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ABOUT MOLECULES IN DISTRIBUTIVE LATTICES (I)

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Scopul acestei lucrări este de a prezenta unele proprietăți ale moleculelor în laticele distributive finite.

The aim of this paper is to present some properties of molecules in finite distributive latices

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1. Definitions. Preliminaries

Let A be a bounded distributive lattice.

Definition 1.1 In A an element x is an atom if $x \neq 0$ and $y \leq x \Rightarrow y = 0$ or $y = x$. An element x is join-irreducible if $x = y \vee z \Rightarrow x = y$ or $x = z$. Note by $J(A)$ the set of nonzero join – irreducible elements and by $At(A)$ the set of atoms of A.

We have $At(A) \subseteq J(A)$

Is A is a Boolean algebra $At(A) = J(A)$, and this equality characterizes finite Boolean algebras.

Another generalization of atoms was introduced by Abian [1].

Definition 1.2 [1] An element m in A is a molecule if $m \neq 0$ and $x, y \leq m$, $x, y \neq 0 \Rightarrow x \wedge y \neq 0$. Note by $M(A)$ the set of molecules. We have $At(A) \subseteq M(A)$ and in a Boolean algebra $At(A) = M(A)$.

There are no relations between molecules and join-irreducible elements in arbitrary lattices.

The notion of molecule was studied by Yaqub [6] in Postalgebras. He proved.

Proposition 1.3 [6] In a Postalgebra A the following conditions are equivalent

(i) m is a molecule

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(ii) $\varphi_{n-1}m \in AtC(A)$

and the principal ideal $(m]$ is prime $\Leftrightarrow \neg m$ is a molecule.

This means $\neg(m] = [\neg m)$ is a prime filter $\Leftrightarrow \neg m$ is a molecule so *m* is a molecule $\Leftrightarrow m \in J(A)$

Hence in Postalgebras molecule and join-irreducible element is a same notion. We have the following generalization.

Proposition 1.4[3] In a *n*-valued Lukasiewicz-Moisil algebra *A* (an algebra without negation) the following are equivalent:

(i) *m* is a molecule;

(ii) $\varphi_{n-1}m \in AtC(A)$

(iii) $m \in J(A)$

But $\varphi_{n-1} : A \to A$ is a Boolean multiplicative closure, that is a multiplicative closure operator and $\mathrm{Im}\,\varphi_{n-1} = \mathrm{C}(\mathrm{A})$. We have the following

Proposition 1.5 [5] If *A* is a bounded distributive lattice, the following are equivalent:

(i) *A* has a Boolean multiplicative closure

(ii) A is a Stone Algebra (That is *A* ia pseudocomplemented lattice such that $x^* \vee x^{**} = 1$, for any $x \in A$). x^* is the pseudocomplement of *x*, and $x \mapsto x^{**}$ is the Boolean multiplicative closure.

Then the Proposition 1.4 allows us the following generalization:

Proposition 1.6: In a Stone algebra *A* the following are equivalent:

(i) *m* is a molecule

(ii) $m^{**} \in AtC(A)$

$m \in J(A) \Rightarrow m$ is a molecule.

Proof :(i)$\Rightarrow$(ii) If m^{**} is a not an atom in $C(A)$ there is $a \in C(A)$, such that $0 < a < m^{**}$, so $m^{**} \not\leq a$, that is $m^{**} \wedge a^* \neq 0$. Hence $\left(m \wedge a^*\right)^{**} \neq 0$. Therefore $m \wedge a^* \neq 0$. But $(m \wedge a)^{**} = m^{**} \wedge a = a \neq 0$ so $m \wedge a \neq 0$. Since *m* is molecule, it follows that $m \wedge a \wedge a^* \neq 0$, a contradiction.

(ii)$\Rightarrow$(i):Consider $x, y \neq 0$, $x, y \leq m$. We have $x^{**} \neq 0$ and $y^{**} \neq 0$, so $x^{**} = y^{**} = m^{**}$. This implies $(x \wedge y)^{**} = x^{**} \wedge y^{**} \neq 0$ so $x \wedge y \neq 0$.

If $m \in J(A)$, $[m)$ is a prime filter. Consider $[m^{**})_{C(A)}$ and $x, y \in C(A)$, such that $x \vee y \geq m^{**}$. Therefore $x \vee y \geq m$. Since $[m)$ is prime it follows $x \geq m$, or $y \geq m$ so $x \geq m^{**}$ or $y \geq m^{**}$. Hence $[m^{**})_{C(A)}$ is prime. So $m^{**} \in J(C(A)) = AtC(A)$.

Therefore in a Stone algebra $J(A) \subseteq M(A)$. In the next section we prove the converse of Propositions 1.4 and 1.6 in the finite case.

2. Molecules in finite distributive lattices

In a infinite distributive lattice there are no relations between molecules and atoms. But in a finite lattice the situation is very different and we have a satisfactory characterization of molecules.

Consider now *A* a finite distributive lattice.

Proposition2.1 $m \in A$ is a molecule iff $(\exists! a)(a \in At(A) \text{and } a \leq m)$

Proof. If $m \in M(A)$ then there exists an atom $a \leq m$, since *A* is finite. If $b \leq m, b \in At(A)$ and $b \neq a$ we have $a \wedge b = 0$, a contradiction.

If *m* is an element such that there exists an unique atom $a, a \leq m$, consider $0 \neq x \leq m$. We have an atom $b \leq x$ so $b \leq m$. Hence $a = b$. If $x, y \leq m, x, y \neq 0$, then $a \leq x \wedge y$, so $x \wedge y \neq 0$.

If $a \in At(A)$ note by $M_a = \{m \in M \mid \operatorname{supp} m = a\}$, where supp *m* is the unique atom of Proposition 2.1

Proposition 2.2 The following properties hold:

(i) $x \in M_a, 0 \neq y \leq x \Rightarrow y \in M_a$

(ii) $x, y \in M_a \Rightarrow x \wedge y \in M_a$

(iii) $x, y \in M_a \Rightarrow x \vee y \in M_a$

(iv) M_a is a sub lattice, where *a* is the least element

(v) $x \in M_a, x \in M_b\ a \neq b \Rightarrow x \wedge y = 0$

(vi) $M = \underset{a \in At(A)}{+} M_a$ (cardinal sum).

Proof :

(i) If $x \in M_a$ then $\operatorname{supp} x = a$, and $b \leq y$, $b \in At(A) \Rightarrow b \leq x \Rightarrow b = a \Rightarrow y \in M_a$

(ii) By (i)

(iii) Consider an atom $b \leq x \vee y$. As *b* is join-irreducible, we have $b \leq x$ or

$b \leq y$, so $b = a$

(iv) By (ii) and (iii)

(v) If $x \wedge y \neq 0$ then $x \wedge y \in M_a, M_b$, a contradiction.

Lema 2.3 We have the following cases:

(a) card $At(A) = 1 \Rightarrow A$ is a Stone algebra.

(b) card $At(A) > 1$. The following are equivalent:

(i) $J(A) \subseteq M(A)$

(ii) $[M]_D = A$

(iii)For any $x \neq 0$ there exists a set $\{x_1, \ldots x_n\} \subseteq M$ such that $i \neq j \Rightarrow x_i \wedge x_j = 0$ and $x = \bigvee_{i=1}^{n} x_i$ or $x \in M$

(iv) $1 = \bigvee_{a \in At(A)} 1_a$, where 1_a is the greatest element in M_a

(v) $A \cong \prod_{a \in At} A | [1_a)$

(vi) A is a Stone algebra

Proof:

If A has an unique atom, A is a dense lattice, that is a Stone algebra. Suppose now, card $At(A) > 1$.

(i)$\Rightarrow$(ii): Any $x \in A, x \neq 0$ is a join of nonzero join-irreducible elements, because A is finite, and $0 = a_1 \wedge a_2, a_1, a_2 \in At(A)$ so $[J(A)]_D = A$

(ii)$\Rightarrow$(iii): By Proposition 2.2 (ii), (iii), (v)

(iii)$\Rightarrow$(iv): By Proposition 2.2 (iv) M_a is a finite lattice, that is it has a greatest element

(iv)$\Rightarrow$(v): The set $\{1_a\}_{a \in At(A)}$, contains disjunct elements with join 1, so we have the decomposition $A = \prod_{a \in At} A | [1_a)$ (By [2], p.68)

(v)$\Rightarrow$(vi): $A | [1_a) \cong (1_a] = 0 \oplus M_a$;as M_a is a lattice (Proposition 2.2 (iv)) it follows that the factors are dense lattices, hence Stone Algebras

(vi)$\Rightarrow$(i): By Proposition 1.6

Corollary 2.4 For a finite distributive lattice, the following are equivalent:

(i) A is a Stone algebra

(ii) $J(A) \subseteq M(A)$

And the decomposition of Lemma 2.3 (iii) is unique

Proposition 2.5 In a finite distributive lattice, $J(A) \supseteq M(A)$ iff M_a is a chain for any $a \in At(A)$

Proof:

If any molecule is join-irreducible, consider $x, y \in M_a$. By Proposition 2.2 (iii) $x \vee y \in M_a$, so $x \vee y \in J(A)$.Hence $x \vee y = x$ or $x \vee y = y$ and *x*, *y* are comparable. If M_a is a chain, consider $x \in M_a$. *x* is a join of nonzero join – irreducible elements. By Proposition 2.2 (i), these elements belong to M_a, and their join is a join – irreducible element.

Theorema 2.6 If *A* is finite distributive lattice, the following are equivalent:

(i) $J(A) = M(A)$

(ii) *A* has a structure of Lukasiewicz-Moisil algebra

(iii) $J(A) = +L_i$ where L_i are chains

Proof:

(i)$\Rightarrow$(ii):By Lemma 2.3 *A* is direct product of factors $0 \oplus M_a$ and by Proposition 2.5 M_a are chains, so *A* is direct product of chains and has a structure of Lukasiewicz-Moisil algebra

(ii)$\Rightarrow$(iii): By [4], p.277

(iii)$\Rightarrow$(i): If $x \in J(A)$, then *x* belongs to a unique maximal chain L_i ;so the least element in L_i is an atom a and $x \geq a$ (*a* is unique by hypothesis). Hence $x \in M_a$.

If $x \in M$, then there exist an unique atom *a*, such that $x \in M_a$, and *x* is a join of nonzero join – irreducible elements in M_a. These elements are in L_i for some *i*, so their join is in L_i.

Remark Proposition 2.1, 2.2 and Lemma 2.3 hold if *A* is a distributive lattice that satisfies the descending chain condition.

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R E F E R E N C E S

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