**. As a result, this feature can lead to simultaneous integrating attraction and repulsion effects related to long and short distances in the nucleon-nucleon interaction. Due to the coupling of lower components in the Dirac equation [7], the observed pseudospin symmetry in the mono-particle levels of the spherical nuclei is understandable through the relativistic mean field theory.

The Dirac equation is one of the most important equations in different physics fields [8-14]. The equation is used to solve many nuclear and high energy problems [15-17]. Recently, remarkable efforts have been made to study relativistic wave equations as well as their relativistic effects, in which solving the Dirac equation with spin and pseudospin symmetry was important. Within the framework of the Dirac equation, the spin symmetry arises if the magnitude of the attractive scalar potential $S(r)$ and repulsive vector potential are nearly equal, $S(r) \sim V(r)$ in the nuclei (i.e., when the potential difference is $\Delta(r) = V(r) - S(r) = C_s$ = constant). However, the pseudospin symmetry occurs if $S(r) \sim -V(r)$ (i.e., when the sum potential is as $\Sigma(r) = V(r) + S(r) = C_{ps}$ = constant) [18-20]. The cases of $\Delta(r) = 0$ and $\Sigma(r) = 0$ correspond to SU(2) symmetries in the Dirac Hamiltonian [21-23]. The spin symmetry is relevant for mesons [24-25]. The pseudospin symmetry concept has been applied to many systems in nuclear physics and related areas [26-29]. It has also been utilized to explain features of deformed nuclei [28] and super-deformation [29] as well as to establish an effective nuclear shell-model scheme [26, 30].

The pseudospin symmetry introduced in the nuclear theory refers to a quasi-degeneracy of the single-nucleon doublets and can be characterized by the non-relativistic quantum numbers $(n, l, j = l+1/2)$ and $(n-1, l+2, j = l+3/2)$, where n, l, and j are the single-nucleon radial, orbital and, total angular momentum quantum numbers, respectively, for a single particle [26, 27]. The total angular momentum is given by $j = l' + s'$ where $l' = l + 1$ is a pseudo-angular momentum and $s' = 1/2$ is a pseudospin angular momentum. In real nuclei, the pseudospin symmetry is only an approximation and the quality of approximation depends on the pseudo-centrifugal potential and pseudo-spin-orbit potential [31].

Other authors adopted different approaches to solve this equation with different potentials [32-39]. Many second order differential equations like Legendre, Hermit, and associated equations have supersymmetry properties and invariance

form. Hence, we can employ an invariant form and some effective methods in the supersymmetry quantum mechanics to solve such equations [40].

In this work, we study the relativistic effects in determining energy levels of weakly bound isotopes of He, Be, C and O near the closed shells. To do so, we describe a two-body system formed through a neutron added to an inert core. We employ a proper interaction between the neutron and inert core, which is a modifed of Yukawa potential [41]

$$V(r) = V_0 \frac{e^{-\alpha r}}{r} + V_1 \frac{e^{-2\alpha r}}{r^2} \tag{1}$$

Where α is range of the potential, and V_0 , V_1 are adjustable parameters. Then we solve the Dirac equation with pseudospin symmetry in the shell model by using the basic concept of the supersymmetric shape-invariance method.

The organization of this paper is as follows. In Section 2 we review relativistic approach briefly. In Section 3 Discussion and results are presented. Conclusion is given in Section 4.

2. Relativistic Approach

The Dirac equation for spin-1/2 particles in an attractive scalar $S(r)$, repulsive vector $V(r)$ potentials reads as (in atomic units ħ=c=1) [42]

$$[\vec{\alpha}.\vec{\mathrm{p}} + \beta(M + S(r))]\psi(\vec{r}) = (E - V(r))\psi(\vec{r}) \tag{2}$$

Where α and β are the Dirac matrices. For spherical nuclei, the nucleon angular momentum J and spin-orbit operator $\boldsymbol{K} = -\boldsymbol{\beta}(\boldsymbol{\sigma}.\boldsymbol{L} + 1)$ commute with the Dirac Hamiltonian.The eigenvalues of $\boldsymbol{K}$ are $\kappa = \pm(\mathrm{j}+1/2)$ with minus for the aligned spin ($s_{1/2}$, $p_{3/2}$, etc.) and plus for the unaligned spin ($p_{1/2}$, $d_{3/2}$, etc.). Then, we use the quantum number κ since it is sufficient to label the orbitals. The wave functions can be classified according to their angular momentum j, κ, and the radial quantum number n. They can be written in the following form

$$\psi_{n\kappa} = \frac{1}{r}\begin{pmatrix} F_{n\kappa}(r)\,\mathrm{Y}^{l}_{j\,m}(\theta,\varphi) \\ iG_{n\kappa}(r)\,\mathrm{Y}^{l'}_{j\,m}(\theta,\varphi) \end{pmatrix} \tag{3}$$

Where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the upper and lower radial functions, and $\mathrm{Y}^{l}_{j\,m}(\theta,\varphi)$ and $\mathrm{Y}^{l'}_{j\,m}(\theta,\varphi)$ are the spinor spherical harmonic functions, respectively. The orbital angular-momentum quantum numbers l and l' are the labels of upper and

lower components, respectively. We substitute Eq. (3) into Eq. (2) and obtain two coupled differential equations for the upper and lower radial wave functions, $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$, respectively:

$$(\frac{d}{dr}-\frac{\kappa}{r})G_{n\kappa}(r)=(M-E_{n\kappa}+\sum(r))F_{n\kappa}(r) \tag{4}$$

$$(\frac{d}{dr}+\frac{\kappa}{r})F_{n\kappa}(r)=(M+E_{n\kappa}-\Delta(r))G_{n\kappa}(r) \tag{5}$$

By substituting $G_{n\kappa}(r)$ (from Eq. (4)) and $F_{n\kappa}(r)$ (from Eq. (5)) into Eq. (5) and Eq. (4), respectively, two following second-order differential equations for the upper and lower components are obtained.

$$[-\frac{d^2}{dr^2}+\frac{\kappa(\kappa+1)}{r^2}-\frac{d\Delta/dr}{E_{n\kappa}+M-\Delta(r)}\times (\frac{d}{dr}+\frac{\kappa}{r})]F_{n\kappa}(r)=[E_{n\kappa}+M-\Delta(r)][M-E_{n\kappa}+\Sigma(r)]F_{n\kappa}(r) \tag{6}$$

$$[-\frac{d^2}{dr^2}+\frac{\kappa(\kappa-1)}{r^2}-\frac{d\Sigma/dr}{M-E_{n\kappa}+\Sigma(r)}\times (\frac{d}{dr}-\frac{\kappa}{r})]G_{n\kappa}(r)=[E_{n\kappa}+M-\Delta(r)][M-E_{n\kappa}+\Sigma(r)]G_{n\kappa}(r) \tag{7}$$

Considering pseudospin symmetry, $\Sigma(r)=0$, and $\Delta(r)$ as the modified potential and [18], we can reach Eq. (8) by substituting Eq. (1) into Eq. (7):

$$[-\frac{d^2}{dr^2}+\frac{\kappa(\kappa-1)}{r^2}+(E_{n\kappa}+M)\times +V_0\frac{e^{-\alpha r}}{r}+V_1\frac{e^{-2\alpha r}}{r^2}]G_{n\kappa}(r)=[E_{n\kappa}^2-M^2(E_{n\kappa}+M)]G_{n\kappa}(r) \tag{8}$$

This equation cannot be solved analytically for $\kappa\neq 0$ with the standard methods such as SUSY and NU because from Eq. (9), it is seen that the effective potential is a combination of exponential and inverse square potentials. Therefore, analytical solutions can be achieved by using an approximation method. We use an approximation for the pseudo-centrifugal term similar to the one used by Dong et al. [43]

$$\frac{1}{r^2}\approx 4\alpha^2\frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \tag{9}$$

This is a good approximation for small values of parameter α, which is shown in Fig 1.

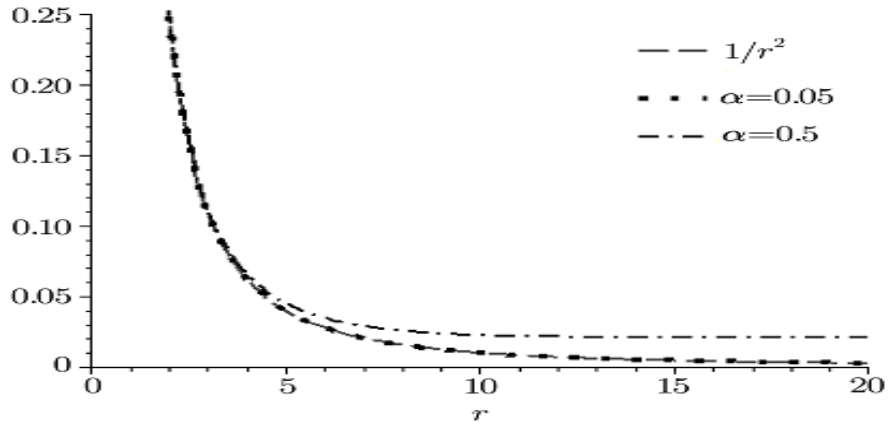


Fig 1: A plot of $1/r^2$ and approximation for various α [44]

Using the approximation given in Eq. (9) and bellow definition,

$$\begin{cases} \tilde{V}_0 = -2V_0\,\alpha\,[E_{n\kappa} - M\,] \\ \tilde{V}_1 = 4\alpha^2\,\kappa(\kappa+1) \\ \tilde{V}_2 = 4V_1\,\alpha^2(E_{n\kappa} - M\,) \\ \tilde{E}_{n\kappa} = E_{n\kappa}^2 - M^2 - (E_{n\kappa} + M\,) \end{cases} \tag{10}$$

We can write down the Schrödinger-like Eq. (8) for the lower spinor component as

$$[-\frac{d^2}{dr^2} + \tilde{V}_0 \frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})} + \tilde{V}_1 \frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} + \tilde{V}_2 \frac{e^{-4\alpha r}}{(1-e^{-2\alpha r})^2}] \times G_{n\kappa}(r) = \tilde{E}_{n\kappa} G_{n\kappa}(r) \tag{11}$$

Eq. (11) can be solved by using the basic concept of the supersymmetric formalism and supersymmetric shape-invariance method [45]. The ground-state lower spinor component $G_{0,\kappa}(r)$ can be given by

$$G_{0,\kappa}(r) = \exp(-\int W(r)\,dr) \tag{12}$$

Where $W(r)$ is called superpotential in the supersymmetric quantum mechanics [46]. Substituting Eq. (12) into Eq. (11) leads to the following equation for $W(r)$

$$W^2(r) - \frac{dW(r)}{dr} =$$

$$\tilde{V}_0 \frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})} + \tilde{V}_1 \frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} + \tilde{V}_2 \frac{e^{-4\alpha r}}{(1-e^{-2\alpha r})^2} - \tilde{E}_{0,\kappa} \tag{13}$$

Where $W(r)$ is called a superpotential supersymmetric quantum mechanics. Eq. (13) is a non-linear Riccati equation. By using the proposed supersymmetric shape-invariance method in ref. [46], the superpotential in Eq. (13) will be rewritten as

$$W(r) = \frac{1}{\sqrt{2M}} (-Q_1 + \frac{Q_2}{1-e^{-2\alpha r}}) \tag{14}$$

Which the coefficients are as bellow in Eq. (14)

$$\begin{cases} Q_1 = \dfrac{1}{2Q_2} [2M\ (-\tilde{V}_0 + 2\tilde{V}_1 + 3\tilde{V}_2) - {Q_2}^2] \\ Q_2 = -\alpha + (\alpha^2 + 2M\ (\tilde{V}_1 + \tilde{V}_2))^{1/2} \end{cases} \tag{15}$$

Taking advantage of the basic concept in the six-parameter exponential-type potential (SPEP) method [47], we can calculate the energy levels as follows

$$E_{n\kappa} = \tilde{V}_2 - \tilde{V}_0 - \frac{1}{8M} [\frac{2M\ (-\tilde{V}_0 + 2\tilde{V}_1 + 3\tilde{V}_2)}{\alpha + 2n\alpha - \sqrt{\alpha^2 + 2M\ (\tilde{V}_1 + \tilde{V}_2)}} - (\alpha + 2n\alpha - \sqrt{\alpha^2 + 2M\ (\tilde{V}_1 + \tilde{V}_2)}\ ]^2$$

$$, \quad n = 0,1,2,... \tag{16}$$

Finally, the ground states and the first excited states in relativistic approach (Eq. (16)) are given in Tables 1 and 2 for some light weakly bound nuclei near closed shells.

3. Result and discussion

The energy levels of the ground states and the first excited states were calculated for the light weakly bound nuclei (He, Be, C, and O) near the closed shells with spot relativistic effects. The characteristics of these nuclei, which are near the closed shells, allowed us to provide a description based on assuming a two-body system formed by a neutron added to an inert core. Furthermore, relativistic analyses based on the Dirac equation have shown that they can achieve better agreement with experimental data than the nonrelativistic analyses based on the Schrödinger equation. One of the merits of the Dirac approach instead of a

nonrelativistic method is that the spin-orbit potential appears inherently in the Dirac approach when the Dirac equation is reduced to a Schrödinger-like second-order differential equation, whereas the spin-orbit potential should be manually inserted in the nonrelativistic Schrödinger approach.

3.1 The Ground State Energy

Numerical results for ground state energy of He, Be, C and O isotopes are given in Table 1. Also, calculated results were compared to the experimental data. The obtained results are reasonably acceptable because according to the numerical results calculated from the ground state energy for these isotopes (Table 1), the difference (ΔE) between the experimental and calculated binding energies is less than 1.2 MeV for all nuclei, which is less than 0.5% of the experimental values. Overall, the relativistic approach using Dirac equation can produce good results when calculating the ground state energy of light weakly bound nuclei near closed shells.

Table 1

The ground state energy of the weakly-bound nuclei. The experimental data are from Ref. [48].

Isotope	Parameter of potential			J_π	Energy (MeV)	
	α	V_0	V_1		E_{our}	$E_{\exp}$
^{5}He	0.021	0.431	-112.113	$3/2^-$	28.015	27.560
^{7}He	0.023	0.453	-112.455	$3/2^-$	29.326	28.861
^{9}Be	0.027	0.525	-112.734	$3/2^-$	58.752	58.158
^{11}Be	0.011	0.567	-113.022	$1/2^+$	66.147	65.472
^{13}C	0.044	0.589	-113.089	$1/2^-$	97.786	97.097
^{15}C	0.047	0.612	-113.253	$1/2^+$	107.342	106.500
^{17}O	0.056	0.654	-113.442	$5/2^+$	132.557	131.750
^{19}O	0.059	0.676	-113.678	$5/2^+$	144.648	143.754

3.2 The First Excited-State Energy

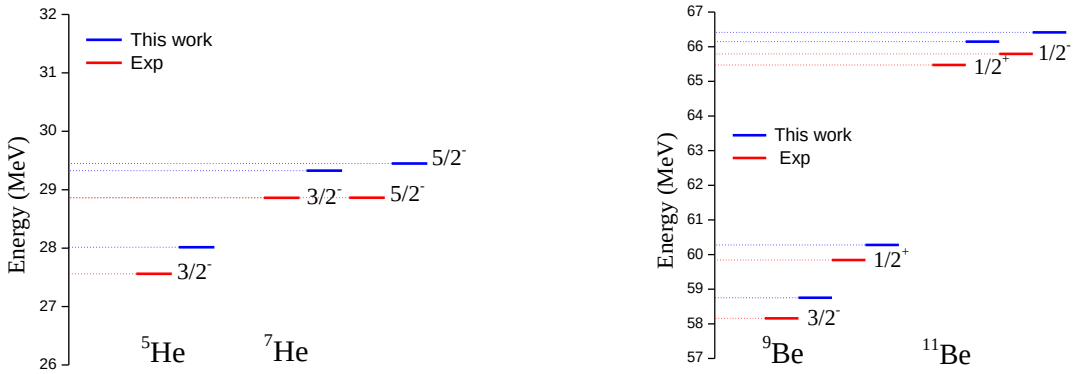
For the first excited energy level of these nuclei (Table 2), calculations show a good agreement with experimental results. According to the numerical results calculated from the first excited energy for these isotopes (Table 2), it can is found that the difference (ΔE) between experimental and calculated binding energies is less than 1.3 MeV for all nuclei. This is less than 0.5% of the experimental values. Since ^{5}He isotope doesn't have any excited state, it was not considered in the calculations.

Table 2

The First excited-state energy of the weakly-bound nuclei.The experimental data are from Ref. [48].

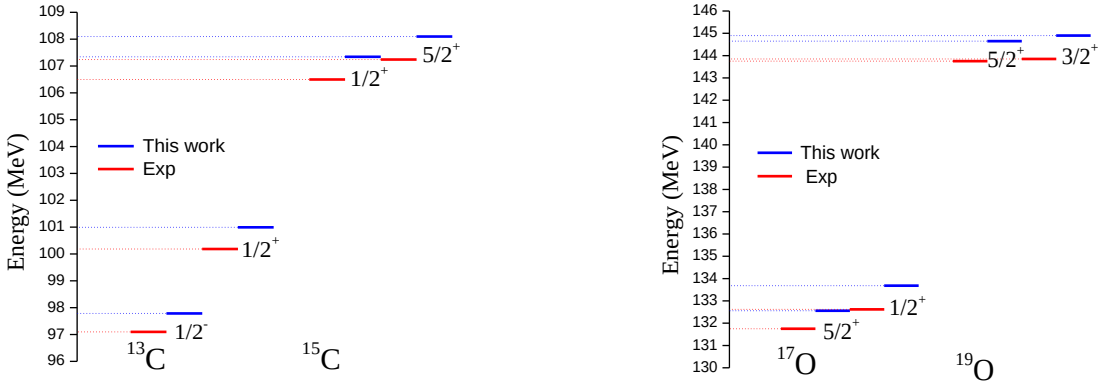
Isotope	Parameter of potential			J_π	Energy (MeV)	
	α	V_0	V_1		E_{our}	E_{exp}
^{7}He	0.023	0.453	-112.455	$5/2^-$	29.448	28.863
^{9}Be	0.027	0.525	-112.734	$1/2^+$	60.276	59.842
^{11}Be	0.031	0.567	-113.022	$1/2^-$	66.413	65.792
^{13}C	0.044	0.589	-113.089	$1/2^+$	100.994	100.186
^{15}C	0.047	0.612	-113.253	$5/2^+$	108.100	107.240
^{17}O	0.056	0.654	-113.442	$1/2^+$	133.681	132.620
^{19}O	0.059	0.676	-113.678	$3/2^+$	144.897	143.850

To well analyze and compare the obtained results for given nuclei, Figs 2-3 have been plotted. On the plot related to isotopes of each element, the ground state and the first excited level of each isotope can be seen for experimental data and calculated results.



(a) (b)

Fig 2: Energy levels of ground states and the first excited states for ^{5}He, ^{7}He, ^{9}Be and ^{11}Be isotopes. In the (a), the total angular moment $(3/2)^-$ belong to the ground state of ^{5}He and ^{7}He and $(5/2)^-$ belong to the first excited state of ^{7}He. In the (b), the total angular moment $(3/2)^-$and $(1/2)^+$ are for the ground state of ^{9}Be and ^{11}Be, respectively, $(1/2)^+$ and $(1/2)^-$ belong to the first excited state of ^{9}Be and ^{11}Be, respectively.



(a) (b)

Fig 3: Energy levels of ground states and the first excited states for ^{13}C, ^{15}C, ^{17}O and ^{19}O isotopes. In the (a), the total angular moment $(1/2)^-$ and $(1/2)^+$ are for the ground state of ^{13}C and ^{15}C, respectively, $(1/2)^+$ and $(5/2)^+$ are for the first excited state of ^{13}C and ^{15}C, respectively. In the (b), the total angular moment $(5/2)^+$ is for the ground state of ^{17}O and ^{19}O , $(1/2)^+$ and $(3/2)^+$ are for the first excited state of ^{17}O and ^{19}O, respectively.

4. Conclusion

In the present work, we calculated energy levels of a series of weakly bound nuclei, using a relativistic method. For this purpose, Dirac equation with an appropriate nuclear potential was solved by a supersymmetric shape-invariance method. Then, the energy levels of the ground states and the first excited states were calculated for the light weakly bound nuclei (He, Be, C, and O) near the closed shells. Results exhibited well matched to the experimental data. Finally, it can be concluded that using relativistic approach can be useful to determine properties of light weakly bound nuclei near closed shells.

R E F E R E N C E S

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