# ON ANALYTICAL SOLUTION OF THE BLACK-SCHOLES EQUATION BY THE FIRST INTEGRAL METHOD

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*The Black-Scholes formula is used as a model for valuing European or American call and put options on a non-dividend paying stock. In option pricing theory, the Black-Scholes equation is one of the most effective models for pricing options. In this paper, the first integral method is employed to obtain a quick and accurate solution to the Black-Scholes equation with boundary condition for a European option pricing problem.*

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## 1. Introduction

The financial mathematics is definitely among the most popular subjects of applied mathematics today from both academic and the industry point of view. The main subjects in financial mathematics is concerned with modeling of evolution of financial processes such as stock prices, interest rates, exchange rates and pricing derivatives on basic underlying. The first basic breakthrough in the financial mathematics was made by Black and Scholes which is indeed found an explicit closed form solution for pricing plain vanilla European options [5, 6]. According to the idea of Black and Scholes, the option price can be modeled as a terminal boundary problem for a partial differential equation. Therefore, it is reasonable to adopt the existing theory and methods of partial differential equation as a fundamental approach to the study of the option pricing. This includes designing efficient algorithms for solving option pricing problems from the viewpoint of numerical solutions of partial differential equation problems.
Many authors have applied several different methods to solve the Black-Scholes equation [1, 4, 7]. In this paper, the first integral method (FIM) is applied to solve the Black-Scholes partial differential equation and boundary conditions for a European option pricing problem. The FIM is a direct algebraic method for obtaining exact solutions of some nonlinear partial differential equations. Recently, this method has been widely used by many researchers [2, 8-10].

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## 2. The Black-Scholes model framework

The underlying asset price follows the geometric Brownian motion [5]

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{1}$$

where the value of the option also depends on the initial price of the stock $S_0$, the expiry date $T$, the volatility of the underlying asset $\sigma$, the exercise (strike) price $E$, and the risk-free interest rate $\mu$. Let $V = V(S,t)$ denote the option price. The value of the option at the expiry (maturity) time $T$ for call option is

$$V(S,T) = (S-E)^+ = max\{0, S-E\},$$

and for put option is as follows

$$V(S,T) = (E-S)^+ = max\{0, E-S\}.$$

Now, we can derive a mathematical model of the option pricing using the $\Delta$-hedging technique [5, 6]. For this purpose, we consider a portfolio

$$\Pi = V - S\Delta, \tag{2}$$

where $\Delta$ denotes shares of the underlying asset. We choose $\Delta$ such that $\Pi$ is risk-free in interval $(t, t+dt)$. If portfolio $\Pi$ starts at time $t$, and $\Delta$ remains unchained in $(t, t+dt)$, then the requirement $\Pi$ be risk-free means the return of the portfolio at $t+dt$ should be

$$\frac{\Pi_{t+dt} - \Pi_t}{\Pi_t} = rdt, \tag{3}$$

or equivalently

$$dV_t - \Delta dS_t = r\Pi_t dt = r(V_t - \Delta S_t)dt. \tag{4}$$

Recall that the stochastic process $S_t$ satisfies the stochastic differential equation (1), hence using Itô formula [5] we conclude that

$$dV_t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S})dt + \sigma S \frac{\partial V}{\partial S} dW_t. \tag{5}$$

Using (4) and (5), we can write

$$(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - \Delta\mu S)dt + (\sigma S \frac{\partial V}{\partial S} - \Delta\sigma S)dW_t = r(V - \Delta S)dt. \tag{6}$$

Since we assume that the change over any time step $(t, t+dt)$ is non-random, the coefficient of the random term $dW_t$ on the left hand side must be zero. For this purpose, we choose $\Delta = \frac{\partial V}{\partial S}$. Therefore, from (6) we get the following partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial t^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{7}$$

This is the Black-Scholes equation that describes the option price movement. Therefore, in order to determine the option value at any time in [0, T], we need to solve the following partial differential equation problem in the domain $\Omega = \{(S,t):\ 0 \leq S < \infty, 0 \leq t \leq T\}$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial t^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{8}$$

with the boundary condition, $V(T,S) = (S-E)^+$ for call option and $V(T,S) = (E-S)^+$, for put option.

By setting $x = lnS$ and $\tau = T - t$, the problem (8) is reduced to a Cauchy problem of a parabolic equation with constant coefficients

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - (r - \frac{\sigma^2}{2})\frac{\partial V}{\partial x} + rV = 0, \tag{9}$$

subject to the boundary condition, $V(T,S) = (e^x - E)^+$ for call option and $V(T,S) = (E - e^x)^+$ for put option.

## 3. The First integral method analysis

Consider the nonlinear partial differential equation in the form

$$\Psi(u, u_x, u_t, u_{xx}, u_{xt}, ...) = 0, \tag{10}$$

where $u = u(x,t)$ is the solution of nonlinear partial differential equation (10). Here, we consider the following transformation

$$u(x,t) = f(\xi), \tag{11}$$

where $\xi = x - ct$. This enables us to use the following changes

$$\frac{\partial}{\partial t}(.) = -c\frac{\partial}{\partial \xi}(.),\ \frac{\partial}{\partial x}(.) = \frac{\partial}{\partial \xi}(.),\ \frac{\partial^2}{\partial x^2}(.) = \frac{\partial^2}{\partial \xi^2}(.),\ \frac{\partial^2}{\partial x \partial t}(.) = -c\frac{\partial^2}{\partial \xi^2}(.), \tag{12}$$

and so on for other derivatives. Using Equation (12) we can convert the nonlinear partial differential equation (10) to the following nonlinear ordinary differential equation

$$G(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, ...) = 0. \tag{13}$$

Let us consider a new independent variable

$$X(\xi) = f(\xi),\ Y = \frac{\partial f(\xi)}{\partial \xi}, \tag{14}$$

which leads a system of nonlinear ordinary differential equations

$$\begin{aligned} \frac{\partial X(\xi)}{\partial \xi} &= Y(\xi) \\ \frac{\partial Y(\xi)}{\partial \xi} &= \varphi(X(\xi), Y(\xi)). \end{aligned} \tag{15}$$

By the qualitative theory of ordinary differential equations, if we can find the integrals to Equation (15) under the same conditions, then the general solutions to Equation (14) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the division theorem to obtain one first integral to Equation (14) which reduces Equation (13) to a first order integrable ordinary differential equation. An exact solution to (10) is then obtained by solving this equation. Now, let us recall the following theorem, i.e. division theorem [8].

**Theorem 3.1.** *Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$ and also $P(w,z)$ is irreducible in $C[w,z]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ such that*

$$Q(w,z) = P(w,z)G(w,z).$$

## 4. The Black-Scholes equation with FIM

Our goal for this section is to solve the following Black-Scholes partial differential equation

$$\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2} + (k-1)\frac{\partial V}{\partial x} - kV, \tag{16}$$

subject to the boundary condition, $V(T,S) = (e^x - E)^+$ for call option and $k = \frac{2r}{\sigma^2}$ is a real constant.
By considering the following transformation

$$V(x,t) = f(\xi),\ \xi = x - ct, \tag{17}$$

the Black-Scholes equation becomes

$$-cf' = f'' + (k-1)f' - kf. \tag{18}$$

Using (14) and (15), we get

$$\dot{X}(\xi) = Y(\xi), \tag{19}$$

$$\dot{Y}(\xi) = (1-c-k)Y(\xi) + kX(\xi). \tag{20}$$

According to FIM, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (19) and (20), and

$$Q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i,$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$Q(X(\xi),Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0, \tag{21}$$

where $a_i(X),\ i = 0,1,...,m$ are polynomials of $X$ and $a_m(X) \neq 0$. Equation (21) is called the first integral to (19) and (20). Using the division theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $C[X,Y]$ such that

$$\frac{dQ}{d\xi} = \frac{dQ}{dX}\frac{dX}{d\xi} + \frac{dQ}{dY}\frac{dY}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i. \tag{22}$$

Suppose that $m = 1$ by comparing with the coefficients of $Y^i,\ i = 2, 1, 0$ on both sides of (22) we have

$$\dot{a_1}(X) = h(X)a_1(X), \tag{23}$$

$$\dot{a_0}(X) + (1 - c - k)a_1(X) = g(X)a_1(X) + h(X)a_0(X), \tag{24}$$

$$a_1(X)(kX) = g(X)a_0(X). \tag{25}$$

Since $a_i(X),\ i = 0, 1$ are polynomials, then from (23) we conclude that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $deg(g(X)) = 0$. Suppose that $g(X) = A_0$, then we find $a_0(X)$ as follows

$$a_0(X) = A_0 + (A_1 + c + k - 1)X, \tag{26}$$

where $A_0$ is arbitrary integration constant.
Substituting $a_0(X)$ and $g(X)$ into (25) and setting all the coefficients of powers $X$ to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = 0,\ A_1 = -\frac{c}{2} - \frac{k}{2} + \frac{1}{2} \pm \sqrt{c^2 + 2ck - 2c + k^2 + 2k + 1}, \tag{27}$$

where $k$ and $c$ are arbitrary constants.
Using the conditions (27) in (21), we obtain

$$Y(\xi) = \{-\frac{c}{2} - \frac{k}{2} + \frac{1}{2} \pm \sqrt{c^2 + 2ck - 2c + k^2 + 2k + 1}\}X(\xi). \tag{28}$$

By combining (28) and (19), we obtain the exact solution to Equation (18) and then the exact solution to the Black-Scholes equation can be written as follows

$$V(x,t) = \exp\{-\frac{c}{2} - \frac{k}{2} + \frac{1}{2} \pm \sqrt{c^2 + 2ck - 2c + k^2 + 2k + 1}(x - ct + \xi_0)\},$$

where $\xi_0$ is an arbitrary constant.

## 5. **Conclusion**

The main goal of this paper is to provide analytical solution of the Black-Scholes option pricing equation by the first integral method. We obtained an efficient and accurate solution to solve the Black-Scholes partial differential equation.

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