# SOME PHYSICAL IMPLICATIONS OF ABSOLUTE GEOMETRIES IN THE DESCRIPTION OF COMPLEX SYSTEMS DYNAMICS

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*Some implications of absolute geometries in the description of complex systems dynamics, at various scale resolutions are highlighted. In such context, by means of an analytic geometry of 2 x 2 matrices, a generalization of the standard velocities space in Fock's sense is obtained. Moreover, in the one-dimensional homographic action case, various chaos transition scenarios (period doubling and intermittences) can be mimed, through the selection of scale resolution.*

**Keywords**: absolute geometries, complex systems dynamics, 2 x 2 matrices, velocities space, chaos transition scenarios

## 1. Introduction

Recent results [1-3] highlight the fundamental role of absolute geometries in the description of complex systems dynamics, both at microscopic scale (e.g., based on SL(2R) invariance of multifractal Schrödinger-type equations [1, 2]) and at macroscopic scale (e.g., through the same SL(2R) invariance in the description of axial symmetry gravitational field, based on the Ernst complex potential [4, 5]).

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Moreover, various mathematical operational procedures, developed in the framework of the Scale Relativity Theory [6-8], imply, by means of the same SL(2R) invariance, several multifractal analyses of complex systems dynamics (at different scale resolutions). Because the SL(2R) invariance of the differential equations that describe complex systems dynamics is implicitly linked to homographic transformations (transformation related to absolute geometries) [8], in the present paper we will adress some physical implications of absolute geometries in the description of complex systems dynamics.

## 2. Metrization Principles

There are two fundamental actions of $2 \times 2$ matrices generally considered in theoretical physics: action in a one-dimensional domain and action in a two-dimensional domain, regardless of whether they are real or complex. The first of these actions is the so-called projective or homographic action. If the variable that characterizes the domain is denoted by $z$ (being real or complex) and the matrix is represented by the table of numbers

$$\boldsymbol{\alpha} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{1}$$

in which the four numbers are, again, real or complex, then the homographic action defines by the following correspondence of the domain $z$ in itself:

$$z \rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(z) \equiv \frac{\alpha z + \beta}{\gamma z + \delta} \tag{2}$$

On the other hand, there is the linear action of the matrix (1) in the two-dimensional domain. If the domain is arithmeticized by the pair of numbers (real or complex) $(x, y)$, then this action is defined by the following correspondence of the domain in itself:

$$(x, y) \rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}(x, y) \equiv (\alpha x + \beta y, \gamma x + \delta y) \tag{3}$$

The main difference between the two actions lies in the number of matrix elements required to fully characterize them. In the case of the homographic action, three numbers are sufficient, while for the linear action all four numbers are necessary to characterize it completely.

As known, these two actions or, more precisely, the correspondences that they induce by equations (2) and (3) are closely related to the algebraic properties of the set $2 \times 2$ matrices. In particular, the composition of functions is related to the multiplication of matrices. The fact is that matrices in general, not only the ones we are talking about now, form a group in relation with the matrix multiplication operation. In our particular case, this property induces a similar property of applications (2) and (3). This fact can be used to define properties of the matrices action described by differential geometry methods. But for this we need a geometry of the matrices themselves, and this refers a priori to a four-dimensional space, since a matrix has four components - its elements.

### 3. Analytic Geometry of 2 × 2 Matrices

If we denote by $X$ a point in this four-dimensional space, then its representation by coordinates is given by a quadruple of numbers representing the elements of the matrix:

$$X \equiv (\alpha, \beta, \gamma, \delta) \tag{4}$$

In the case of the homographic action where only three of these numbers are sufficient to characterize it, the elements of the matrix can be considered as homogeneous coordinates in a three-dimensional space. Only for the linear action the space has dimension four. The quadratic form

$$(X, X) \equiv \alpha\delta - \beta\gamma \tag{5}$$

which represents the determinant of the matrix, is an algebraic surface of the second degree in space, i.e., a quadric of the space. The points in space, located on this quadric, represent singular matrices, which are not reversible if they do not specifically refer to an action: they have an inverse, which makes sense if we refer to the homographic action, but they do not have an inverse to the linear action. Here is the point where the equivalence between the group of matrices and the group of induced morphisms breaks for certain correspondences.

The above quadric plays an important role in our space, in the sense that it can be taken as the absolute of space in Cayley's sense and can be used in the construction of a geometry absolute or Cayleyene [9, 10]. This geometry helps us, in turn, in the construction of some metric structures that lead to conservation laws which characterize the various actions, approximately in the spirit of the classic Hamilton-Jacobi equation. Our immediate goal is to discover those laws.

To construct that absolute geometry, we consider the quadratic form (5) as the norm in our space. It induces a scalar product through the doubling procedure:

$$(X_1, X_2) = \frac{1}{2}(\alpha_1\delta_2 + \alpha_2\delta_1 - \beta_1\gamma_2 - \beta_2\gamma_1) \tag{6}$$

where the correspondence between the indices of the points and those of the coordinates makes the notation obvious. This scalar product helps us to characterize the straight line in space, this being the essential concept in producing absolute metrics. A straight line between two points can be drawn, usually, as a linear combination of those points:

$$X = \lambda X_1 + \mu X_2 \tag{7}$$

with $\lambda$ and $\mu$ variable numbers that represent the homogeneous coordinates on the line. This straight line intersects the absolute in two points that have homogeneous parameters on the line, partially determined by the quadratic equation:

$$(X, X) \equiv \lambda^2 (X_1, X_1) + 2\lambda\mu(X_1, X_2) + \mu^2 (X_2, X_2) = 0 \tag{8}$$

Indeed, from here we can only determine the ratios of these two parameters, as the roots of the equation, that is

$$t \equiv \frac{\lambda}{\mu} = -\frac{(X_1, X_2)}{(X_1, X_1)} \pm \frac{\sqrt{(X_1, X_2)^2 - (X_1, X_1)(X_2, X_2)}}{(X_1, X_1)} \tag{9}$$

It turns out, however, that these ratios are sufficient for our purpose. Operationally, the metric is the distance between two infinitely neighboring points in space, so we need the distance between several two-points. The quantity that reduces to the known distance in the Euclidean limit is the anharmonic ratio that two points have to two reference points to their right. More precisely, the distance is, up to a numerical factor, the natural logarithm of this anharmonic ratio.

Given two points $X_{1,2}$, the straight line can be taken in the form of the inhomogeneous equation, i.e., $X = tX_1 + X_2$, because the parameter *t* intervenes in the form of the roots in equation (9). In order to define the distance between the two points, we choose two other reference points on the right $X_{3,4}$, let's say, and we construct the anharmonic ratio of the four points. It is defined by the anharmonic ratio of the inhomogeneous parameters of the points:

$$(X_1, X_2, X_3, X_4) = \frac{t_1 - t_3}{t_1 - t_4} : \frac{t_2 - t_3}{t_2 - t_4} \tag{10}$$

The absolute distance is simply proportional to the logarithm of this quantity. It depends, of course, on the pair of reference points chosen, but in our case this ambiguity can be substantially reduced if we refer the construction of the line to the absolute of space. Let us note first that in the equation of the line we have the values $t_2 = 0$ for the point $X_2$ and $t_1 = \infty$ for the point $X_1$. With this, the anharmonic ratio (10) takes the simple form

$$(X_1, X_2; X_3, X_4) = \frac{t_4}{t_3} \tag{11}$$

Furthermore, a unique choice for the points $X_3$ and $X_4$ is that of intersection points of the right with the absolute. This choice has the advantage of allowing a standardization of construction, because any pair of points in space has a corresponding pair of points on the absolute: the points where the line determined by the respective pair intersects the absolute. The corresponding parameters of the two points on the absolute are then given by the pair of roots from equation (9), so that equation (11) becomes

$$(X_1,X_2;X_3,X_4) = \frac{(X_1,X_2)+\sqrt{(X_1,X_2)^2-(X_1,X_1)(X_2\cdot X_2)}}{(X_1,X_2)-\sqrt{(X_1,X_2)^2-(X_1,X_1)(X_2\cdot X_2)}} \qquad (12)$$

With this expression of the anharmonic ratio, we can construct an infinitesimal version of a distance, which is precisely the metric we need. Assuming that our points $X_1$ and $X_2$ are infinitely close, and noting generically $X_1 = X, X_2 = X + dX$, we can calculate the quantities required in equation (12) in the form

$$\begin{aligned}
&(X_1,X_2) = (X,X) + (X,dX)\\
&(X_2,X_2) = (X,X) + 2(X < dX) + (dX,dX)\\
&(X_1,X_2)^2 - (X_1,X_1)(X_2,X_2) = (X,dX)^2 - (X,X)(dX,dX)
\end{aligned} \qquad (13)$$

Now, in the real domain we can accept that the quantity $(X,dX)/(X,X)$ is an infinitesimal of the first order, while $(dX,dX)/(X,X)$ is an infinitesimal of the second order. Therefore, the anharmonic ratio (12) can be developed to the first order in the form

$$(X_1,X_2;X_3,X_4) = 1 + 2\sqrt{\left(\frac{(X,dX)}{(X,X)}\right)^2 - \frac{(dX,dX)}{(X,X)}} \qquad (14)$$

In the same infinitesimal order, the logarithm of this expression is the part containing the radical, so that the absolute metric can be written, up to a factor that fixes its physical dimensions, as:

$$(ds)^2 = \left(\frac{(X,dX)}{(X,X)}\right)^2 - \frac{(dX,dX)}{(X,X)} \qquad (15)$$

It turns out that this expression is also valid under broader conditions of definition for our space: complex points, the general functional definition of the absolute, etc. We will refer here only to those definitions that have at least some physical meaning.

For the moment, let's make an observation on the absolute given by equation (5). When the coordinates of the points are real, this quadric is a hyperboloid with a canvas, as we know it from analytic geometry. The description can be extended for complex coordinates, in which case we are dealing with a space with five real dimensions, which can describe, for example, the conics in ordinary space. But regardless of these details, a calculation of the metric (15) gives us that metric in the form

$$(ds)^2 = \frac{(\alpha d\delta + \delta d\alpha - \beta d\gamma - \gamma d\beta)^2}{4(\alpha\delta - \beta\gamma)^2} - \frac{d\alpha d\delta - d\beta d\gamma}{\alpha\delta - \beta\gamma} \qquad (16)$$

This metric can be related to several important physical situations, especially related to homographic action of the matrices, which we will now consider in detail.

An immediate physical consequence of the above written formalism refers to a generalization of the velocities space from Einstein's Theory.

In such context, if we denote by $X$ a point in this space of velocities, then a coordinate representation is given by a quadruple of (until further notice, real) numbers:

$$X \equiv (c, \boldsymbol{v}) \tag{17}$$

The norm of these points is then given by the quadratic form:

$$(X, X) \equiv c^2 - \boldsymbol{v}^2 \tag{18}$$

Thus, the points which satisfy the condition $(X, X) = 0$ are geometrically shaping an absolute for this geometry. Among other things, physically they represent the propagation of light if the limit velocity $c$ is a constant. Specifically, the points with positive norm represent *inertial motions*, while the points with negative norm represent, for instance, *de Broglie waves* in the regular case of special relativity, or some other ensembles of Hertz material particles in the case of the Kepler motion [10].

The norm (18) induces an internal multiplication of the points by the known 'polarization' procedure (see, for instance, [10]):

$$(X_1, X_2) \equiv c_1 c_2 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \tag{19}$$

Here, an obvious correspondence is understood, between indices of points and indices of coordinates. This internal product helps in describing a straight line in space, which is the essential concept necessary in constructing a metric, at least from the differential geometric point of view.

Now, let us go back to our specific case, as given by equation (18). If the coordinates of points are real, then the absolute quadric is a two-sheeted hyperboloid, as we know it from analytic geometry. This description can also be used for the case of complex coordinates. However, in that instance, the geometrical image is not quite as simple as in the real case: there, we have to do with the intersection of two real quadrics, therefore with a real conic in space. Regardless of these details, momentarily we shall refer to a direct calculation of the metric given by equation (15). This will be based on the identification $dX \equiv (dc, d\boldsymbol{v})$, which gives the following quadratic form as a metric based on this representation:

$$(ds)^2 = \frac{c^2(d\boldsymbol{v})^2 - (\boldsymbol{v} \times d\boldsymbol{v})^2}{(c^2 - \boldsymbol{v}^2)^2} + \frac{\boldsymbol{v}^2(dc)^2 - 2c(\boldsymbol{v} \cdot d\boldsymbol{v})(dc)}{(c^2 - \boldsymbol{v}^2)^2} \tag{20}$$

This metric can be related to some known, and physically important situations. Let us, therefore, consider the equation just obtained in greater detail.

We wrote here the absolute metric as a sum of two terms for a particular reason: in the framework of special relativity, the first term from equation (20) is the metric of the velocity space, and it is a direct consequence of the law of composition of relativistic velocities [11]. In this way, one can say that the whole metric (20), for instance in cases where the first component of the point $X$ is a constant: the differential of a constant $c$ is always zero, and the second term in (20) vanishes. In other words, there *is no intrafinite four-vector velocity in relativistic physics* as we inherited it from Maxwell electrodynamics [12-16].

In order to understand the method, let us assume the case of constant $c$: only after analyzing this case, we can properly improve on it. The absolute metric (20) turns out to be:

$$(ds)^2 = \left[1 - \left(\frac{\boldsymbol{v}}{c}\right)^2\right]^{-2} \left[\left(\frac{d\boldsymbol{v}}{c}\right)^2 - \left(\frac{\boldsymbol{v}}{c} \times \frac{d\boldsymbol{v}}{c}\right)^2\right] \tag{21}$$

In the three-dimensional velocity space of relativity, this metric can be written in the form:

$$(ds)^2 = \left(\frac{d\boldsymbol{q}}{1 - \boldsymbol{q}^2}\right)^2 - \left(\frac{\boldsymbol{q} \times d\boldsymbol{q}}{1 - \boldsymbol{q}^2}\right)^2, \boldsymbol{q} \equiv \frac{\boldsymbol{v}}{c} \tag{22}$$

## 4. The One-Dimensional Homographic Action

Consider the equation

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta} \tag{23}$$

which represents the homographic action of the generic matrix that we denote by $\boldsymbol{\alpha}$. The problem what we want to solve is the following: to find the relationship between the set of matrices and a set of values of $t$ for which $t$ remains constant. This problem has important applications in theoretical statistics and relativity. From a geometrical point of view this means finding the set of points $(\alpha, \beta, \gamma, \delta)$ that uniquely corresponds to the values parameter *t*. Using equation (23), our problem is solved by a Riccati differential equation which is obtained as a consequence of the constancy of $t$: $dt = 0$ (for details see [17-23]):

$$dt + \omega_1 t^2 + \omega_2 t + \omega_3 = 0 \tag{24}$$

where we use the following notation:

$$\begin{aligned} \omega_1 &= \frac{\gamma d\alpha - \alpha d\gamma}{\alpha\delta - \beta\gamma}, \\ \omega_2 &= \frac{\delta d\alpha - \alpha d\delta + \gamma d\beta - \beta d\gamma}{\alpha\delta - \beta\gamma} \\ \omega_3 &= \frac{\delta d\beta - \beta d\delta}{\alpha\delta - \beta\gamma} \end{aligned} \quad (25)$$

It is then easy to see that the metric (16) is directly related to the discriminant of the quadratic polynomial from equation (24):

$$(ds)^2 = \frac{1}{4}\omega_2^2 - 4\omega_1\omega_3\Big) \quad (26)$$

The three differential forms in equation (25) constitute what is commonly known as a coframe [18, 21]) at any point of absolute space. This coframe allows us to translate the geometric properties of absolute space into algebraic properties related to differential equation (24).

The simplest of these properties refer to the motion on geodesics of the metric, which translates directly into statistical properties. In this case the 1-forms $\omega_1, \omega_2, \omega_3$ are exact differentials in the same parameter the length of the arc of the geodesic, let's say. Along these geodesics, equation (24) turns into an ordinary differential equation of the Riccati type:

$$\frac{dt}{ds} = a_1 t^2 + 2a_2 t + a_3 \quad (27)$$

Here the parameters $a_{1,2,3}$ are constants that characterize a certain geodesic of the family.

In the following, let us rewrite the Riccati equation (27), in the form

$$\dot{w} - \frac{1}{M}w^2 + 2\frac{R}{M}w - K = 0 \quad (28)$$

where we used the notations

$$\frac{dt}{ds} = \dot{w}, a_1 = \frac{1}{M}, a_2 = -\frac{R}{M}, a_3 = K \quad (29)$$

For obvious physical reasons it is therefore important to find the most general solution of equation (28). José Carineña and Arturo Ramos offer us a pass in short but modern and pertinent review of the integrability of Riccati's equation [24-27]. For our current needs it is enough to note that the complex numbers

$$w_0 \equiv R + iM\Omega,\ w_0^* \equiv R - iM\Omega;\ \Omega^2 = \frac{K}{M} - \left(\frac{R}{M}\right)^2 \quad (30)$$

roots of the quadratic polynomial on the right side of equation (28), are two solutions (constants, that's right) of the equation: being constants, their derivative is zero, being roots of the right-hand polynomial, it cancels. So, first we do the homographic transformation:

$$z = \frac{w - w_0}{w - w_0^*} \tag{31}$$

and now it can easily be seen by direct calculation that $z$ is a solution of the linear and homogeneous equation of the first order

$$\dot{z} = 2i\Omega z \;\therefore\; z(t) = z(0)e^{2i\Omega t} \tag{32}$$

Therefore, if we conveniently express the initial condition $z(0)$, we can give the general solution of the equation (28) by simply inverting the transformation (31), with the result

$$w = \frac{w_0 + re^{2i\Omega(t-t_r)}w_0^*}{1 + re^{2i\Omega(t-t_r)}} \tag{33}$$

where $r$ and $t_r$ are two real constants that characterize the solution. Using equation (30) we can put this solution in real terms, i.e.

$$\begin{aligned} z = \quad & R + M\Omega\left(\frac{2r\sin\left[2\Omega(t-t_r)\right]}{1 + r^2 + 2r\cos\left[2\Omega(t-t_r)\right]}\right. \\ & \left. +i\frac{1-r^2}{1 + r^2 + 2r\cos\left[2\Omega(t-t_r)\right]}\right) \end{aligned} \tag{34}$$

which highlights a frequency modulation through what we would call a Stoler transformation [15, 28, 29] which leads us to a complex form of this parameter. More than that, if we make the notation

$$r \equiv \coth\,\tau \tag{35}$$

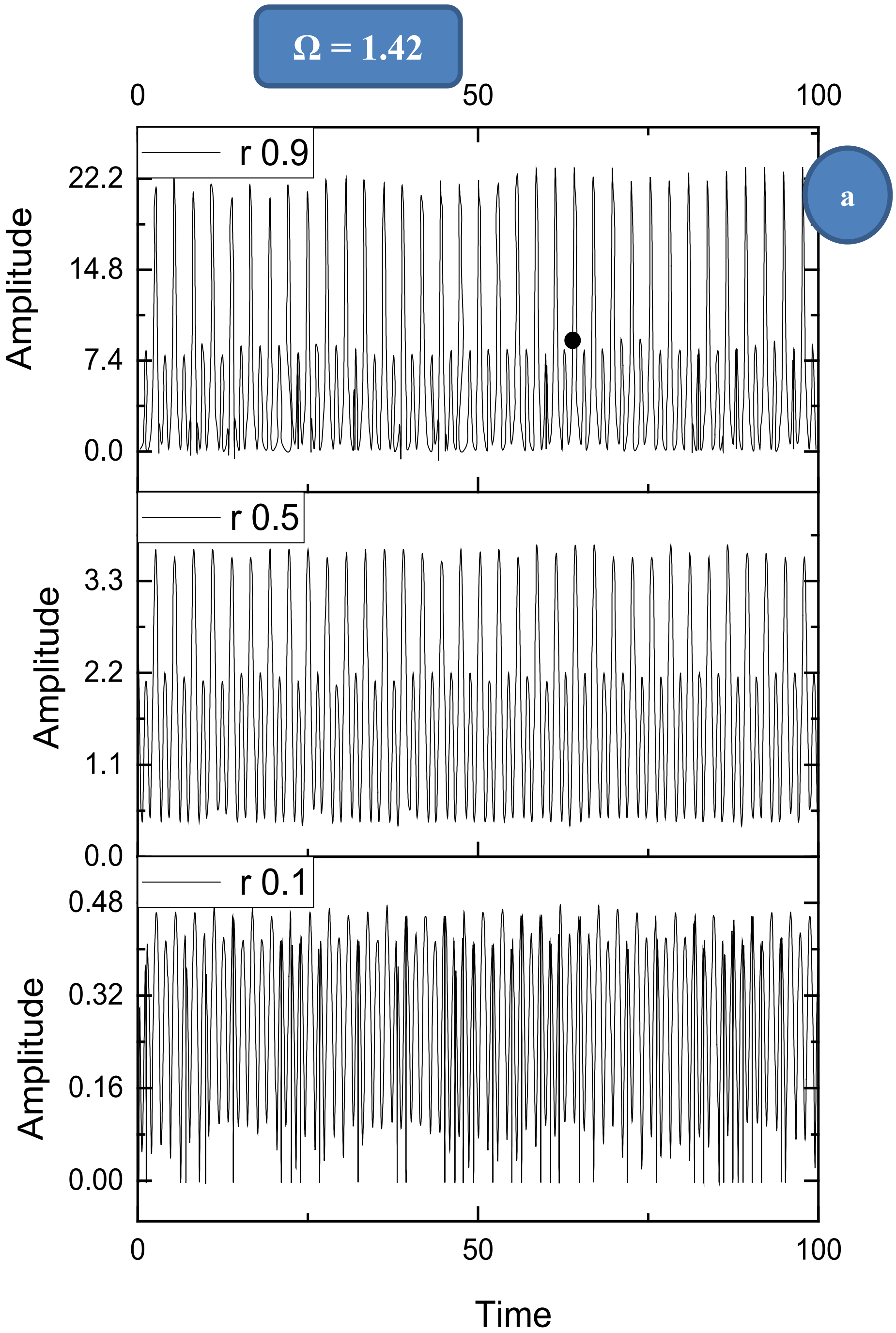
equation (34) becomes

$$z = R + M\Omega h \tag{36}$$

where $h$ is given by

$$h = -i\frac{\cosh\,\tau - e^{-2i\Omega(t-t_m)}\sinh\,\tau}{\cosh\,\tau + e^{-2i\Omega(t-t_m)}\sinh\,\tau} \tag{37}$$

We present in Figure 1 a, b the dependences of Rez on Ω and t, at various scale resolutions. In this way, several chaos transition scenarios (period-doubling (a) and intermittences (b)) can be mimed. Such a situation is verified by means of the Fourier transformation.


Ω = 1.42
a
r 0.9
r 0.5
r 0.1
Amplitude
Time
22.2
14.8
7.4
0.0
3.3
2.2
1.1
0.0
0.48
0.32
0.16
0.00
0
50
100

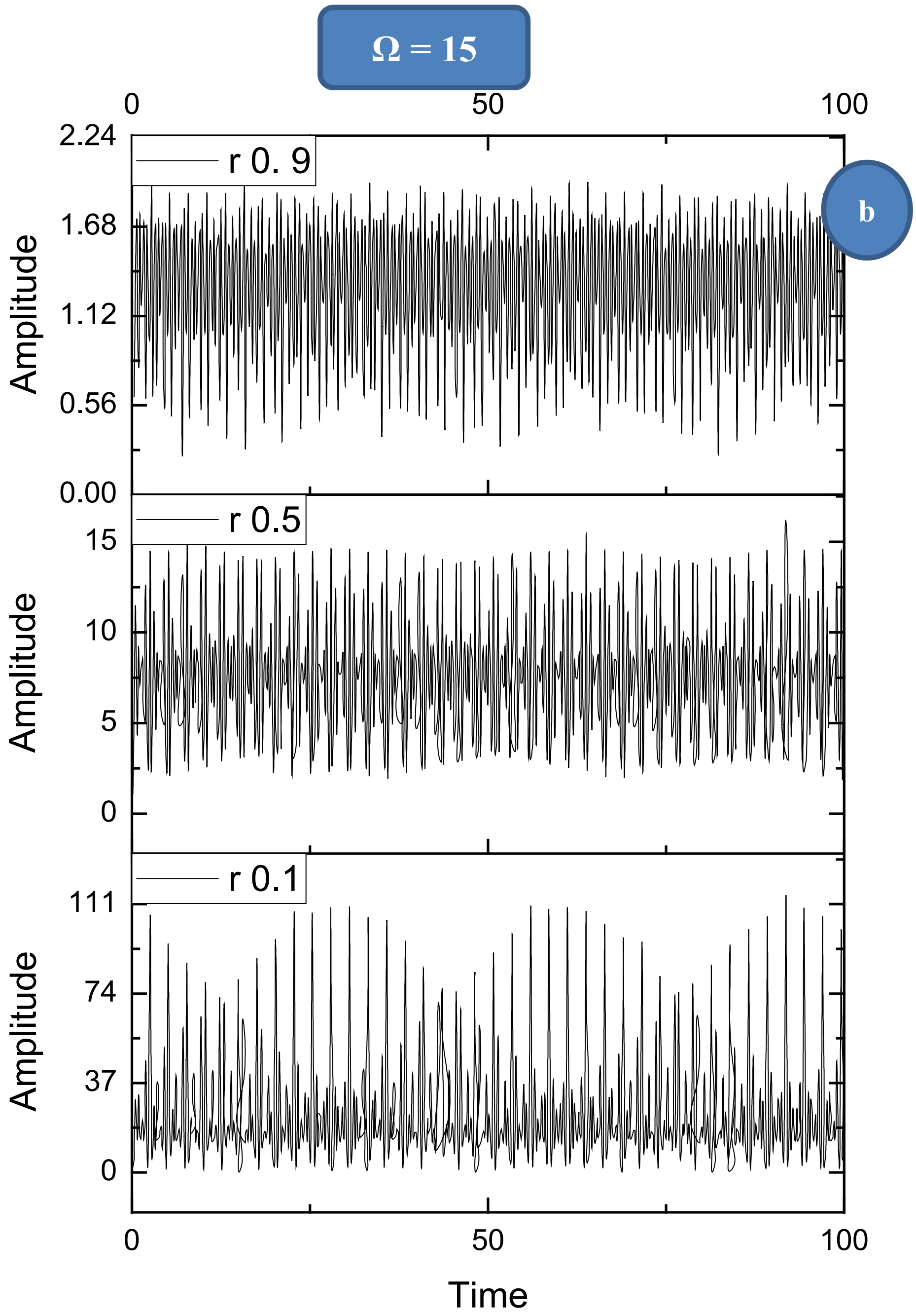



Fig. 1a, b. The time variation of the Rez solution amplitude for different values of Ω (a): 1.42 and b): 15) and for three different values of *r*: 0.1, 0.5, 0.9

To obtain these graphic representations based on the time series theoretically developed in the presented model, the same software programs as in the bibliographic references [30, 31] were used.

## 5. Conclusions

Several implications of absolute geometries in fundamental physics have been presented:

i. a generalization of the standard velocities space in Fock's sense;

ii. chaos transition scenarios (period doubling and intermittences) mimed through the selection of scale resolutions.

Our constructed velocities space refers not only to the standard space in Fock's sense (in which the vacuum speed of light is a constant) but also to a more general space (in which the limit velocity, in particular the speed of light, can also be variable).

We must highlight the fact that the Riccati-type gauge (see eq. (24)) proves to be fundamental in non-linear dynamics, in miming various chaos transition scenarios, as presented in the paper.

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