ON VARIATIONAL INEQUALITY PROBLEM AND FIXED POINT PROBLEM OF NONEXPANSIVE SEMIGROUPS

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In this paper, we introduce an iterative method for approximating a common solution of variational inequality problem for pseudomonotone and Lipschitz continuous operators and fixed point of nonexpansive semigroups in the setting of real Hilbert spaces. The proposed method does not need any Armijo-type line search techniques but rather uses an easily implementable self-adaptive step size technique that generates non-monotonic sequence of step sizes. The strong convergence of the sequence generated by the proposed method is proved. The iterative algorithm and results presented in this paper improve the previously known results of this area.

Keywords: Variational inequality; fixed point problem; strong convergence; nonexpansive semigroup; Lipschitz continuous.

2000 AMS Subject Classification: 49J30, 47H09, 47J20.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H, let $F : H \to H$ be a mapping. The variational inequality problem is to find a point $x^* \in C$ such that

$$\langle x - x^*, F(x^*)\rangle \geq 0, \quad \forall x \in C. \tag{1}$$

The solution of (1) is denoted by $VI(C, F)$. It is easy to observe that

$$x^* \in VI(C, F) \Longleftrightarrow x^* = P_C[x^* - \rho F(x^*)], \quad \text{where} \quad \rho > 0.$$

It is well-known that problem (1) covers not only diverse theoretical disciplines such as optimization problems, complementarity problems, equilibrium problems, and fixed point problems but also practical ones such as the power control problem, signal recovery problem, engineering, economics, mathematical programming, and many more. Many authors have studied and proposed several iterative algorithms for solving (1); see for example [6, 7, 8, 9, 11, 12, 18].

In 1964, Polyak [22] introduced the following inertial extrapolation process as a useful tool for speeding up the convergence rate of iterative methods.

$$x_{n+1} = x_n + \alpha_1(x_n - x_{n-1}) - \alpha_2 F x_n, \quad n \geq 0, \tag{2}$$

where α_1 and α_2 are two real numbers. In recent years, many researchers have extensively used this beneficial concept to combine their algorithms with an inertial term in order to accelerate the speed of convergence; see for example [1, 34]. By combination of the inertial

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type method, the viscosity method and the modified Tseng extra-gradient method Thong et al. [31] introduced a Tseng's extra-gradient method. It only needs to compute two values of the inequality mapping and one projection onto the feasible set per iteration. However, the control parameter requires to depend on the Lipschitz constant of F, which cause to limits the applicability of the algorithm. Recently, many authors used a new iteration step size rule in their algorithms to address variational inequality problems. For example, M. Tian and M. Tong [30] introduced an algorithm with self-adaptive method for finding a solution of the variational inequality problem involving monotone operator and the fixed point problem of a quasi-nonexpansive mapping in real Hilbert spaces. Their algorithm is described as follows:
compute

$$\begin{aligned} w_n &= x_n + \alpha_n(x_n - x_{n-1}) \\ y_n &= P_C[w_n - \lambda_n F(w_n)], \\ T_n &= \{x \in H : \langle w_n - \lambda_n F(w_n) - y_n, x - y_n\rangle \leq 0\}, \\ z_n &= P_{T_n}[w_n - \lambda_n F(y_n)], \\ x_{n+1} &= (1-\beta_n)w_n + \beta_n T(z_n), \end{aligned}$$

where the stepsize λ_n is generated by

$$\lambda_{n+1} := \begin{cases} \min\left(\frac{\mu\|w_n - u_n\|}{\|F(w_n) - F(u_n)\|}, \lambda_n\right) & \text{if } F(w_n) \neq F(u_n), \\ \lambda_n & \text{otherwise.} \end{cases} \tag{3}$$

In 2024, K. Wang et al. [33] proposed a projection and contraction method with a double inertial extrapolation step and self-adaptive step sizes to solve variational inequalities with quasi-monotonicity in real Hilbert spaces. Let $x_0, x_1 \in C$ be arbitrary and choose $\lambda_1 > 0, \mu \in (0,1), \alpha, \beta, \theta_n \in (0,1)$. Set

$$\begin{aligned} z_n &= x_n + \beta(x_n - x_{n-1}), \\ w_n &= x_n + \theta_n(x_n - x_{n-1}), \\ y_n &= P_C[w_n - \lambda_n F(w_n)] \end{aligned}$$

where

$$\lambda_{n+1} := \begin{cases} \min\left(\frac{\mu\|w_n - y_n\|}{\|F(w_n) - F(y_n)\|}, \lambda_n\right) & \text{if } F(w_n) \neq F(y_n), \\ \lambda_n & \text{otherwise.} \end{cases}$$

If $w_n = y_n = x_n$ then stop. Otherwise, compute

$$v_n = w_n - y_n - \lambda_n(F(w_n) - F(y_n))$$

and

$$x_{n+1} = (1-\alpha)z_n + \alpha(w_n - \gamma d_n v_n)$$

where

$$d_n := \begin{cases} \frac{\langle w_n - y_n, v_n\rangle}{\|v_n\|^2}, & \text{if } v_n \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

It should be mentioned that the step size criterion (3) used in [30] and [33] generates a non-increasing sequence of steps, which may affect the execution efficiency of this algorithm. To overcome this drawback, a modified version of the step size criterion (3), which generates a non-monotonic sequence of step sizes, was recently introduced in [16, 28, 29, 32]. Very recently, B. Tan and S.Y. Cho [29] proposed two extragradient iterative algorithms using inertial techniques to solve (1). Their algorithm is designed as follows: let $x_0, x_1 \in C$ be arbitrary and choose $\chi \in (0,1), \theta > 0, \tau_1 > 0$, and a sequence $\{\tau_n\}$ satisfying $\sum_{n=1}^{\infty} \tau_n < \infty$, $\{\alpha_n\} \subset (0,1)$ which satisfies the following conditions: $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1})$$

with θ_n such that

$$\theta_n := \begin{cases} \min\left(\frac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta\right) & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{otherwise.} \end{cases}$$

where ϵ_n is satisfying the condition $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.

Compute

$$y_n = P_C[w_n - \tau_n F(w_n)],$$

and

$$z_n = P_{T_n}[w_n - \tau_n F(y_n)],$$

where

$$T_n = \{x \in H : \langle w_n - \tau_n F(w_n) - y_n, x - y_n\rangle \leq 0\}.$$

Compute

$$x_{n+1} = (1 - \alpha_n - \beta_n)w_n + \beta_n z_n$$

and update the step size τ_{n+1} by

$$\tau_{n+1} := \begin{cases} \min\left(\frac{\mu\|w_n - y_n\|}{\|F(w_n) - F(y_n)\|}, \tau_n + \xi_n\right) & \text{if } F(w_n) \neq F(y_n), \\ \tau_n + \xi_n & \text{otherwise.} \end{cases} \tag{4}$$

We introduce the following definitions which are useful in the following analysis.

Definition 1.1. *The mapping $T : C \to H$ is said to be*

(a) *monotone if*

$$\langle Tx - Ty, x - y\rangle \geq 0, \qquad \forall x, y \in C;$$

(b) *pseudomonotone if*

$$\langle Tx, y - x\rangle \geq 0 \Rightarrow \langle Ty, y - x\rangle \geq 0, \qquad \forall x, y \in C;$$

(c) *nonexpansive if*

$$\|Tx - Ty\| \leq \|x - y\|, \qquad \forall x, y \in C;$$

(d) *L-Lipschitz continuous if there exists a constant $L > 0$ such that*

$$\|Tx - Ty\| \leq L\|x - y\|, \qquad \forall x, y \in C;$$

(e) *contraction if there exists a constant $0 \leq k < 1$ such that*

$$\|Tx - Ty\| \leq k\|x - y\|, \qquad \forall x, y \in C;$$

Another important problem is the fixed point problem for the mapping T, which is defined as follows:

$$\text{find} \quad x \in C \quad \text{such that} \quad Tx = x.$$

For several years, the study of fixed point theory has attracted considerable attention and continues to generate great interest in the field of mathematics. By investigating the properties and behavior of fixed points of mappings, many researchers have gained a deeper understanding of these areas and have been able to establish connections and develop new techniques; see for example [2, 3, 4, 5, 13, 25].

A family $\Gamma_a := \{T(s) : 0 \leq s < \infty\}$ of mappings from C into itself is called nonexpansive semigroup on C if it satisfies the following conditions:
(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;
(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(iv) for all $x \in C$ and $s \geq 0$, $s \mapsto T(s)x$ is continuous.
The set of all the common fixed points of a family Γ_a is denoted by $\mathrm{Fix}(\Gamma_a)$, *i.e.*,

$$\mathrm{Fix}(\Gamma_a) := \{x \in C : T(s)x = x, s \geq 0\}.$$

The fixed point problem for a nonexpansive semigroup Γ_a is:

$$\text{find} \quad x \in C \quad \text{such that} \quad x \in \mathrm{Fix}(\Gamma_a). \tag{5}$$

A nonexpansive semigroup Γ_a on C is said to be uniformly asymptotically regular (u.a.r) on C if for all $h > 0$ and any bounded subset E of C,

$$\lim_{t\to\infty} \sup_{x \in E} \|T(h)(T(t)x) - T(t)x\| = 0.$$

The fixed point problem for a nonexpansive semigroup becomes a multi-disciplinary subject, which has received much more attention nowadays. Many authors have analyzed and studied iterative algorithms for approximating the solution of the (5); see for example [10, 15, 24].

In this paper, motivated by the above results in the literature and other related results in this direction, by the combinations of the extragradient method, viscosity method, projection and contraction method, we propose an inertial extragradient algorithms with non-monotone step sizes for approximating a common solution of variational inequality problem (1) for pseudomonotone and Lipschitz continuous operators and fixed point (5) of nonexpansive semigroups in the setting of real Hilbert spaces. Our contributions to this paper include the following:

(i) Our method only requires that the underlying operator for the (1) be pseudomonotone, Lipschitz continuous and without the weak sequential continuity condition often used in the literature.
(ii) Our algorithm does not need any Armijo-type line search techniques but rather uses an easily implementable self-adaptive step size technique that generates non-monotonic sequence of step sizes. This step size is formulated such that it reduces the dependence of the algorithms on the initial step size.
(iii) Our step size properly includes those in [16, 28, 29, 21, 31].
(iv) The proof of our strong convergence result does not rely on the usual "two cases approach" widely used in many papers to prove strong convergence results.
(v) We do not need to assume the u.a.r condition employed by authors in the literature to obtain our strong convergence result.

2. Preliminaries

In this section, we give some useful preliminary results which shall be used in establishing the convergence of our method in the sequel.

Lemma 2.1. [14] *Let C be a closed convex of a real Hilbert space H. Then, the following assertions hold:*
(i) $\langle u - P_C[u], v - P_C[u]\rangle \leq 0, \qquad \forall u \in H, v \in C.$

(ii) $\|P_C[u] - P_C[v]\|^2 \le \langle P_C[u] - P_C[v], u - v\rangle, \qquad \forall u, v \in H.$
(iii) $\|P_C[u] - v\|^2 \le \|u - v\|^2 - \|u - P_C[u]\|^2, \qquad \forall u, v \in H.$

Lemma 2.2. [20] *Each Hilbert space H satisfies the Opial conditions, i.e., for any sequence $\{u_n\}$ with $u_n \rightharpoonup u$ the inequality*

$$\lim\inf_{n\to\infty} \|u_n - u\| < \lim\inf_{n\to\infty} \|u_n - v\| \tag{6}$$

holds for every $v \in H$ with $v \neq u$.

Lemma 2.3. [26] *Let $\{b_n\}$ and $\{\vartheta_n\}$ be two nonnegative real sequences such that*

$$b_{n+1} \le b_n + \vartheta_n, \quad \forall n \ge 1.$$

If $\sum_{n=0}^{\infty} \vartheta_n < \infty$, then $\lim_{n\to\infty} b_n$ exists.

Lemma 2.4. [23] *Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\kappa_n\}$ be a sequence of real numbers in $(0,1)$ with conditions $\sum_{n=1}^{\infty} \kappa_n = \infty$ and φ_n be a sequence of real numbers. Assume that*

$$a_{n+1} \le (1 - \kappa_n)a_n + \kappa_n\varphi_n, \quad \forall n \ge 1.$$

If $\limsup_{k\to\infty} \varphi_{n_k} \le 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. [27] *Let C be a nonempty bounded closed and convex subset of a real Hilbert space H. Let $\Gamma_a := \{T(s) : 0 \le s < \infty\}$ from C be a nonexpansive semigroup on C. Then for all $h \ge 0$,*

$$\lim_{t\to\infty} \sup_{x\in C} \Big\| \frac{1}{t}\int_0^t T(s)xds - T(h)\Big(\frac{1}{t}\int_0^t T(s)xds\Big)\Big\| = 0.$$

Lemma 2.6. [19] *Let H be a real Hilbert space. Then, the following assertions hold:*
(i) $\|u + v\|^2 \le \|u\|^2 + 2\langle v, u + v\rangle, \qquad \forall u, v \in H.$
(ii) $2\langle u, v\rangle = \|u\|^2 + \|v\|^2 - \|u - v\|^2 = \|u + v\|^2 - \|u\|^2 - \|v\|^2, \ \forall u, v \in H.$

Lemma 2.7. [35] *For each $u_1, \cdots, u_m \in H$ and $\eta_1, \cdots, \eta_m \in [0,1]$ with $\sum_{i=1}^{m} \eta_i = 1$, the following equality holds*

$$\|\eta_1 u_1 + \cdots + \eta_m u_m\|^2 = \sum_{i=1}^{m} \eta_i \|u_i\|^2 - \sum_{1\le i\le j\le m} \eta_i\eta_j \|u_i - u_j\|^2.$$

3. The proposed method and some properties

In this section, we suggest and analyze our method for finding the common solutions of (1) for pseudomonotone operators and common fixed point (5) of nonexpansive semigroups. Let H be real Hilbert space, Let C be nonempty closed and convex subset of H. Let $\Gamma := \{T(t) : 0 \le t < \infty\}$ be one-parameter nonexpansive semigroups on H. Let $f : H \to H$ a contraction mapping with constant $k \in [0,1)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\epsilon_n\}$, $\{t_n\}$, $\{\tau_n\}$ and $\{\rho_n\}$ are nonnegative sequences satisfying the following conditions:
(a) $\alpha_n + \beta_n + \delta_n = 1$, and $\liminf_{n\to\infty}\beta_n\delta_n > 0$;
(b) Let $\{\epsilon_n\}$ be positive sequence such that $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$;
(c) $0 < t_n < \infty$;

(d) $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(e) $\sum_{n=1}^{\infty} \tau_n < \infty$, $\lim_{n\to\infty} \rho_n = 0$.

Assuming the usual conditions:

(C1) The feasible set C is nonempty closed and convex.

(C2) The operator $F : H \to H$ be pseudomonotone and L-Lipschitz continuous on H and satisfies the following property: whenever $\{x_n\} \in C, x_n \rightharpoonup x^*$, one has $\|F(x^*)\| \leq \liminf_{n\to\infty} \|F(x_n)\|$.

(C3) The solution set $\Omega = \text{Fix}(\Gamma_a) \bigcap VI(C,F) \neq \emptyset$.

We propose the following algorithm for finding the common solutions of (1) and (5).

Algorithm 3.1.

Step 0. The initial step:
Given $\chi \in (0,1), \gamma \in (0,2), \lambda_1 > 0, \theta > 0, \varpi > 0$, and let $x_{-1}, x_0, x_1 \in H$ be arbitrary.
Given x_{n-2}, x_{n-1}, x_n.

Step 1. Choose θ_n and ϖ_n such that

$$\theta_n := \begin{cases} \min\Big(\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\Big) & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{7}$$

And

$$\varpi_n := \begin{cases} \max\Big(-\varpi, \frac{-\epsilon_n}{\|x_{n-1} - x_{n-2}\|}\Big) & \text{if } x_{n-1} \neq x_{n-2}, \\ \varpi, & \text{otherwise.} \end{cases} \tag{8}$$

Step 2. Set

$$a_n = x_n + \theta_n(x_n - x_{n-1}) + \varpi_n(x_{n-1} - x_{n-2}),$$

and compute

$$z_n = P_C[a_n - \lambda_n F(a_n)],$$

if $z_n = a_n$ then stop, a_n is a solution of (1). Else, do Step 3.

Step 3. Compute

$$u_n = P_{Q_n}[a_n - \gamma\lambda_n d_n F(z_n)],$$

where

$$Q_n = \{x \in H : \langle a_n - \lambda_n F(a_n) - z_n, x - z_n\rangle \leq 0\},$$

$$d_n := \begin{cases} \frac{\langle a_n - z_n, v_n\rangle}{\|v_n\|^2}, & \text{if } v_n \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

$$v_n = a_n - z_n - \lambda_n(F(a_n) - F(z_n)),$$

Step 4. Compute

$$x_{n+1} = \alpha_n f(x_n) + \beta_n u_n + \delta_n \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds.$$

Update

$$\lambda_{n+1} := \begin{cases} \min\Big(\frac{(\rho_n + \chi)\|a_n - z_n\|}{\|F(a_n) - F(z_n)\|}, \lambda_n + \tau_n\Big) & \text{if } F(a_n) \neq F(z_n), \\ \lambda_n + \tau_n & \text{otherwise.} \end{cases} \tag{10}$$

Set $n := n + 1$ and go to Step 1.

Remark 3.1. *By condition (b), from (7) we we obtain*

$$\theta_n \|x_n - x_{n-1}\| \leq \epsilon_n,$$

then

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0 \tag{11}$$

and

$$\lim_{n\to\infty} \frac{|\varpi_n|}{\alpha_n} \|x_{n-1} - x_{n-2}\| = 0. \tag{12}$$

Thus, there exist $N_1 > 0$ and $N_2 > 0$ such that

$$\frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq N_1, \forall n \in \mathbf{N} \tag{13}$$

and

$$\frac{|\varpi_n|}{\alpha_n} \|x_{n-1} - x_{n-2}\| \leq N_2, \forall n \in \mathbf{N}. \tag{14}$$

Lemma 3.1. [17] *Assume that (C1)-(C3) hold and $\{a_n\}$ and $\{z_n\}$ are sequences generated by Algorithm 3.1. If there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ convergent weakly to a point $\tilde{x} \in H$ and $\lim_{k\to\infty} \|a_{n_k} - z_{n_k}\| = 0$, then $\tilde{x} \in VI(C, B)$.*

To prove the global convergence for the proposed method, we need the following lemmas.

Lemma 3.2. *Let $\{\lambda_n\}$ be a sequence defined by (10). then, we have $\lim_{n\to\infty} \lambda_n = \lambda$, where $\lambda \in \left[\min\left(\frac{\chi}{L}, \lambda_1\right), \lambda_1 + \sum_{n=1}^{\infty} \tau_n\right]$.*

Proof. Since F is L-Lipschitz continuous, when $F(a_n) - F(z_n) \neq 0$ we have

$$\frac{(\rho_n + \chi)\|a_n - z_n\|}{\|F(a_n) - F(z_n)\|} \geq \frac{(\rho_n + \chi)\|a_n - z_n\|}{L\|a_n - z_n\|} \geq \frac{\chi}{L}.$$

Hence, from the definition of λ_{n+1}, the sequence $\{\lambda_{n+1}\}$ is bounded below by $\min\left(\frac{\chi}{L}, \lambda_1\right)$ and we have

$$\lambda_{n+1} \leq \lambda_n + \tau_n \leq \lambda_1 + \sum_{n=1}^{\infty} \tau_n.$$

It implies that

$$\min\left(\frac{\chi}{L}, \lambda_1\right) \leq \lambda_n \leq \lambda_1 + \sum_{n=1}^{\infty} \tau_n.$$

By Lemma 2.3, it follows that $\lim_{n\to\infty} \lambda_n$ denoted by $\lambda = \lim_{n\to\infty} \lambda_n$ exists. Clearly, we have $\lambda \in \left[\min\left(\frac{\chi}{L}, \lambda_1\right), \lambda_1 + \sum_{n=1}^{\infty} \tau_n\right]$. □

Remark 3.2. *It follows from Lemma 3.2 and condition (e) that*

$$\lim_{n\to\infty} \left(1 - \frac{(\rho_n + \chi)\lambda_n}{\lambda_{n+1}}\right) = 1 - \chi > 0, \tag{15}$$

there exists $n_0 > 0$ such that for all $n \geq n_0$, we have $1 - \frac{(\rho_n+\chi)\lambda_n}{\lambda_{n+1}} > \frac{1-\chi}{2} > 0$.

Lemma 3.3. *Let $\{x_n\}$ be a sequence generated by the Algorithm 3.1 and $x^* \in \Omega$. Then we have $\forall n \geq n_0$*

$$\begin{aligned} \|u_n - x^*\|^2 &\leq \|a_n - x^*\|^2 - \|a_n - u_n - \gamma d_n v_n\|^2 \\ &\quad -\gamma(2-\gamma)\left(\frac{1-\frac{\rho_n+\chi}{\lambda_{n+1}}}{1+\frac{\rho_n+\chi}{\lambda_{n+1}}}\right)^2 \|a_n - z_n\|^2. \end{aligned} \tag{16}$$

Proof. From (10) and Lemma 3.2 we have

$$\begin{aligned} \|v_n\| &= \|a_n - z_n - \lambda_n(F(a_n) - F(z_n))\| \\ &\geq \|a_n - z_n\| - \lambda_n\|F(a_n) - F(z_n)\| \\ &\geq \|a_n - z_n\| - \frac{(\rho_n+\chi)\lambda_n}{\lambda_{n+1}}\|a_n - z_n\| \\ &= \left(1 - \frac{(\rho_n+\chi)\lambda_n}{\lambda_{n+1}}\right)\|a_n - z_n\|. \end{aligned} \tag{17}$$

From Remark 3.2 for all $n \geq n_0$, we have

$$\|v_n\| > \frac{(1-\chi)}{2}\|a_n - z_n\|. \tag{18}$$

Since $x^* \in C \subset Q_n$, then by lemma 2.1 we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|P_{Q_n}[a_n - \gamma\lambda_n d_n F(z_n)] - P_{Q_n}[x^*]\|^2 \\ &\leq \langle u_n - x^*, a_n - \gamma\lambda_n d_n F(z_n) - x^*\rangle \\ &= \frac{1}{2}(\|u_n - x^*\|^2 + \|a_n - \gamma\lambda_n d_n F(z_n) - x^*\|^2 - \|u_n - a_n + \gamma\lambda_n d_n F(z_n)\|^2) \\ &= \frac{1}{2}(\|u_n - x^*\|^2 + \|a_n - x^*\|^2 - \|u_n - a_n\|^2 - 2\gamma\lambda_n d_n\langle u_n - x^*, F(z_n)\rangle) \end{aligned}$$

which implies that

$$\|u_n - x^*\|^2 \leq \|a_n - x^*\|^2 - \|u_n - a_n\|^2 - 2\gamma\lambda_n d_n\langle u_n - x^*, F(z_n)\rangle. \tag{19}$$

Since $z_n \in C, x^* \in VI(C,F)$ we have $\langle F(x^*), z_n - x^*\rangle \geq 0$, and using the pseudomonotonicity of F, we get

$$\langle F(z_n), z_n - x^*\rangle \geq 0.$$

This implies that

$$\langle F(z_n), u_n - x^*\rangle \geq \langle F(z_n), u_n - z_n\rangle. \tag{20}$$

On the other hand, from $u_n \in Q_n$, we have

$$\langle a_n - \lambda_n(F(a_n) - F(z_n)) - z_n, u_n - z_n\rangle \leq \langle \lambda_n F(z_n), u_n - z_n\rangle.$$

Thus

$$\langle v_n, u_n - z_n\rangle \leq \lambda_n\langle F(z_n), u_n - z_n\rangle. \tag{21}$$

From (18) for all $n \geq n_0$, we have $v_n \neq 0$. This implies that

$$\langle a_n - z_n, v_n\rangle = d_n\|v_n\|^2, \forall n \geq n_0. \tag{22}$$

Then for all $n \geq n_0$, we obtain

$$\begin{aligned} -2\gamma d_n\langle v_n, u_n - z_n\rangle &= -2\gamma d_n\langle v_n, a_n - z_n\rangle + 2\gamma d_n\langle v_n, a_n - u_n\rangle \\ &= -2\gamma d_n^2\|v_n\|^2 + \|a_n - u_n\|^2 + \gamma^2 d_n^2\|v_n\|^2 - \|a_n - u_n - \gamma d_n v_n\|^2 \\ &= \|a_n - u_n\|^2 - \|a_n - u_n - \gamma d_n v_n\|^2 - \gamma(2-\gamma)d_n^2\|v_n\|^2. \end{aligned} \tag{23}$$

Using (20), (21) and (23), we get

$$\begin{aligned}-2\gamma d_n\lambda_n\langle F(z_n), u_n - x^*\rangle &\leq -2\gamma d_n\lambda_n\langle F(z_n), u_n - z_n\rangle \\ &\leq \|a_n - u_n\|^2 - \|a_n - u_n - \gamma d_n v_n\|^2 - \gamma(2-\gamma)d_n^2\|v_n\|^2.\end{aligned} \tag{24}$$

Putting (24) into (19), then for all $n \geq n_0$, we have

$$\|u_n - x^*\|^2 \leq \|a_n - x^*\|^2 - \|a_n - u_n - \gamma d_n v_n\|^2 - \gamma(2-\gamma)d_n^2\|v_n\|^2. \tag{25}$$

From (10), we get

$$\begin{aligned}\|v_n\| &= \|a_n - z_n - \lambda_n(F(a_n) - F(z_n)\| \\ &\leq \|a_n - z_n\| + \lambda_n\|F(a_n) - F(z_n)\| \\ &\leq \|a_n - z_n\| + \frac{\lambda_n(\rho_n + \chi)}{\lambda_{n+1}}\|a_n - z_n\| \\ &= \left(1 + \frac{\lambda_n(\rho_n + \chi)}{\lambda_{n+1}}\right)\|a_n - z_n\|.\end{aligned} \tag{26}$$

Using the definition of v_n, we have

$$\begin{aligned}\langle a_n - z_n, v_n\rangle &= \|a_n - z_n\|^2 - \lambda_n\langle a_n - z_n, F(a_n) - F(z_n)\rangle \\ &\geq \|a_n - z_n\|^2 - \lambda_n\|a_n - z_n\|\|F(a_n) - F(z_n)\| \\ &\geq \|a_n - z_n\|^2 - \frac{\lambda_n(\rho_n + \chi)}{\lambda_{n+1}}\|a_n - z_n\|^2 \\ &= \left(1 - \frac{\lambda_n(\rho_n + \chi)}{\lambda_{n+1}}\right)\|a_n - z_n\|^2.\end{aligned} \tag{27}$$

From (9), (26) and (27), then for all $n \geq n_0$, we obtain

$$d_n^2\|v_n\|^2 \geq \frac{\left(1 - \frac{\lambda_n(\rho_n+\chi)}{\lambda_{n+1}}\right)^2}{\left(1 + \frac{\lambda_n(\rho_n+\chi)}{\lambda_{n+1}}\right)^2}\|a_n - z_n\|^2. \tag{28}$$

Combining (25) and (28), we get the assertion of this lemma. □

Lemma 3.4. *Let $\{x_n\}$ be a sequence generated by the Algorithm 3.1 and $x^* \in \Omega$. Then $\{x_n\}$ is bounded.*

Proof. Since $x^* \in \Omega$ by using condition (a), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\| \alpha_n f(x_n) + \beta_n u_n + \delta_n \frac{1}{t_n} \int_0^{t_n} T(s) u_n ds - x^* \right\| \\
&\leq \alpha_n \|f(x_n) - x^*\| + \beta_n \|u_n - x^*\| + \delta_n \left\| \frac{1}{t_n} \int_0^{t_n} T(s) u_n ds - x^* \right\| \\
&\leq \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|u_n - x^*\| \\
&\quad + \delta_n \left\| \frac{1}{t_n} \int_0^{t_n} T(s) u_n ds - x^* \right\| \\
&\leq \alpha_n k \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|u_n - x^*\| \\
&\quad + \delta_n \left\| \frac{1}{t_n} \int_0^{t_n} T(s) u_n ds - \frac{1}{t_n} \int_0^{t_n} T(s) x^* ds \right\| \\
&\leq \alpha_n k \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n) \|u_n - x^*\| \\
&\leq \alpha_n k \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n) \|a_n - x^*\|. \qquad (29)
\end{aligned}$$

On the other hand, from (13) and (14), we have

$$\begin{aligned}
\|a_n - x^*\| &= \|x_n + \theta_n (x_n - x_{n-1}) + \varpi_n (x_{n-1} - x_{n-2}) - x^*\| \\
&\leq \|x_n - x^*\| + \alpha_n \Big(\frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| + \frac{|\varpi_n|}{\alpha_n} \|x_{n-1} - x_{n-2}\| \Big) \\
&\leq \|x_n - x^*\| + \alpha_n (N_1 + N_2). \qquad (30)
\end{aligned}$$

From (29) and (30), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \alpha_n k \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n)(\|x_n - x^*\| + \alpha_n (N_1 + N_2)) \\
&= (1 - (1 - k)\alpha_n) \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + (1 - \alpha_n)\alpha_n (N_1 + N_2) \\
&\leq (1 - (1 - k)\alpha_n) \|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \alpha_n (N_1 + N_2) \\
&= (1 - (1 - k)\alpha_n) \|x_n - x^*\| + \alpha_n (1 - k) \frac{\|f(x^*) - x^*\| + (N_1 + N_2)}{1 - k} \\
&\leq \max\Big(\|x_n - x^*\|, \frac{\|f(x^*) - x^*\| + (N_1 + N_2)}{1 - k} \Big). \qquad (31)
\end{aligned}$$

By induction on n, we obtain

$$\|x_n - x^*\| \leq \max\Big(\|x_{n_0} - x^*\|, \frac{\|f(x^*) - x^*\| + (N_1 + N_2)}{1 - k} \Big), \forall n \geq n_0.$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{a_n\}, \{z_n\}, \{u_n\}, \{f(x_n)\}$ are bounded. □

4. Convergence Analysis

We now begin to analyze the strong convergence of the proposed method. Note that the proof of strong convergence result does not need the two cases approach used in the literature.

Theorem 4.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then, the sequence $\{x_n\}$ converges strongly to $\tilde{x} \in \Omega$, where $\tilde{x} = P_\Omega[f(\tilde{x})]$.*

Proof. Let $\tilde{x} \in \Omega$. From the definition of a_n, we get

$$
\begin{aligned}
\|a_n - \tilde{x}\|^2 &\leq \|x_n - \tilde{x}\|^2 + \|\theta_n(x_n - x_{n-1}) + \varpi_n(x_{n-1} - x_{n-2})\|^2 + 2\theta_n\|x_n - \tilde{x}\|\|x_n - x_{n-1}\| \\
&\quad +2|\varpi_n|\|x_n - \tilde{x}\|\|x_{n-1} - x_{n-2}\| \\
&\leq \|x_n - \tilde{x}\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + \varpi_n^2\|x_{n-1} - x_{n-2}\|^2 + 2\theta_n\|x_n - \tilde{x}\|\|x_n - x_{n-1}\| \\
&\quad +2|\varpi_n|\|x_n - \tilde{x}\|\|x_{n-1} - x_{n-2}\| + 2\theta_n|\varpi_n|\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\| \\
&= \|x_n - \tilde{x}\|^2 + \alpha_n\theta_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - \tilde{x}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&\quad +\alpha_n|\varpi_n|\frac{|\varpi_n|}{\alpha_n}\|x_{n-1} - x_{n-2}\|^2 + 2\alpha_n\|x_n - \tilde{x}\|\frac{|\varpi_n|}{\alpha_n}\|x_{n-1} - x_{n-2}\| \\
&\quad +2\alpha_n\theta_n\frac{|\varpi_n|}{\alpha_n}\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\| \\
&= \|x_n - \tilde{x}\|^2 + \alpha_n qq_n \qquad (32)
\end{aligned}
$$

where

$$
\begin{aligned}
qq_n &= \theta_n\|x_n - x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + 2\|x_n - \tilde{x}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&\quad +|\varpi_n|\|x_{n-1} - x_{n-2}\|\frac{|\varpi_n|}{\alpha_n}\|x_{n-1} - x_{n-2}\| + 2\|x_n - \tilde{x}\|\frac{|\varpi_n|}{\alpha_n}\|x_{n-1} - x_{n-2}\| \\
&\quad +2\theta_n\|x_n - x_{n-1}\|\frac{|\varpi_n|}{\alpha_n}\|x_{n-1} - x_{n-2}\|.
\end{aligned}
$$

From (11) and (12) it easy to prove that

$$
\lim_{n\to\infty} qq_n = 0. \qquad (33)
$$

Then from Lemma 2.7, Lemma 3.3 and (32), we have

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &= \left\|\alpha_n f(x_n)+\beta_n u_n+\delta_n\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds-\tilde{x}\right\|^2 \\
&\le \alpha_n\|f(x_n)-\tilde{x}\|^2+\beta_n\|u_n-\tilde{x}\|^2+\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds-\tilde{x}\right\|^2 \\
&\quad -\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds-u_n\right\|^2 \\
&\le \alpha_n\Big(\|f(x_n)-f(\tilde{x})\|+\|f(\tilde{x})-\tilde{x}\|\Big)^2+\beta_n\|u_n-\tilde{x}\|^2 \\
&\quad +\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds-\frac{1}{t_n}\int_0^{t_n}T(s)\tilde{x}ds\right\|^2-\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds-u_n\right\|^2 \\
&\le \alpha_n\Big(k\|x_n-\tilde{x}\|+\|f(\tilde{x})-\tilde{x}\|\Big)^2+(1-\alpha_n)\|u_n-\tilde{x}\|^2 \\
&\quad -\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds-u_n\right\|^2 \\
&\le \alpha_n\|x_n-\tilde{x}\|^2+\alpha_n\Big(2\|x_n-\tilde{x}\|\|f(\tilde{x})-\tilde{x}\|+\|f(\tilde{x})-\tilde{x}\|^2\Big)+(1-\alpha_n)\|u_n-\tilde{x}\|^2 \\
&\quad -\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds-u_n\right\|^2 \\
&\le \alpha_n\|x_n-\tilde{x}\|^2+\alpha_n\Big(2\|x_n-\tilde{x}\|\|f(\tilde{x})-\tilde{x}\|+\|f(\tilde{x})-\tilde{x}\|^2\Big)+(1-\alpha_n)\|a_n-\tilde{x}\|^2 \\
&\quad -(1-\alpha_n)\|a_n-u_n-\gamma d_n v_n\|^2-\gamma(2-\gamma)(1-\alpha_n)\left(\frac{1-\frac{\rho_n+\chi}{\lambda_{n+1}}}{1+\frac{\rho_n+\chi}{\lambda_{n+1}}}\right)^2\|a_n-z_n\|^2 \\
&\quad -\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds-u_n\right\|^2 \\
&\le \alpha_n\|x_n-\tilde{x}\|^2+\alpha_n\Big(2\|x_n-\tilde{x}\|\|f(\tilde{x})-\tilde{x}\|+\|f(\tilde{x})-\tilde{x}\|^2\Big) \\
&\quad +(1-\alpha_n)(\|x_n-\tilde{x}\|^2+\alpha_n q q_n)-(1-\alpha_n)\|a_n-u_n-\gamma d_n v_n\|^2 \\
&\quad -\gamma(2-\gamma)(1-\alpha_n)\left(\frac{1-\frac{\rho_n+\chi}{\lambda_{n+1}}}{1+\frac{\rho_n+\chi}{\lambda_{n+1}}}\right)^2\|a_n-z_n\|^2-\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds-u_n\right\|^2 \\
&\le \|x_n-\tilde{x}\|^2+\alpha_n\Big(2\|x_n-\tilde{x}\|\|f(\tilde{x})-\tilde{x}\|+\|f(\tilde{x})-\tilde{x}\|^2+q q_n\Big) \\
&\quad -(1-\alpha_n)\|a_n-u_n-\gamma d_n v_n\|^2-\gamma(2-\gamma)(1-\alpha_n)\left(\frac{1-\frac{\rho_n+\chi}{\lambda_{n+1}}}{1+\frac{\rho_n+\chi}{\lambda_{n+1}}}\right)^2\|a_n-z_n\|^2 \\
&\quad -\beta_n\delta_n\left\|\frac{1}{t_n}\int_0^{t_n}T(s)u_n ds-u_n\right\|^2 \qquad (34)
\end{aligned}$$

Suppose that $\{\|x_{n_k}-\tilde{x}\|^2\}$ is a subsequence of $\{\|x_n-\tilde{x}\|^2\}$ satisfying

$$\liminf_{k\to\infty}\Big(\|x_{n_k+1}-\tilde{x}\|^2-\|x_{n_k}-\tilde{x}\|^2\Big)\ge 0. \qquad (35)$$

From (34), we obtain

$$\begin{aligned} &(1-\alpha_{n_k})\|a_{n_k}-u_{n_k}-\gamma d_{n_k}v_{n_k}\|^2+\gamma(2-\gamma)(1-\alpha_{n_k})\left(\frac{1-\frac{\rho_{n_k}+\chi}{\lambda_{n_k+1}}}{1+\frac{\rho_{n_k}+\chi}{\lambda_{n_k+1}}}\right)^2\|a_{n_k}-z_{n_k}\|^2 \\ &+\beta_{n_k}\delta_{n_k}\left\|\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)u_{n_k}ds-u_{n_k}\right\|^2 \\ \leq\ & \|x_{n_k}-\tilde{x}\|^2-\|x_{n_k+1}-\tilde{x}\|^2+\alpha_{n_k}\Big(2\|x_{n_k}-\tilde{x}\|\|f(\tilde{x})-\tilde{x}\|+\|f(\tilde{x})-\tilde{x}\|^2+qq_{n_k}\Big). \end{aligned}$$

From above inequality and (35), and since $\lim_{n\to\infty}\alpha_n=0$, we have

$$\begin{aligned} &\limsup_{k\to\infty}\Big((1-\alpha_{n_k})\|a_{n_k}-u_{n_k}-\gamma d_{n_k}v_{n_k}\|^2+\gamma(2-\gamma)(1-\alpha_{n_k})\left(\frac{1-\frac{\rho_{n_k}+\chi}{\lambda_{n_k+1}}}{1+\frac{\rho_{n_k}+\chi}{\lambda_{n_k+1}}}\right)^2\|a_{n_k}-z_{n_k}\|^2 \\ &+\beta_{n_k}\delta_{n_k}\left\|\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)u_{n_k}ds-u_{n_k}\right\|^2\Big) \\ \leq\ & \limsup_{k\to\infty}\Big(\|x_{n_k}-\tilde{x}\|^2-\|x_{n_k+1}-\tilde{x}\|^2+\alpha_{n_k}\Big(2\|x_{n_k}-\tilde{x}\|\|f(\tilde{x})-\tilde{x}\|+\|f(\tilde{x})-\tilde{x}\|^2+qq_{n_k}\Big)\Big) \\ =\ & -\liminf_{k\to\infty}\Big(\|x_{n_k+1}-\tilde{x}\|^2-\|x_{n_k}-\tilde{x}\|^2\Big) \\ \leq\ & 0. \end{aligned}$$

Recalling (15) and condition (a) we have

$$\begin{aligned} \lim_{k\to\infty}\|a_{n_k}-u_{n_k}-\gamma d_{n_k}v_{n_k}\|&=0, \\ \lim_{k\to\infty}\|a_{n_k}-z_{n_k}\|&=0, \\ \lim_{k\to\infty}\left\|\frac{1}{t_{n_k}}\int_0^{t_{n_k}}T(s)u_{n_k}ds-u_{n_k}\right\|&=0. \end{aligned}\tag{36}$$

From (9) by using Cauchy Schwartz inequality, we have $d_{n_k}\|v_{n_k}\|\leq\|a_{n_k}-z_{n_k}\|$. Then

$$\begin{aligned} \|a_{n_k}-u_{n_k}\| &\leq \|a_{n_k}-u_{n_k}-\gamma d_{n_k}v_{n_k}\|+\gamma d_{n_k}\|v_{n_k}\| \\ &\leq \|a_{n_k}-u_{n_k}-\gamma d_{n_k}v_{n_k}\|+\gamma\|a_{n_k}-z_{n_k}\|. \end{aligned}$$

This implies that

$$\lim_{k\to\infty}\|a_{n_k}-u_{n_k}\|=0.\tag{37}$$

On the other hand, from (13) and (14), we have

$$\begin{aligned} \|a_{n_k}-x_{n_k}\| &= \|\theta_{n_k}(x_{n_k}-x_{n_k-1})+\varpi_{n_k}(x_{n_k-1}-x_{n_k-2})\| \\ &\leq \alpha_{n_k}\Big(\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|+\frac{|\varpi_{n_k}|}{\alpha_{n_k}}\|x_{n_k-1}-x_{n_k-2}\|\Big) \\ &\leq \alpha_{n_k}(N_1+N_2). \end{aligned}$$

Thus

$$\lim_{k\to\infty}\|a_{n_k}-x_{n_k}\|=0.\tag{38}$$

Since

$$\|u_{n_k} - x_{n_k}\| \le \|u_{n_k} - a_{n_k}\| + \|a_{n_k} - x_{n_k}\|$$

It follows from (37) and (38) that

$$\lim_{k\to\infty} \|u_{n_k} - x_{n_k}\| = 0. \tag{39}$$

Further, for all $h \ge 0$, we see that

$$\begin{aligned}\|u_{n_k} - T(h)u_{n_k}\| &\le \left\|u_{n_k} - \frac{1}{t_{n_k}}\int_0^{t_{n_k}} T(s)u_{n_k}ds\right\| \\ &+ \left\|\frac{1}{t_{n_k}}\int_0^{t_{n_k}} T(s)u_{n_k}ds - T(h)\big(\frac{1}{t_{n_k}}\int_0^{t_{n_k}} T(s)u_{n_k}ds\big)\right\| \\ &+ \left\|T(h)\Big(\frac{1}{t_{n_k}}\int_0^{t_{n_k}} T(s)u_{n_k}ds\Big) - T(h)u_{n_k}\right\|. \end{aligned} \tag{40}$$

Using (36) and Lemma 2.5, we have

$$\lim_{k\to\infty} \|u_{n_k} - T(h)u_{n_k}\| = 0. \tag{41}$$

From the definition of x_{n_k+1}, we have

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &\le \alpha_{n_k}\|f(x_{n_k}) - x_{n_k}\| + \beta_{n_k}\|u_{n_k} - x_{n_k}\| + \delta_{n_k}\left\|\frac{1}{t_{n_k}}\int_0^{t_{n_k}} T(s)u_{n_k}ds - x_{n_k}\right\| \\ &\le \alpha_{n_k}\|f(x_{n_k}) - x_{n_k}\| + (\beta_{n_k} + \delta_{n_k})\|u_{n_k} - x_{n_k}\| \\ &\quad +\delta_{n_k}\left\|\frac{1}{t_{n_k}}\int_0^{t_{n_k}} T(s)u_{n_k}ds - u_{n_k}\right\|.\end{aligned}$$

It follows from (36) and (39) that

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{42}$$

Now, we prove that $\omega_w(x_n) \subset \Omega$, where

$$\omega_w(x_n) = \{x \in H : x_{n_i} \rightharpoonup x \quad \text{for some subsequences} \quad \{x_{n_i}\} \quad \text{of} \quad \{x_n\}\}.$$

Since the sequence $\{x_n\}$ is bounded we have $\omega_w(x_n)$ is nonempty. Let $\tilde{x} \in \omega_w(x_n)$. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \tilde{x}$ as $k \to \infty$. Since $\lim_{k\to\infty} \|a_{n_k} - x_{n_k}\| = 0$, we have that $a_{n_k} \rightharpoonup \tilde{x}$ as $k \to \infty$. In view of $\lim_{k\to\infty} \|a_{n_k} - z_{n_k}\| = 0$, it follows from Lemma 3.1 that $\tilde{x} \in VI(C, F)$. Next, we show that $\tilde{x} \in \text{Fix}(\Gamma)$. Since $\lim_{k\to\infty} \|u_{n_k} - x_{n_k}\| = 0$ (see(39)), we have $u_{n_k} \rightharpoonup \tilde{x}$ as $k \to \infty$. Now, for all $r \ge 0$ we have

$$\|u_{n_k} - T(r)\tilde{x}\| \le \|u_{n_k} - T(r)u_{n_k}\| + \|T(r)u_{n_k} - T(r)\tilde{x}\| \le \|u_{n_k} - T(r)u_{n_k}\| + \|u_{n_k} - \tilde{x}\|.$$

It follows from (41) that

$$\liminf_{k\to\infty} \|u_{n_k} - T(r)\tilde{x}\| \le \liminf_{k\to\infty} \|u_{n_k} - \tilde{x}\|.$$

By the Opial property of the Hilbert space H_1 (see(Lemma 2.2)), we obtain that $T(r)\tilde{x} = \tilde{x}$ for all $r \ge 0$, which implies that $\tilde{x} \in \text{Fix}(\Gamma)$. Since $\tilde{x} \in \omega_w(x_n)$, it follows that $\omega_w(x_n) \subset \Omega$. Next, we show that

$$\limsup_{k\to\infty} \langle f(\tilde{x}) - \tilde{x}, x_{n_k+1} - \tilde{x}\rangle \le 0.$$

Let a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to some $\hat{x} \in \Omega$, and such that

$$\lim_{j\to\infty}\langle f(\tilde{x})-\tilde{x}, x_{n_{k_j}}-\tilde{x}\rangle = \limsup_{k\to\infty}\langle f(\tilde{x})-\tilde{x}, x_{n_k}-\tilde{x}\rangle.$$

Since $\{x_{n_{k_j}}\}$ converges weakly to $\hat{x} \in \Omega$ and $\tilde{x} = P_\Omega[f(\tilde{x})]$, it follows that

$$\limsup_{k\to\infty}\langle f(\tilde{x})-\tilde{x}, x_{n_k+1}-\tilde{x}\rangle = \limsup_{k\to\infty}\langle f(\tilde{x})-\tilde{x}, x_{n_k}-\tilde{x}\rangle = \langle f(\tilde{x})-\tilde{x}, \hat{x}-\tilde{x}\rangle \leq 0. \quad (43)$$

On the other hand, since $\tilde{x} \in \Omega$ it follows from Lemma 2.6 that

$$\begin{aligned}
\|x_{n_k+1}-\tilde{x}\|^2 &= \Big\|\alpha_{n_k}(f(x_{n_k})-f(\tilde{x}))+\beta_{n_k}(u_{n_k}-\tilde{x})+\delta_{n_k}\left(\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T(s)u_{n_k}ds-\tilde{x}\right)\\
&\quad +\alpha_{n_k}(f(\tilde{x})-\tilde{x})\Big\|^2\\
&\leq \Big\|\alpha_{n_k}(f(x_{n_k})-f(\tilde{x}))+\beta_{n_k}(u_{n_k}-\tilde{x})+\delta_{n_k}\left(\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}T(s)u_{n_k}ds-\tilde{x}\right)\Big\|^2\\
&\quad +2\alpha_{n_k}\langle f(\tilde{x})-\tilde{x}, x_{n_k+1}-\tilde{x}\rangle\\
&\leq \alpha_{n_k}\|f(x_{n_k})-f(\tilde{x})\|^2+\beta_{n_k}\|u_{n_k}-\tilde{x}\|^2\\
&\quad +\delta_{n_k}\left(\frac{1}{t_{n_{k,1}}}\int_0^{t_{n_{k,1}}}\|T(s)u_{n_k}-T(s)\tilde{x}\|ds\right)^2+2\alpha_{n_k}\langle f(\tilde{x})-\tilde{x}, x_{n_k+1}-\tilde{x}\rangle\\
&\leq \alpha_{n_k}k\|x_{n_k}-\tilde{x}\|^2+(1-\alpha_{n_k})\|u_{n_k}-\tilde{x}\|^2+2\alpha_{n_k}\langle f(\tilde{x})-\tilde{x}, x_{n_k+1}-\tilde{x}\rangle. \quad (44)
\end{aligned}$$

Using Lemma 3.3, (32) and (44), we get

$$\begin{aligned}
\|x_{n_k+1}-\tilde{x}\|^2 &\leq \alpha_{n_k}k\|x_{n_k}-\tilde{x}\|^2+(1-\alpha_{n_k})\|a_{n_k}-\tilde{x}\|^2+2\alpha_{n_k}\langle f(\tilde{x})-\tilde{x}, x_{n_k+1}-\tilde{x}\rangle\\
&\leq \alpha_{n_k}k\|x_{n_k}-\tilde{x}\|^2+(1-\alpha_{n_k})\Big(\|x_{n_k}-\tilde{x}\|^2+\alpha_{n_k}qq_{n_k}\Big)\\
&\quad +2\alpha_{n_k}\langle f(\tilde{x})-\tilde{x}, x_{n_k+1}-\tilde{x}\rangle\\
&\leq (1-(1-k)\alpha_{n_k})\|x_{n_k}-\tilde{x}\|^2+(1-k)\alpha_{n_k}\left(\frac{qq_{n_k}}{1-k}+\frac{2\langle f(\tilde{x})-\tilde{x}, x_{n_k+1}-\tilde{x}\rangle}{1-k}\right)\\
&= (1-\sigma_{n_k})\|x_{n_k}-\tilde{x}\|^2+\sigma_{n_k}\left(\frac{qq_{n_k}}{1-k}+\frac{2\langle f(\tilde{x})-\tilde{x}, x_{n_k+1}-\tilde{x}\rangle}{1-k}\right)
\end{aligned}$$

where $\sigma_{n_k} = (1-k)\alpha_{n_k}$. Let $\varphi_{n_k} = \frac{qq_{n_k}}{1-k} + \frac{2\langle f(\tilde{x})-\tilde{x}, x_{n_k+1}-\tilde{x}\rangle}{1-k}$, it is easy to see that

$$\sum_{n_k=1}^{\infty}\sigma_{n_k} = \infty, \lim_{k\to\infty}\sigma_{n_k} = 0$$

and from (33), (43), we obtain

$$\limsup_{k\to\infty}\varphi_{n_k} \leq 0.$$

Thus from (35) all the conditions of Lemma 2.4 are satisfied.
Hence we deduce that $\lim_{n\to\infty}\|x_n-\tilde{x}\|^2 = 0$. Consequently, $\lim_{n\to\infty}\|x_n-\tilde{x}\| = 0$. Therefore, x_n converges strongly to $\tilde{x}$. This completes the proof. □

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