AN APPROACH TO ACCESS TO A CONCEPT LATTICE VIA THE IDEA OF LATTICE THEORY

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A concept lattice is defined by a binary relation on a context and indeed a lattice. In this paper, for a given context, firstly, we present a way to obtain all the cover elements of the smallest element in the given context ordered by a naturally relation. After that, we define a binary relation on a new context produced by one of elements in the concept lattice for the given context. A new concept lattice is derived from just the new binary relation. By the help of lattice theory, we explain the isomorphic relation between the new concept lattice and an interval in the concept lattice produced by the given context. Henceforth, using this isomorphic relation and the way presented previously to obtain all the cover elements of the smallest element, we give an approach to earn all the elements in the concept lattice for the given context. Simultaneously, the Hasse diagram of the concept lattice is got.

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1. Introduction

We know that concept lattices are used in several areas of database managing and the theory of concept lattices is an efficient tool for knowledge representation and knowledge discovery (cf. [1, 2, 3, 5]). As Berry and Sigayret said in [2], one of the important challenges in data handling is generating or navigating the concept lattice of a binary relation. In this paper, we present an approach for generating and understanding a concept lattice by the intervals and the cover elements' properties in the concept lattice for a given context. The approach generates all the concepts as well as the edges of the Hasse diagram of the lattice, without requiring a data structure.

A. Berry et.al. in [6] also present an algorithmic process which encounters all the concepts without requiring an exponential-size data structure. The

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algorithm in [6] is a Depth-First fashion. To get the cover of a concept (A, B), it needs to compute the partition into maxmods. After that, it computes the set ND of non-dominating maxmods. Finally, using the set ND of non-domination maxmods of $I(G\backslash A, B)$ (Notice: $I(G\backslash A, B)$ is a sub-relation of I, cf. [6]), it obtains the cover of (A, B). The method in this paper is different from [6] and the others such as [2, 7, 8] and so on. It is a Breadth-First fashion. We notice that though [6] informs us that in a sublattice of a given context, if (A, B) is the least, then this sublattice is exactly the lattice of the sub-relation $I(G, B)$ and the atoms of this lattice are defined by the properties of $I(G\backslash A, B)$. It does not show the relationship between $[(A, B), (G, \varnothing)]$ and $\doteq(G\backslash A, B, I_B)$ (Notice: the definition of $\doteq(G\backslash A, B, I_B)$ can be found in the following). This paper just deals with the relationship between the interval $[(A, B), (G, \varnothing)]\subseteq\doteq(G, M, I)$ and $\doteq(G\backslash A, B, I_B)$. We may hope that the relationship is not only useful in this paper but also important to the discussion with concept lattices in the future. Using this relationship, for finding the cover of a given concept (A, B) in a concept lattice, we only need the method computing the cover of the least element in this concept lattice. The method computing the cover of the least element is visual and simple.

Firstly, we declare that all the discussions in this paper are finite. Secondly, some knowledge needed in the sequel are now recalled on.

Definition 1 (1) [1, &2] A triple (G, M, I) is called a *formal context*, if G and M are sets and $I \subseteq G\times M$ is a binary relation between G and M. For $A\subseteq G$ and $B\subseteq M$, we define $A'=\{m\in M \mid (g, m)\in I \text{ for all } g\in A\}$ and $B'=\{g\in G \mid (g, m)\in I \text{ for all } m\in B\}$. (A, B) is a *formal concept* of (G, M, I) if and only if $A'=B$ and $A=B'$.

The concepts of a given context are naturally ordered by $(A_1, B_1) \le (A_2, B_2) \Leftrightarrow A_1\subseteq A_2$ $(\Leftrightarrow B_2\subseteq B_1)$. The ordered set of all formal concepts of (G, M, I) is denoted by $\doteq(G, M, I)$ and is called the *concept lattice* of (G, M, I).

(2) [3, p.12, &4, p.4] In the poset $(P, \le)$, *a covers b* or b is *covered* by a (in notation, $b\rightleftarrows a$) if and only if $b<a$ and, for no x, $b<x<a$.

[4, p.5] In a poset P of finite length with the least element 0, the *height* $h[x]$ of $x\in P$ is, by definition, the l.u.b. of the lengths of the chains $0=x_0<x_1<\ldots<x_l=x$ between 0 and x.

[3, p.21] Given $a \le b$ in a lattice L, the *interval* $[a, b]=\{x\in L \mid a \le x \le b\}$.

Lemma 1 Let (G, M, I) be a formal context, $A_j, A\subseteq G$ and $B_j, B\subseteq M$ ($j\in J$). Then

(I)[1] (1) (i) $A_1\subseteq A_2 \Rightarrow A_2' \subseteq A_1'$; (i)' $B_1\subseteq B_2 \Rightarrow B_2' \subseteq B_1'$.

(ii) $A\subseteq A''$ and $A'=A'''$; (ii)' $B\subseteq B''$ and $B'=B'''$.

(iii) $A\subseteq B' \Leftrightarrow B\subseteq A'$.

(2) $\doteq(G, M, I)$ is a complete lattice in which infimumand supremum are given by

$\wedge_{j\in J}(A_j, B_j)=(\cap_{j\in J}A_j, (\cup_{j\in J}B_j)'')$; $\vee_{j\in J}(A_j, B_j)=((\cup_{j\in J}A_j)'', \cap_{j\in J}B_j)$.

(II)[5] (iv) $(\cup_{j\in J}A_j)' =\cap_{j\in J}A_j'$; (iv)' $(\cup_{j\in J}B_j)' =\cap_{j\in J}B_j'$.

Remark (1) [4, p.5] In a poset P, $h[x]=1$ if and only if x covers 0.

[4, p.7] Given $a\le b$ in a lattice L, the interval is a sublattice.

(2) In this paper, a formal context (G, M, I) and a formal concept of (G, M, I) is simply said a context and a concept respectively. We just use gIm to express $(g, m)\in I$.

2. Computing the cover elements of the smallest element

Let $(G, M=\{m_1, m_2, \ldots, m_n\}, I)$ be a context. No harm to suppose, in what follows, in $\doteq(G, M, I)$, $(\varnothing, M)$ and $(G, \varnothing)$ is the smallest element and the greatest element respectively. Actually, if not, we can easily change the original context to be that as the supposition and the original concept lattice is easier to be obtained from the new concept lattice. In this section, we just give a way to obtain the whole cover elements of $(\varnothing, M)$ in $\doteq(G, M, I)$.

Let $\triangleq_1=\{M\backslash m_j \mid (M\backslash m_j)''=M\backslash m_j, j\in\{1, 2, \ldots, n\}\}$, $ƛ_1=\{(M\backslash m_j)' \mid M\backslash m_j \in\triangleq_1\}$ and $\bigcap_1=\{ j \mid M\backslash m_j \in\triangleq_1\}$. In addition, $\coprod_1=\{((M\backslash m_j)', M\backslash m_j) \mid j\in\bigcap_1\}$.

Suppose we have got $\triangleq_j$ ($j=1, 2, \ldots, p-1$) such that

$\triangleq_j =\{M\backslash\{ m_{i_1}, \ldots, m_{i_j} \} \mid (M\backslash\{ m_{i_1}, \ldots, m_{i_j} \})''= M\backslash\{ m_{i_1}, \ldots, m_{i_j} \}$ and for any

$S\subset\{i_1, \ldots, i_j\}, S\notin\bigcap_{|S|}\}$,

$ƛ_j=\{(M\backslash\{m_{i_1}, \ldots, m_{i_j}\})' \mid M\backslash\{m_{i_1}, \ldots, m_{i_j}\}\in\triangleq_j\}$,

$\bigcap_j=\{\{i_1, \ldots, i_j\} \mid M\backslash\{m_{i_1}, \ldots, m_{i_j}\}\in\triangleq_j\}$,

$\coprod_j=\{((M\backslash\{m_{i_1}, \ldots, m_{i_j}\})', M\backslash\{m_{i_1}, \ldots, m_{i_j}\}) \mid \{i_1, \ldots, i_j\}\in\bigcap_j\}$.

Let $\triangleq_p=\{M\backslash\{m_{i_1}, \ldots, m_{ip}\} \mid (M\backslash\{m_{i_1}, \ldots, m_{ip}\})''=M\backslash\{m_{i_1}, \ldots, m_{ip}\}$ and for any $S\subset\{i_1, \ldots, i_p\}$, $M\backslash\{m_{s_1}, \ldots, m_{s_{|S|}}\}\notin\triangleq_{|S|}$ $\}$, $ƛ_p=\{(M\backslash\{m_{i_1}, \ldots, m_{ip}\})' \mid M\backslash\{m_{i_1}, \ldots, m_{ip}\}\in\triangleq_p\}$, $\bigcap_p=\{\{i_1, \ldots, i_p\} \mid M\backslash\{m_{i_1}, \ldots, m_{ip}\}\in\triangleq_p\}$ and $\coprod_p=\{((M\backslash\{m_{i_1}, \ldots, m_{ip}\})', M\backslash\{m_{i_1}, \ldots, m_{ip}\}) \mid \{i_1, \ldots, i_p\}\in\bigcap_p\}$.

Since n, $|G|<\infty$ impel that the above process is stopped after k steps where $k\leq n$.

We may observe that by Definition 1, $(A, B)\in\doteq(G, M, I) \Leftrightarrow B'=A$ and $A'=B$. This reveals that A is uniquely determined by B. Next to discuss with the cover elements of $(\varnothing, M)$.

Theorem 1 Let $(A, B)\in\doteq(G, M, I)$. Then

$$(A, B) \text{ covers } (\varnothing, M) \Leftrightarrow B\in\triangleq_p \text{ for some } p\in\{1, 2, \ldots, k\}.$$

Proof ($\Leftarrow$) Let $B\in\triangleq_p$ for some $p\in\{1, 2, \ldots, k\}$.

Suppose (A, B) does not cover $(\varnothing, M)$. This implies that there is $(X, Y)\in\doteq(G, M, I)$ satisfying $(\varnothing, M) \rightleftarrows (X, Y) < (A, B)$. By Definition 1, it follows $M\supset Y\supset B$.

Let $Y=M\backslash\{m_{y_1}, \ldots, m_{y_t}\}$ where $y_1<y_2<\ldots<y_t$. $B\subset Y\subset M$ tells us $t\neq n$. In

addition, $M\backslash\{m_{i_1}, \ldots, m_{i_p}\}=B=Y\backslash\{m_{b_1}, \ldots, m_{b_w}\}=M\backslash\{m_{y_1}, \ldots, m_{y_t}, m_{b_1}, \ldots, m_{b_w}\}$ where $b_1, \ldots, b_w \notin \{y_1, \ldots, y_t\}$ and $b_\alpha \neq b_\beta$ for any $\alpha\neq\beta$ ($\alpha, \beta\in\{1, \ldots, w\}\subset\{1, \ldots, n\}$).

$(\varnothing, M)\rightleftarrows(X, Y)$ expresses that for any $S\subset\{y_1, \ldots, y_t\}$, $(M\backslash\{m_{s_1}, \ldots, m_{s_{|S|}}\})'' \neq M\backslash\{m_{s_1}, \ldots, m_{s_{|S|}}\}$. Thus $Y\in\triangleq_t$ is correct for some $t\in\{1, \ldots, k\}$. But there is $(M\backslash\{m_{t_1}, \ldots, m_{t_{|T|}}\})''\neq M\backslash\{m_{t_1}, \ldots, m_{t_{|T|}}\}$ for any $T\subset\{i_1, \ldots, i_p\}=\{y_1, \ldots, y_t\}\cup\{b_1, \ldots, b_w\}$ according to the definition of $\triangleq_p$ and $B\in\triangleq_p$. This follows a contradiction to $\{y_1, \ldots, y_t\}\subset\{i_1, \ldots, i_p\}$ and $Y\in\triangleq_t$.

Hence, (A, B) covers $(\varnothing, M)$.

$(\Rightarrow)$ Let (A, B) cover $(\varnothing, M)$. Then $\{i_1, \ldots, i_t\}\subset\{1, \ldots, n\}$.

Suppose $B\notin\triangleq_t$ for $\forall t\in\{1, \ldots, k\}$. This indicates that there exists $\{j_1, \ldots, j_{|S|}\}=S\subset\{i_1, \ldots, i_t\}$ satisfying $M\backslash\{m_{j_1}, \ldots, m_{j_{|S|}}\}\in\triangleq_{|S|}$. Further, $1\leq|S|$ and $(M\backslash\{m_{j_1}, \ldots, m_{j_{|S|}}\})'\in\bar{\lambda}_{|S|}$. Therefore, it follows

$$((M\backslash\{m_{j_1}, \ldots, m_{j_{|S|}}\})', M\backslash\{m_{j_1}, \ldots, m_{j_{|S|}}\})\in\doteq(G, M, I).$$

Namely, $(\varnothing, M)\leq((M\backslash\{m_{j_1}, \ldots, m_{j_{|S|}}\})', M\backslash\{m_{j_1}, \ldots, m_{j_{|S|}}\})<(A, B)$. By the cover property of (A, B), one has $M\backslash\{m_{j_1}, \ldots, m_{j_{|S|}}\}=M$, and so $|S|=0$, a contradiction.

That is to say, $B\in\triangleq_p$ holds for some $p\in\{1, 2, \ldots, k\}$.

Corollary 1 Let $B\subseteq M$. Then $B\in\triangleq_p$ for some $p\in\{1, \ldots, k\}$ $\Leftrightarrow$ $(B', B)\in\doteq(G, M, I)$ and (B', B) covers $(\varnothing, M)$.

Proof By the selection of $\triangleq_p$ ($p=1, \ldots, k$) and Theorem 1.

The following is to introduce an algorithm to get $\triangleq_1$, ƛ_1, $\coprod_1$ and $\bigcap_1$.

Step 1. Let $\triangleq_1=\varnothing$, $\text{ƛ}_1=\varnothing$, $\coprod_1=\varnothing$, $\bigcap_1=\varnothing$ and $j=0$.

Step 2. If $j=n$, then go to Step 5. If $j<n$, go to Step 3.
Step 3. $j=j+1$.

Step 4. If $(M\backslash m_j)''= M\backslash m_j$, then $\triangleq_1=\triangleq_1\cup(M\backslash m_j)$, $\text{ƛ}_1=\text{ƛ}_1\cup(M\backslash m_j)'$, $\coprod_1= \coprod_1\cup((M\backslash m_j)', M\backslash m_j)$, $\bigcap_1=\bigcap_1\cup j$. Otherwise, go to Step 2.

Step 5. Stop.

Next to introduce an algorithm to approach $\triangleq_p$ and so on. Suppose for $1\le t<p$, $\triangleq_t$ is got, certainly, ƛ_t, $\bigcap_t$ are got, too. Herein, $\coprod_t=\{(B', B) \mid B\in\triangleq_t$ for some $t<p\}$ is obtained.

Step 1. Let $\triangleq_p=\text{ƛ}_p=\bigcap_p=\coprod_p=\lhd=\varnothing$, and $\quad=\{\{i_1, i_2, \ldots, i_p\}\mid$ for any $S\subset\{i_1, i_2, \ldots, i_p\}$, $S\notin\bigcap_{|S|}\}$.

Step 2. If $\quad\backslash\lhd=\varnothing$, then go to Step 6. Otherwise, go to Step 3.

Step 3. Select $\{i_1, i_2, \ldots, i_p\}\in\quad\backslash\lhd$.

Step 4. $\lhd=\lhd\cup\{i_1, i_2, \ldots, i_p\}$.

Step 5. If $(M\backslash\{m_{i_1}, \ldots, m_{i_p}\})''=M\backslash\{m_{i_1}, \ldots, m_{i_p}\}$, then $\triangleq_p=\triangleq_p\cup(M\backslash\{m_{i_1}, \ldots, m_{i_p}\})$, $\text{ƛ}_p=\text{ƛ}_p\cup(M\backslash\{m_{i_1}, \ldots, m_{i_p}\})'$, $\bigcap_p=\bigcap_p\cup(\{i_1, \ldots, i_p\})$ and $\coprod_p=\coprod_p\cup((M\backslash\{m_{i_1}, \ldots, m_{i_p}\})', M\backslash\{m_{i_1}, \ldots, m_{i_p}\})$.

Otherwise, go to Step 2.
Step 6. Stop.

By the selection of $\triangleq_p$ and $ƛ_p$, we see that for $(A, B)\in \doteq(G, M, I)$, if $(\varnothing, M)\rightleftarrows(A, B)$, then $B\in\triangleq_p$, simultaneously, $A\in ƛ_p$. Because of Theorem 1, we may use the above algorithms to obtain all the cover elements $\cup_{p=1}^{k} \coprod_p$ of $(\varnothing, M)$ in $\doteq(G, M, I)$. In view of the knowledge of lattice theory, if every $\triangleq_p=\{\varnothing\}$, $(p\in\{1, \ldots, k\})$, then $\triangleq_k=\{\varnothing\}$, $ƛ_k=G$ and $\bigcap_k=\{\{1, 2, \ldots, n\}\}$. So $(G, \varnothing)$ covers $(\varnothing, M)$. Therefore, under this case, $\doteq(G, M, I)=\{(\varnothing, M), (G, \varnothing)\}$ holds.

3. Properties of an interval

Let $(A, B)\in\doteq(G, M, I)$. In this section, we firstly research on the properties of the interval $[(A, B), (G, \varnothing)]$ in $\doteq(G, M, I)$, followed by giving an approach to obtain the whole elements in $\doteq(G, M, I)$, and meanwhile, getting the Hasse diagram of $\doteq(G, M, I)$. (The *Hasse diagram* of a poset is seen [3, p.13 &4, p.4]).

According to Definition 1, we may indicate $(X, Y)\in[(A, B), (G, \varnothing)] \Rightarrow A\subseteq X\subseteq G$ and $\varnothing\subseteq Y\subseteq B$. Based on this, next we define a binary relation I_B between $G\backslash A$ and B as: $\forall x\in G\backslash A, y\in B, \quad xI_By \Leftrightarrow xIy$ and for any $a\in A$, aIy.

In light of Definition 1 and Lemma 1, $\doteq(G\backslash A, B, I_B)$ is a concept lattice. To state more clearly and to be different from the signs in $\doteq(G, M, I)$, in $\doteq(G\backslash A, B, I_B)$, for $U\subseteq G\backslash A$ and $V\subseteq B$, we just present $U'^{B}=\{m\in B|\forall x\in U, xI_Bm\}$ and $V'^{B}=\{g\in G\backslash A|\forall y\in V, gI_By\}$.

Theorem 2 Let $(A, B)\in\doteq(G, M, I)$. Then theinterval $[(A, B), (G, \varnothing)]$ of $\doteq(G, M, I)$ is isomorphic to the lattice $\doteq(G\backslash A, B, I_B)$.

Proof Distinguished three steps to finish the proof.

Step 1. Let $(X, Y)\in[(A, B), (G, \varnothing)]$. Let $X_{\text{new}}=X\backslash A$ and $Y_{\text{new}}=Y$. Then $X_{\text{new}}\subseteq G\backslash A$ and $Y_{\text{new}}\subseteq B$. We will prove $(X_{\text{new}}, Y_{\text{new}})\in\doteq(G\backslash A, B, I_B)$.

Since $(X, Y)\in\doteq(G, M, I)$, $A\subseteq X$ and $X'=\{m\in M \mid \forall x\in X, xIm\}=\{m\in M \mid \forall x\in A, xIm$, and, $\forall x\in X\backslash A, xIm\}=Y\subseteq B$. This follows

$X'=\{m\in B \mid \forall x\in X_{\text{new}}, xI_Bm\}=X_{\text{new}}'^B$.

Thus, $Y=X'=X_{\text{new}}'^B$, i.e. $Y_{\text{new}}=X_{\text{new}}'^B$.

On the other hand, $Y'=\{g\in G \mid \forall y\in Y, gIy\}=X\supseteq A$ implies that $\forall y\in Y$ induces aIy for any $a\in A$, and hence, $Y'=\{g\in G\backslash A \mid \forall y\in Y, gIy$, and besides, aIy for $\forall a\in A\}$. But $Y_{\text{new}}'^B=\{g\in G\backslash A \mid \forall y\in Y_{\text{new}}, gI_By\}=\{g\in G\backslash A \mid \forall y\in Y, gI_By\}$

$=\{g\in G\backslash A \mid \forall y\in Y, gIy$, and simultaneously, $\forall a\in A, aIy\}$.

Thus, $Y'=Y_{\text{new}}'^B$.

So, combining the above two hands, it obtains $(X_{\text{new}}, Y_{\text{new}})\in\doteq(G\backslash A, B, I_B)$.

Step 2. Let $(C_{\text{new}}, D_{\text{new}})\in\doteq(G\backslash A, B, I_B)$, $C=C_{\text{new}}\cup A$ and $D=D_{\text{new}}$. Then $A\subseteq C_{\text{new}}\cup A=C\subseteq G$, and $D\subseteq B$. We just prove $(C, D)\in\doteq(G, M, I)$.

Since $C_{\text{new}}'^B=D_{\text{new}}$ and $D_{\text{new}}'^B=C_{\text{new}}$. In addition,

$C_{\text{new}}'^B=\{b\in B \mid \forall x\in C_{\text{new}}, xI_Bb\}$

$=\{b\in B \mid \forall x\in C_{\text{new}}, xIb$, and besides, $\forall a\in A, aIb\}$

$=\{b\in B \mid \forall x\in C_{\text{new}}\cup A, xIb\}=\{b\in B \mid \forall x\in C, xIb\}=D_{\text{new}}=D$,

and $D_{\text{new}}'^B=\{c\in G\backslash A \mid \forall y\in D_{\text{new}}, cI_By\}$

$=\{c\in G\backslash A \mid \forall y\in D_{\text{new}}=D, cIy$, and besides, $\forall a\in A, aIy\}=C_{\text{new}}$.

However, $D'=\{z\in G \mid \forall y\in D, zIy\}$

$=\{z\in G\backslash A \mid \forall y\in D, zIy\}\cup\{z\in A \mid \forall y\in D, zIy\}$.

Moreover, $D\subseteq B$ and Lemma 1 together follows $A=B'\subseteq D'$. Thus for any $a\in A$ and $\forall y\in D$, it must have aIy.

Furthermore, for any $z\in D'\backslash A$, we assure zIy for any $y\in D$, and so $z\in C_{\text{new}}$. Conversely, for any $c\in C_{\text{new}}$, we own cIy for any $y\in D$, and so $c\in D'$.

In one word, there is $D'\backslash A=C_{\text{new}}$, i.e. $D'=C_{\text{new}}\cup A=C$. So $C=D'$ holds.

On the other hand, $C'=\{m\in M \mid \forall x\in C, xIm\}=\{m\in M \mid \forall x\in C_{\text{new}}, xIm\}\cap\{m\in M \mid \forall x\in A, xIm\}=\{m\in M \mid \forall x\in C_{\text{new}}, xI_Bm\}=C_{\text{new}}'^B=D_{\text{new}}=D\subseteq B$.

Summing up, $(C, D)\in\doteq(G, M, I)$.

Step 3. Combining the above two steps, we may state that the map defined as f : $[(A, B), (G, \emptyset)] \rightarrow \doteq(G\backslash A, B, I_B)$ satisfying f : $(X, Y) \rightarrow (X\backslash A, Y)$ is a bijection between $[(A, B), (G, \emptyset)]$ and $\doteq(G\backslash A, B, I_B)$.

Let $(X_j, Y_j) \in[(A, B), (G, \emptyset)]$, $X_{j\text{new}}=X_j\backslash A$, and $Y_{j\text{new}}=Y_j$ (j=1, 2). Then $Y_j \subseteq B$, (j=1, 2), and further, $Y_1\cap Y_2\subseteq B$, and so $(Y_1\cap Y_2)' \supseteq B'$and $B'=A$ by Lemma 1. In view of Lemma 1, it causes

$(X_1, Y_1)\vee(X_2, Y_2)=((X_1\cup X_2)'', Y_1\cap Y_2)=((Y_1\cap Y_2)', Y_1\cap Y_2)$;

$(X_{1\text{new}} , Y_{1\text{new}})\vee(X_{2\text{new}}, Y_{2\text{new}})=((X_{1\text{new}}\cup X_{2\text{new}})'^{B'B}, Y_{1\text{new}}\cap Y_{2\text{new}})$

$=((Y_{1\text{new}}\cap Y_{2\text{new}})'^{B}, Y_{1\text{new}}\cap Y_{2\text{new}})$.

By the definition, it follows $f((X_j, Y_j))=(X_{j\text{new}}, Y_{j\text{new}})$, ($j$=1,2);

$f((X_1, Y_1)\vee(X_2, Y_2))= ((Y_1\cap Y_2)'\backslash A, Y_1\cap Y_2)$;

$f((X_1, Y_1))\vee f((X_2, Y_2))=(X_{1\text{new}}, Y_{1\text{new}})\vee(X_{2\text{new}}, Y_{2\text{new}})$

$= ((Y_{1\text{new}} \cap Y_{2\text{new}})'^{B}, Y_1\cap Y_2)$.

On the other hand, by the above, we firmly believe $(Y_1\cap Y_2)'\supseteq A$. Additionally, $(Y_1\cap Y_2)'=\{g\in G \mid \forall y\in Y_1\cap Y_2,\ gIy\}\supseteq A$ shows that for $\forall y\in Y_1\cap Y_2$ and $\forall a\in A$, aIy holds.

But $(Y_{1\text{new}}\cap Y_{2\text{new}})'^{B}=\{g\in G\backslash A \mid \forall y\in Y_{1\text{new}}\cap Y_{2\text{new}}, gI_By\}$

$=\{g\in G\backslash A \mid \forall y\in Y_1\cap Y_2, gIy$, and simultaneously, $\forall a\in A, aIy\}$.

These point out $(Y_1\cap Y_2)'\backslash A =(Y_{1\text{new}}\cap Y_{2\text{new}})'^{B}$.

Hence $f((X_1, Y_1)\vee(X_2, Y_2))=f((X_1, Y_1))\vee f((X_2, Y_2))$.

Dually, we may prove $f((X_1, Y_1)\wedge(X_2, Y_2))=f((X_1, Y_1))\wedge f((X_2, Y_2))$.

Therefore, $[(A, B), (G, \emptyset)] \cong \doteq(G\backslash A, B, I_B)$.

We may sum up the beyond discussion to state that after using Theorem 1 to obtain all the cover elements of $(\emptyset, B)$ in $\doteq(G\backslash A, B, I_B)$, via Theorem 2, we will obtain all the cover elements of (A, B) in $\doteq(G, M, I)$.

Next we express the sketch of an algorithm to approach $\doteq(G, M, I)$.

In $\doteq(G, M, I)$, $h[(\emptyset, \text{M})]=0$ where h is the height function of $\doteq(G, M, I)$.

Let $x\in\doteq(G, M, I)$ with $h[x]=1$. By Remark in Section 1, it follows that x covers $(\varnothing, M)$. Thus, x is got by the algorithm in Section 2.

In light of Definition 1, $y\in\doteq(G, M, I)$ covers some $z\in\doteq(G, M, I)$ with $h[z]=1$ if and only if $h(y)=2$.

Hence, we may use the algorithm in Section 2 to obtain the cover elements of $(\varnothing, B)$ in $\doteq(G\backslash A, B, I_B)$ where $(A, B)\in\doteq(G, M, I)$ and $h[(A, B)]=1$. After that, using Theorem 2, we find out all the cover elements of (A, B) in $\doteq(G, M, I)$. That is, we earn all the elements of height 2 in $\doteq(G, M, I)$. Simultaneously, owing a cover relation or not between x and y is shown, where $\forall x, y\in\doteq(G, M, I)$, $h[x]=1$ and $h[y]=2$.

Analogously, for $h[z]=t \geq 2$, the cover elements of z in $\doteq(G, M, I)$ are found. The height of any cover element of z is t+1. Thus, all the elements of height t+1 will be searched out. Meanwhile, the cover relations between the set of elements of height t and the set of elements of height t+1 are shown clearly.

Because $(G, \varnothing)$ is the greatest element in $\doteq(G, M, I)$ and $|M|, |G|<\infty$, we may state that the height $h[(G, \varnothing]$ of $\doteq(G, M, I)$ is finite. Moreover, we may find out all the elements in $\doteq(G, M, I)$. Therefore, the above process will be stopped after finite steps. At the same time, the relationships among the elements in $\doteq(G, M, I)$ are obtained. Furthermore, the Hasse diagram of $\doteq(G, M, I)$ is also obtained.

Next, we provide an example to show how to use the approach above to find out the concept lattice $\doteq(G, M, I)$ and its Hasse diagram for a given context (G, M, I).

Example 1 Let M={1, 2, 3, 4, 5, 6} and G={a, b, c, d, e, f}.
The table below describes binary relation I.

Table 1

Describe the relation between *G* and *M*

	1	2	3	4	5	6
a		×	×			×
b	×	×	×			
c	×	×			×	
d	×			×	×	
e	×			×		
f			×			

By using the following steps, we find out $\doteq(G, M, I)$.

Step 1. $(\varnothing, M)$ is the smallest.

Step 2. By Theorem 1, the cover elements $\coprod$ of $(\varnothing, M)$ are $\{(A_{j_t}, B_{j_t}) \mid A_{j_t} \in \triangleq_j, B_{j_t} \in \lambdabar_j,\ \ t=1, 2,\ldots, |\triangleq_j|;\ \ j=1, 2, \ldots, k\}$. Using the algorithm in Section 2 for $(\varnothing, M)$, one gets $\{\triangleq_j \mid j=1, \ldots, k\}$, $\{\lambdabar_j \mid j=1, \ldots, k\}$ and $\coprod$ as follows.

Since $M\backslash\{1\}=\{2, 3, 4, 5, 6\}$. From Table 1, we easily receive $(M\backslash\{1\})'=\{g\in G \mid \forall x\in M\backslash\{1\}, gIx\}=\{a, b, c\}\cap\varnothing=\varnothing$.

Similarly, $(M\backslash\{j\})'=\varnothing$, ($j$=2, 3, 4, 5, 6). Hence, $\triangleq_1=\lambdabar_1=\bigcap_1=\varnothing$.

By Table 1, obviously, $(M\backslash\{i, j\})'=\varnothing$, ($i\neq j$; i, j=1, …,6), that is, $\triangleq_2=\lambdabar_2=\bigcap_2=\varnothing$.

Additionally, $M\backslash\{1, 2, 3\}=\{4, 5, 6\}$, $(M\backslash\{1, 2, 3\})'=\varnothing$. However,
$M\backslash\{1, 4, 5\}=\{2, 3, 6\}$,
$(M\backslash\{1, 4, 5\})'=\{g\in G \mid \forall x\in\{2, 3, 6\}, gIx\}=\{a\}$,
$(M\backslash\{1, 4, 5\})''=(\{a\})'=\{2, 3, 6\}=M\backslash\{1, 4, 5\}$.

Thus $M\backslash\{1, 4, 5\}\in\triangleq_3$, $\{a\}\in\lambdabar_3$, $\{1, 4, 5\}\in\bigcap_3$ and

$((M\backslash\{1, 4, 5\})'=\{a\}, M\backslash\{1, 4, 5\}=\{2, 3, 6\})\in\doteq(G, M, I)$.

Repeated application of the algorithm in Section 2 to $\{i, j, t\}$, $(i, j, t \in \{1, \ldots, 6\}; i \neq j \neq t)$, it yields out

$\triangleq_3=\{M\backslash\{1, 4, 5\}=\{2, 3, 6\}, M\backslash\{4, 5, 6\}=\{1, 2, 3\}, M\backslash\{3, 4, 6\}=\{1, 2, 5\},$

$M\backslash\{2, 3, 6\}=\{1, 4, 5\}\};$

ƛ$_3=\{(M\backslash\{1, 4, 5\})'=\{a\}, (M\backslash\{4, 5, 6\})'=\{b\}, (M\backslash\{3, 4, 6\})'=\{c\},$

$(M\backslash\{2, 3, 6\})'=\{d\}\};$

$\bigcap_3=\{\{1, 4, 5\}, \{4, 5, 6\}, \{3, 4, 6\}, \{2, 3, 6\}\}.$

At the end, all the cover elements of $(\varnothing, M)$ in $\doteq(G, M, I)$ are

$\coprod=\{(\{a\}, \{2, 3, 6\}), (\{b\}, \{1, 2, 3\}), (\{c\}, \{1, 2, 5\}), (\{d\}, \{1, 4, 5\})\}.$

Step 3. For $(\{a\}, \{2, 3, 6\})\in\coprod$.

Let $A=\{a\}$ and $B=\{2, 3, 6\}$. $[(A, B), (G, \varnothing)]$ is an interval in $\doteq(G, M, I)$. Put $X_{\text{new}}=X\backslash A=X\backslash\{a\}$ and $Y_{\text{new}}=Y$ for $(X, Y)\in[(A, B), (G, \varnothing)]$. Then by the definition at the beginning of this section, it causes $\doteq(G\backslash A, B, I_B)=\doteq(\{b, c, d, e, f\}, \{2, 3, 6\}, I_B)$.

By the method in the proof of Theorem 2, in $\doteq(G\backslash A, B, I_B)$, we obtain that the smallest element is $(\varnothing, \{2, 3, 6\})$, and all of its cover elements are $(\{b\}, \{2, 3\})$. Further, in light of Theorem 2, we may indicate that in $\doteq(G, M, I)$, all the cover elements of $(\varnothing\cup A, \{2, 3, 6\})=(A, \{2, 3, 6\})$ are $(\{b\}\cup A, \{2, 3\})=(\{a, b\}, \{2, 3\})$. Analogously, in $\doteq(G, M, I)$,

the cover elements of $(\{b\}, \{1, 2, 3\})$ are $(\{a, b\}, \{2, 3\})$ and $(\{b, c\}, \{1, 2\})$;
the cover elements of $(\{c\}, \{1, 2, 5\})$ are $(\{b, c\}, \{1, 2\})$ and $(\{c, d\}, \{1, 5\})$;
the cover elements of $(\{d\}, \{1, 4, 5\})$ are $(\{c, d\}, \{1, 5\})$ and $(\{d, e\}, \{1, 4\})$.

That is, all the elements of height 2 in $\doteq(G, M, I)$ are $(\{a, b\}, \{2, 3\})$, $(\{b, c\}, \{1, 2\})$, $(\{c, d\}, \{1, 5\})$ and $(\{d, e\}, \{1, 4\})$.

Step 4. Similarly to the closely Step 3, for $x\in\doteq(G, M, I)$ with $h[x]=t$, all the cover elements of x are generated (t=2, 3) and $h[(G, \varnothing)]=4$. Finally, we receive the Hasse diagram of $\doteq(G, M, I)$ as Figure 1.

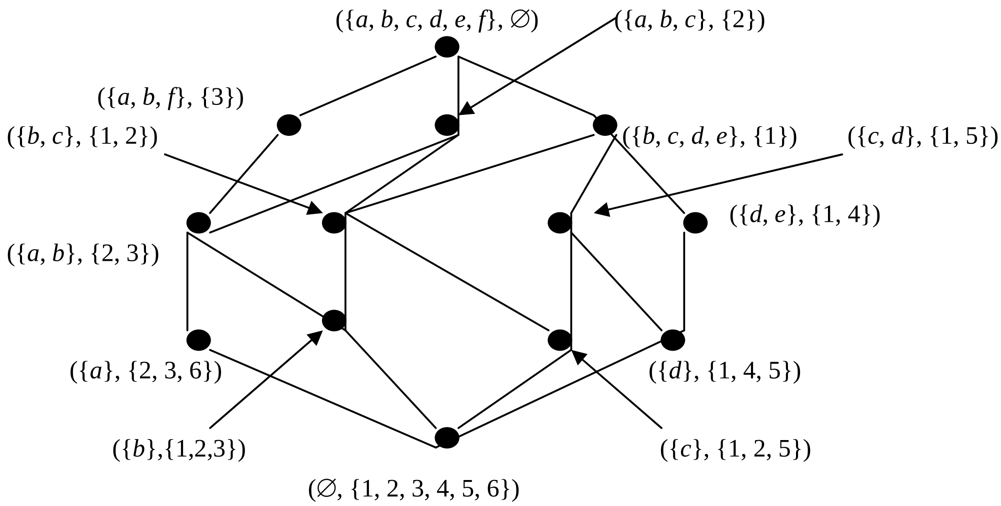

Fig. 1. Hasse diagram of concept lattice relative to the context in Table 1

We still consider that [2] is good and valuable to discuss concept lattices with the way of graph theory. The result in Example 1 is the same as that in [2, Example 2.2]. This also shows the truth of the algorithm produced from Theorem 1 and Theorem 2. However, [2, Example 2.2] is obtained directly from the definition of concept lattice. Certainly, the context in [2, Example 2.2] are also discussed with the methods of graph theory in [2, 4 and 5], and [2] gives the same diagram in [2, 4] and [2, 5] as that in [2, Figure 1] for the context in [2, Example 2.2] respectively.

The approach in this paper is different from others because it is just using the isomorphic and interval's properties of lattice theory. It is visible and accessible. As Ganter and Wille said in [1], for concept lattice, "Much of the mathematics required for the applications comes from lattice theory. ... The new goals made it necessary to extend this theory." We may infer that the potential of the results in this paper is extending and enriching the theory of concept lattices.

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