BOUNDS OF FRACTIONAL INTEGRAL OPERATORS CONTAINING MITTAG-LEFFLER FUNCTION

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Recently, an extended Mittag-Leffler function has been utilized in the extension of fractional integral operators. This paper investigates bounds of the generalized fractional integral operators. The bounds of extended generalized fractional integral operators are calculated by using (α, m)-convex functions. By fixing parameters involved in the Mittag-Leffler function bounds of various well known fractional integral operators have been obtained. Furthermore, Hadamard type inequalities have been established by imposing an additional condition to a differentiable function f such that $|f'|$ is (α, m)-convex.

Keywords: (α, m)-convex functions, generalized fractional integral operators, Mittag-Leffler function, bounds.

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1. Introduction

Mittag-Leffler was a Swedish mathematician. In 1903, he introduced a function which is now well known as Mittag-Leffler function [10]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}; z \in \mathbb{C}$$

where $\Gamma(.)$ is the Gamma function and $\alpha \in \mathbb{C}, \Re(\alpha) > 0$.

Mittag-Leffler function arises in the solution of fractional order differential and integral equations same as exponential function appears in the solution of integer order differential equations. In the recent past, due to rapid development in the subject of fractional calculus, the Mittag-Leffler function gets much importance and fame on account of its wide applications in the diverse fields of sciences. For the last two decades scientist and engineers have paid their great interest in this function due to its wide applications in various problems of applied nature such as; fluid flow, control systems, electric networks, archeology, statistical distribution. For a detailed and comprehensive study of this function and its further consequences, readers are suggested [6] and references therein.

Recently, in [1] Andrić et al. defined an extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(.;p)$ as follows:

Definition 1.1. [1, p. 1381, Eq. (2.8)] *Let $\mu, \sigma, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function is defined by*

$$E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \sigma)} \frac{t^n}{(l)_{n\delta}}, \tag{1}$$

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where β_p is the generalized beta function defined as follows:

$$\beta_p(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}\,dt$$

and $(c)_{nk}$ is the Pochhammer symbol, $(c)_{nk}=\frac{\Gamma(c+nk)}{\Gamma(c)}$.

Lemma 1.1. [1, p. 1384, Eq. (2.12)] *If $m\in\mathbb{N},\omega,\mu,\alpha,l,\gamma,c\in\mathbb{C},\Re(\mu),\Re(\alpha),\Re(l)>0,\Re(c)>\Re(\gamma)>0$ with $p\geq 0,\delta>0$ and $0<k<\delta+\Re(\mu)$, then*

$$\left(\frac{d}{dt}\right)^m[t^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^{\mu};p)]=t^{\alpha-m-1}E^{\gamma,\delta,k,c}_{\mu,\alpha-m,l}(\omega t^{\mu};p)\quad \Re(\alpha)>m. \tag{2}$$

Remark 1.1. *(1) is a generalization of the following functions:*
(i) setting $p=0$, it reduces to the Salim-Faraj function $E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(t)$ defined in [13, p. 2, Eq. (6)],
(ii) setting $l=\delta=1$, it reduces to the function $E^{\gamma,k,c}_{\mu,\sigma}(t;p)$ defined by Rahman et al. in [12, p. 4247, Eq. (2.1)],
(iii) setting $p=0$ and $l=\delta=1$, it reduces to the Shukla-Prajapati function $E^{\gamma,k}_{\mu,\sigma}(t)$ defined in [15, p. 798, Eq. (1.4)] *see also* [16, p. 3, Eq. (1.13)],
(iv) setting $p=0$ and $l=\delta=k=1$, it reduces to the Prabhakar function $E^{\gamma}_{\mu,\sigma}(t)$ defined in [11, p. 7, Eq. (1.3)].

The corresponding left-sided and right-sided generalized fractional integral operators $\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}f$ and $\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,b^-}f$ are defined as follows:

Definition 1.2. [1, p. 1385, Eq. (2.13)] *Let $\omega,\mu,\sigma,l,\gamma,c\in\mathbb{C}$, $\Re(\mu),\Re(\sigma),\Re(l)>0$, $\Re(c)>\Re(\gamma)>0$ with $p\geq 0$, $\delta>0$ and $0<k\leq\delta+\Re(\mu)$. Let $f\in L_1[a,b]$ and $x\in[a,b]$. Then the generalized fractional integral operators are defined by:*

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}f\right)(x;p)=\int_a^x(x-t)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(x-t)^{\mu};p)f(t)dt, \tag{3}$$

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,b^-}f\right)(x;p)=\int_x^b(t-x)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(t-x)^{\mu};p)f(t)dt. \tag{4}$$

Remark 1.2. *(3) and (4) are the generalizations of the following fractional integral operators:*
(i) setting $p=0$, they reduce to the fractional integral operators defined by Salim-Faraj in [13, p. 2, Eq. (8)],
(ii) setting $l=\delta=1$, they reduce to the fractional integral operators defined by Rahman et al. in [12, p. 4247, Eq. (2.2)],
(iii) setting $p=0$ and $l=\delta=1$, they reduce to the fractional integral operators defined by Srivastava-Tomovski in [16, p. 5, Eq. (2.12)],
(iv) setting $p=0$ and $l=\delta=k=1$, they reduce to the fractional integral operators defined by Prabhakar in [11, p. 7, Eqs. (1.2) and (1.4)],
(v) setting $p=\omega=0$, they reduce to the left-sided and right-sided Riemann-Liouville fractional integrals.

Fractional integral and differential operators have played a key role in the development of the theory and applications of fractional differential equations and other subjects of sciences and engineering. In [1], several properties of the extended generalized Mittag-Leffler function and corresponding generalized fractional operators have been studied. In particular in [5], it is noted that

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}1\right)(x;p)=(x-a)^{\sigma}E^{\gamma,\delta,k,c}_{\mu,\sigma+1,l}(w(x-a)^{\mu};p) \tag{5}$$

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-}1\right)(x;p)=(b-x)^{\tau}E^{\gamma,\delta,k,c}_{\mu,\tau+1,l}(w(b-x)^{\mu};p). \tag{6}$$

Furthermore, the following notations have been used:

$$C^{\sigma}_{\omega,a^+}(x;p)=\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}1\right)(x;p) \tag{7}$$

$$C^{\tau}_{\omega,b^-}(x;p)=\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-}1\right)(x;p). \tag{8}$$

These notations will be also used frequently in this paper.
After introducing the extended Mittag-Lefler function and corresponding fractional integral operators, in the following convex function, and one of its generalization named (α,m)-convex function are given:

Definition 1.3. *Let I ba an interval of real numbers. Then a function $f:I\to\mathbb{R}$ is said to be convex, if the following inequality holds:*

$$f(ta+(1-t)b)\le tf(a)+(1-t)f(b),$$

for all $a,b\in I$ and $t\in(0,1)$.

Definition 1.4. [9] *A function $f:[0,b]\subset\mathbb{R}\to\mathbb{R}$, $b>0$ is said to be (α,m)-convex, where $(\alpha,m)\in[0,1]^2$, if the following inequality holds:*

$$f(tx+m(1-t)y)\le t^{\alpha}f(x)+m(1-t^{\alpha})f(y), \tag{9}$$

for all $x,y\in[0,b]$ and $t\in(0,1)$.

Remark 1.3. *(i) If $(\alpha,m)=(1,m)$, then (9) provides the definition of m-convex function.*
(ii) If $(\alpha,m)=(1,1)$, then (9) provides the definition of convex function.
(iii) If $(\alpha,m)=(1,0)$, then (9) provides the definition of star-shaped function.

Convex functions have proved a catalyst in the development of various subjects of pure and applied fields of mathematics and statistics, like optimization theory, mathematical analysis, functional analysis, graph theory, probability etc. A convex function is also defined equivalently by the well known Hadamard inequality. The Hadamard inequality has been studied by various researchers using generalized convex functions. For example, for (α,m)-convex functions see [2, 7, 14, 17]. We are interested to produce a generalized fractional integral inequality of Hadamard type for (α,m)-convex functions (see Theorem 2.3) and study its special cases.

The aim of this research is to establish the bounds of fractional integrals defined in Definition 1.2. For getting these bounds, the definition of (α,m)-convex function has been utilized. By the simultaneous use of Definition 1.2 and (α,m)-convexity, the obtained results are much general and provide the all possible outcomes of fractional integrals (deduced in Remark 1.2) and functions (deduced in Remark 1.3).
The rest of the paper is organized as follows:

In Section 2, the first result provides the bounds of fractional integral operators defined in (3) and (4) for (α,m)-convex functions and further their consequences are discussed. Then by using (α,m)-convexity of a differentiable function f in absolute (i.e. $|f'|$ is (α,m)-convex), bounds of fractional integrals in modulus form are established and some implications are identified. The last result of Section 2 provides estimations of Hadamard type which produce some interesting Hadamard type fractional inequalities. In Section 3, the present research has been concluded.

2. Bounds of generalized fractional integral operators via (α,m)-convex function

Firstly, bounds of sum of left and right generalized fractional operators (3) and (4) are studied as follows:

Theorem 2.1. *Let $f:[a,b]\longrightarrow\mathbb{R}$, $a<b$, be a real valued function. If f is positive and (α,m)-convex, then for $(\alpha,m)\in[0,1]\times(0,1]$, the following fractional inequality holds:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}f\right)(x;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-}f\right)(x;p) \\ &\leq\frac{1}{\alpha+1}\Big((x-a)C^{\sigma-1}_{\omega,a^+}(x;p)f(a)+(b-x)C^{\tau-1}_{\omega,b^-}(x;p)f(b) \\ &+\alpha mf\left(\frac{x}{m}\right)\Big((x-a)C^{\sigma-1}_{\omega,a^+}(x;p)+(b-x)C^{\tau-1}_{\omega,b^-}(x;p)\Big)\Big)\end{aligned}\tag{10}$$

for all $x\in(a,b)$ and $\sigma,\tau\geq1$.

Proof. By applying (α,m)-convexity of the function f, the following inequality can be obtained:

$$f(t)\leq\left(\frac{x-t}{x-a}\right)^{\alpha}f(a)+m\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)f\left(\frac{x}{m}\right).\tag{11}$$

For $x\in[a,b]$ and $\sigma>1$, the following inequality holds true:

$$(x-t)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(x-t)^{\mu};p)\leq(x-a)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(x-a)^{\mu};p),\quad t\in[a,x].\tag{12}$$

From (11) and (12), one can obtain the following integral inequality:

$$\begin{aligned}&\int_a^x(x-t)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(x-t)^{\mu};p)f(t)dt \\ &\leq(x-a)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(x-a)^{\mu};p)\left(f(a)\int_a^x\left(\frac{x-t}{x-a}\right)^{\alpha}dt\right. \\ &\left.+mf\left(\frac{x}{m}\right)\int_a^x\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)dt\right),x\in(a,b).\end{aligned}$$

By using (3) of Definition 1.2, (5) and (7), bound of left fractional operator (3) is obtained

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}f\right)(x;p)\leq(x-a)C^{\sigma-1}_{\omega,a^+}(x;p)\left(\frac{f(a)+\alpha mf\left(\frac{x}{m}\right)}{\alpha+1}\right).\tag{13}$$

Again by using (α,m)-convexity of the function f, the following inequality can be obtained:

$$f(t)\leq\left(\frac{t-x}{b-x}\right)^{\alpha}f(b)+m\left(1-\left(\frac{t-x}{b-x}\right)^{\alpha}\right)f\left(\frac{x}{m}\right).\tag{14}$$

Further for $x\in[a,b]$ and $\tau>1$, the following inequality holds true:

$$(t-x)^{\tau-1}E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(t-x)^{\mu};p)\leq(b-x)^{\tau-1}E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(b-x)^{\mu};p),\quad t\in[x,b].\tag{15}$$

From (14) and (15), one can obtain the following integral inequality:

$$\begin{aligned}&\int_x^b(t-x)^{\tau-1}E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(t-x)^{\mu};p)f(t)dt \\ &\leq(b-x)^{\tau-1}E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(b-x)^{\mu};p)\left(f(b)\int_x^b\left(\frac{t-x}{b-x}\right)^{\alpha}dt\right. \\ &\left.+mf\left(\frac{x}{m}\right)\int_x^b\left(1-\left(\frac{t-x}{b-x}\right)^{\alpha}\right)dt\right).\end{aligned}$$

By using (4) of Definition 1.2, (6) and (8), bound of right fractional operator (4) is obtained

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-}f\right)(x;p)\leq(b-x)C^{\tau-1}_{\omega,b^-}(x;p)\left(\frac{f(b)+\alpha mf\left(\frac{x}{m}\right)}{\alpha+1}\right).\tag{16}$$

By adding (13) and (16), bound required in (10) can be achieved. □

Corollary 2.1. *If $\sigma=\tau$ in (10), then we get the following fractional inequality:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}f\right)(x;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,b^-}f\right)(x;p)\\&\leq\frac{1}{\alpha+1}\Big((x-a)C^{\sigma-1}_{\omega,a^+}(x;p)f(a)+(b-x)C^{\sigma-1}_{\omega,b^-}(x;p)f(b)\\&+\alpha mf\left(\frac{x}{m}\right)\Big((x-a)C^{\sigma-1}_{\omega,a^+}(x;p)+(b-x)C^{\sigma-1}_{\omega,b^-}(x;p)\Big)\Big).\end{aligned}\tag{17}$$

Remark 2.1. *(i) By setting $\omega=p=0$ and $\alpha=m=1$ in (10), one can obtain* [4, Theorem 1].
(ii) By setting $\omega=p=0$ and $\alpha=m=1$ in (17), one can obtain [4, corollary 1].

In the next result, bounds of sum of left and right fractional operators (3) and (4) are studied in modulus form.

Theorem 2.2. *Let $f:[a,b]\longrightarrow\mathbb{R}$, $a<b$, be a real valued function. If f is differentiable and $|f'|$ is (α,m)-convex, then for $(\alpha,m)\in[0,1]\times(0,1]$, the following fractional integral inequality holds:*

$$\begin{aligned}&\Big|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma-1,l,\omega,a^+}f\right)(x;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau-1,l,\omega,b^-}f\right)(x;p)\\&-\left(C^{\sigma-1}_{\omega,a^+}(x;p)f(a)+C^{\tau-1}_{\omega,b^-}(x;p)f(b)\right)\Big|\\&\leq\frac{1}{\alpha+1}\Big((x-a)C^{\sigma-1}_{\omega,a^+}(x;p)|f'(a)|+(b-x)C^{\tau-1}_{\omega,b^-}(x;p)|f'(b)|\\&+\alpha m\left|f'\left(\frac{x}{m}\right)\right|\Big((x-a)C^{\sigma-1}_{\omega,a^+}(x;p)+(b-x)C^{\tau-1}_{\omega,b^-}(x;p)\Big)\Big)\end{aligned}\tag{18}$$

for all $x\in(a,b)$ and $\sigma,\tau\geq1$.

Proof. By using (α,m)-convexity of the function $|f'|$, it follows that:

$$|f'(t)|\leq\left(\frac{x-t}{x-a}\right)^{\alpha}|f'(a)|+m\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|.\tag{19}$$

From (19), one can obtain

$$f'(t)\leq\left(\frac{x-t}{x-a}\right)^{\alpha}|f'(a)|+m\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|.\tag{20}$$

From (12) and (20), one can obtain the following integral inequality:

$$\begin{aligned}&\int_a^x(x-t)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(x-t)^{\mu};p)f'(t)dt\\&\leq(x-a)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(x-a)^{\mu};p)\left(|f'(a)|\int_a^x\left(\frac{x-t}{x-a}\right)^{\alpha}dt\right.\\&\left.+m\left|f'\left(\frac{x}{m}\right)\right|\int_a^x\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)dt\right)\\&=(x-a)^{\sigma}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(x-a)^{\mu};p)\left(\frac{|f'(a)|+\alpha m\left|f'\left(\frac{x}{m}\right)\right|}{\alpha+1}\right).\end{aligned}\tag{21}$$

The left hand side calculated as follows:

$$\int_a^x(x-t)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(x-t)^{\mu};p)f'(t)dt\tag{22}$$

by putting $x-t=u$, that is $t=x-u$, and applying the derivative property (2) of Mittag-Leffler function, one obtains

$$\begin{aligned}&\int_0^{x-a} u^{\sigma-1}E_{\mu,\sigma,l}^{\gamma,\delta,k,c}(\omega u^{\mu};p)f'(x-u)du\\&=-(x-a)^{\sigma-1}E_{\mu,\sigma,l}^{\gamma,\delta,k,c}(\omega(x-a)^{\mu};p)f(a)+\int_0^{x-a}u^{\sigma-2}E_{\mu,\sigma-1,l}^{\gamma,\delta,k,c}(\omega u^{\mu};p)f(x-u)du.\end{aligned}$$

Using the change of variable $x-u=t$, in the second term on the right hand side of the above equality, by (3) one obtains

$$\begin{aligned}&\int_0^{x-a} u^{\sigma-1}E_{\mu,\sigma,l}^{\gamma,\delta,k,c}(\omega u^{\mu};p)f'(x-u)du\\&=\left(\epsilon_{\mu,\sigma-1,l,\omega,a^+}^{\gamma,\delta,k,c}f\right)(x;p)-(x-a)^{\sigma-1}E_{\mu,\sigma,l}^{\gamma,\delta,k,c}(\omega(x-a)^{\mu};p)f(a).\end{aligned}$$

Therefore, by using (5) and (7), (21) takes the following form:

$$\begin{aligned}&\left(\epsilon_{\mu,\sigma-1,l,\omega,a^+}^{\gamma,\delta,k,c}f\right)(x;p)-C_{\omega,a^+}^{\sigma-1}(x;p)f(a)\\&\leq (x-a)C_{\omega,a^+}^{\sigma-1}(x;p)\left(\frac{|f'(a)|+\alpha m\left|f'\left(\frac{x}{m}\right)\right|}{\alpha+1}\right).\end{aligned}\tag{23}$$

Also from (19), one can obtain

$$f'(t)\geq -\left(\left(\frac{x-t}{x-a}\right)^{\alpha}|f'(a)|+m\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|\right).\tag{24}$$

Following the same procedure as we did for (20), the following inequality can be obtained:

$$\begin{aligned}&C_{\omega,a^+}^{\sigma-1}(x;p)f(a)-\left(\epsilon_{\mu,\sigma-1,l,\omega,a^+}^{\gamma,\delta,k,c}f\right)(x;p)\\&\leq (x-a)C_{\omega,a^+}^{\sigma-1}(x;p)\left(\frac{|f'(a)|+\alpha m\left|f'\left(\frac{x}{m}\right)\right|}{\alpha+1}\right).\end{aligned}\tag{25}$$

From (23) and (25), the following modulus inequality can be obtained:

$$\begin{aligned}&\left|\left(\epsilon_{\mu,\sigma-1,l,\omega,a^+}^{\gamma,\delta,k,c}f\right)(x;p)-C_{\omega,a^+}^{\sigma-1}(x;p)f(a)\right|\\&\leq (x-a)C_{\omega,a^+}^{\sigma-1}(x;p)\left(\frac{|f'(a)|+\alpha m\left|f'\left(\frac{x}{m}\right)\right|}{\alpha+1}\right).\end{aligned}\tag{26}$$

By using (α,m)-convexity of $|f'|$, the following inequality can be obtained:

$$|f'(t)|\leq\left(\frac{t-x}{b-x}\right)^{\alpha}|f'(b)|+m\left(1-\left(\frac{t-x}{b-x}\right)^{\alpha}\right)\left|f'\left(\frac{x}{m}\right)\right|.\tag{27}$$

On the same lines as we have done for (12), (20) and (24), one can obtain from (15) and (27), the following modulus inequality:

$$\begin{aligned}&\left|\left(\epsilon_{\mu,\tau-1,l,\omega,b^-}^{\gamma,\delta,k,c}f\right)(x;p)-(b-x)^{\tau-1}E_{\mu,\tau,l}^{\gamma,\delta,k,c}(\omega(b-x)^{\mu};p)f(b)\right|\\&\leq (b-x)C_{\omega,b^-}^{\tau-1}(x;p)\left(\frac{|f'(b)|+\alpha m\left|f'\left(\frac{x}{m}\right)\right|}{\alpha+1}\right).\end{aligned}\tag{28}$$

From inequalities (26) and (28) via triangular inequality, (18) can be achieved. □

In the following corollary and remark consequences of above theorem have been discussed in detail:

Corollary 2.2. *If $\sigma=\tau$ in (18), then we get the following fractional inequality:*

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma-1,l,\omega,a^+}f\right)(x;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma-1,l,\omega,b^-}f\right)(x;p)\right.\\&\left.-\left(C^{\sigma-1}_{\omega,a^+}(x;p)f(a)+C^{\sigma-1}_{\omega,b^-}(x;p)f(b)\right)\right|\\&\leq\frac{1}{\alpha+1}\left((x-a)C^{\sigma-1}_{\omega,a^+}(x;p)|f'(a)|+(b-x)C^{\sigma-1}_{\omega,b^-}(x;p)|f'(b)|\right.\\&\left.+\alpha m\left|f'\left(\frac{x}{m}\right)\right|\left((x-a)C^{\sigma-1}_{\omega,a^+}(x;p)+(b-x)C^{\sigma-1}_{\omega,b^-}(x;p)\right)\right).\end{aligned}\tag{29}$$

Remark 2.2. *(i) By setting $\omega=p=0$, $\alpha=m=1$, and replacing σ with $\sigma+1$ in (18), one can obtain* [4, Theorem 2].
(ii) By setting $\omega=p=0$, $\alpha=m=1$ and replacing σ with $\sigma+1$ in (29), one can obtain [4, Corollary 4].
(iii) By setting $\omega=p=0,\sigma=\alpha=m=1$ and $x=\frac{a+b}{2}$, in (29), one can obtain [4, Corollary 5].
(iv) By setting $\omega=p=0$, $\sigma=1$ in (29), and f' passes through $x=\frac{a+b}{2}$, one can obtain [3, Theorem 2.2].

For the sake of Hadamard type bounds of generalized fractional operators, the following lemma is useful.

Lemma 2.1. *Let $f:[0,\infty)\to R$ be a (α,m)-convex function. If $f(x)=f\left(\frac{a+b-x}{m}\right)$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2^\alpha}(1+m(2^\alpha-1))f(x)\quad x\in[a,b].\tag{30}$$

Theorem 2.3. *Let $f:[a,b]\longrightarrow\mathbb{R}$, $a<b$, be a real valued function. If f is positive, (α,m)-convex and $f(x)=f\left(\frac{a+b-x}{m}\right)$, then for $(\alpha,m)\in[0,1]\times(0,1]$, the following inequality holds:*

$$\begin{aligned}&\frac{2^\alpha}{(1+m(2^\alpha-1))}f\left(\frac{a+b}{2}\right)\left(C_{\tau+1,b^-}(a;p)+C_{\sigma+1,a^+}(b;p)\right)\\&\leq\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-}f\right)(a;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(b;p)\\&\leq(b-a)^2\left(C_{\tau-1,b^-}(a;p)+C_{\sigma-1,a^+}(b;p)\right)\left(\frac{f(a)+\alpha mf\left(\frac{b}{m}\right)}{\alpha+1}\right)\end{aligned}\tag{31}$$

for all $x\in[a,b]$ and $\sigma,\tau>0$.

Proof. By using (α,m)-convexity of the function f, the following inequality can be obtained:

$$f(x)\leq\left(\frac{b-x}{b-a}\right)^\alpha f(a)+m\left(1-\left(\frac{b-x}{b-a}\right)^\alpha\right)f\left(\frac{b}{m}\right).\tag{32}$$

For $x\in[a,b]$ and $\tau>0$, the following inequality holds true:

$$(x-a)^\tau E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(x-a)^\mu;p)\leq(b-a)^\tau E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(b-a)^\mu;p).\tag{33}$$

From (32) and (33), one can obtain the following integral inequality:

$$\begin{aligned}&\int_a^b(x-a)^\tau E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(x-a)^\mu;p)f(x)dx\\&\leq(b-a)^\tau E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(b-a)^\mu;p)\left(f(a)\int_a^b\left(\frac{b-x}{b-a}\right)^\alpha dx\right.\\&\left.+mf\left(\frac{b}{m}\right)\int_a^b\left(1-\left(\frac{b-x}{b-a}\right)^\alpha\right)dx\right).\end{aligned}$$

By using (4) of Definition 1.2, (6) and (8), the following bound of right fractional operator is obtained:

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-}f\right)(a;p) \le (b-a)^2 C_{\tau-1,b^-}(a;p)\left(\frac{f(a)+\alpha m f\left(\frac{b}{m}\right)}{\alpha+1}\right). \tag{34}$$

Now for $x \in [a,b]$ and $\sigma > 0$, the following inequality holds true:

$$(b-x)^\sigma E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(b-x)^\mu;p) \le (b-a)^\sigma E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(b-a)^\mu;p). \tag{35}$$

From (32) and (35), one can obtain the following integral inequality:

$$\begin{aligned}
&\int_a^b (b-x)^\sigma E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(b-x)^\mu;p)f(x)dx \\
&\le (b-a)^\sigma E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(b-a)^\mu;p)\left(f(a)\int_a^b\left(\frac{b-x}{b-a}\right)^\alpha dx\right. \\
&\left.+mf\left(\frac{b}{m}\right)\int_a^b\left(1-\left(\frac{b-x}{b-a}\right)^\alpha\right)dx\right).
\end{aligned}$$

By using (3) of Definition 1.2, (5) and (7), the following bound of left fractional operator is obtained:

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(b;p) \le (b-a)^2 C_{\sigma-1,a^+}(b;p)\left(\frac{f(a)+\alpha m f\left(\frac{b}{m}\right)}{\alpha+1}\right). \tag{36}$$

By adding (34) and (36), following bound for sum of left and right fractional operators holds:

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-}f\right)(a;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(b;p) \\
&\le (b-a)^2\left(C_{\tau-1,b^-}(a;p)+C_{\sigma-1,a^+}(b;p)\right)\left(\frac{f(a)+\alpha m f\left(\frac{b}{m}\right)}{\alpha+1}\right).
\end{aligned} \tag{37}$$

Multiplying (30) with $(x-a)^\tau E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(x-a)^\mu;p)$, one can obtain

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\int_a^b (x-a)^\tau E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(x-a)^\mu;p)dx \\
&\le \frac{1}{2^\alpha}(1+m(2^\alpha-1))\int_a^b (x-a)^\tau E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(x-a)^\mu;p)f(x)dx.
\end{aligned} \tag{38}$$

By using (4) and (8), one can obtain a bound for right fractional operator

$$\frac{2^\alpha}{(1+m(2^\alpha-1))}f\left(\frac{a+b}{2}\right)C_{\tau+1,b^-}(a;p) \le \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-}f\right)(a;p). \tag{39}$$

Multiplying (30) with $(b-x)^\sigma E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(b-x)^\mu;p)$, and then integrating over $[a,b]$, and using (3), one can obtain the bound for left fractional operator:

$$\frac{2^\alpha}{(1+m(2^\alpha-1))}f\left(\frac{a+b}{2}\right)C_{\sigma+1,a^+}(b;p) \le \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(b;p). \tag{40}$$

By adding (39) and (40), the bound for sum of fractional operators is obtained:

$$\begin{aligned}
&\frac{2^\alpha}{(1+m(2^\alpha-1))}f\left(\frac{a+b}{2}\right)\left(C_{\tau+1,b^-}(a;p)+C_{\sigma+1,a^+}(b;p)\right) \\
&\le \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-}f\right)(a;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(b;p).
\end{aligned} \tag{41}$$

From inequalities (37) and (41), inequality (31) can be achieved. □

Corollary 2.3. *If $\sigma = \tau$ in (31), then we get the following fractional inequality:*

$$\frac{2^{\alpha}}{(1+m(2^{\alpha}-1))} f\left(\frac{a+b}{2}\right)\left(C_{\sigma+1,b^-}(a;p)+C_{\sigma+1,a^+}(b;p)\right) \tag{42}$$
$$\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,b^-}f\right)(a;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(b;p)$$
$$\leq (b-a)^2\left(C_{\sigma-1,b^-}(a;p)+C_{\sigma-1,a^+}(b;p)\right)\left(\frac{f(a)+\alpha m f\left(\frac{b}{m}\right)}{\alpha+1}\right).$$

Remark 2.3. *(i) By setting $\omega = p = 0$ and $\alpha = m = 1$ in (31), one can obtain* [4, Theorem 3].
(ii) By setting $\omega = p = 0$, $\alpha = m = 1$ in (42), one can obtain [4, Corollary 6].
(iii) By taking $\sigma \to 0$, $\omega = p = 0$ and $\alpha = m = 1$, one can obtain the Hadamard inequality.

3. Concluding Remarks

The results obtained in this work provide bounds of various fractional integral operators simultaneously, which have been independently defined by various authors of recent decades. For example in Theorem 2.1 selecting $p = 0$, bounds for fractional integral operators defined by Salim and Faraj in [13], selecting $l = \delta = 1$, bounds for fractional integral operators defined by Rahman et al. in [12], selecting $p = 0$ and $l = \delta = 1$, bounds for fractional integral operators defined by Shukla and Prajapati in [15] and see also [16], selecting $p = 0$ and $l = \delta = k = 1$, bounds for fractional integral operators defined by Prabhakar in [11], selecting $p = \omega = 0$, bounds for Riemann-Liouville fractional integrals are achieved for (α, m)-convex, m-convex, convex, star-shaped functions. Moreover, Theorems 2.2 and 2.3 are applicable for all fractional integrals comprised in Remark 1.2 and for all functions comprised in Remark 1.3.

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