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LOCALIZATION OPERATORS RELATED TO α-WINDOWED FOURIER TRANSFORM

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In this paper, we introduce the localization operators $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ related to α-WFT, where $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ are two window functions and $\sigma \in L^p(\mathbb{R}^2), 1 \le p \le \infty$ is a symbol. We study the L^2-boundedness, compactness and Schatten-von Neumann properties for this class of linear operators. We establish a two sided estimate for the trace class norm of localization operators when $\sigma \in L^1(\mathbb{R}^2)$. Moreover, we can prove that those inequalities are sharp when $\phi = \psi$ and σ is a real-valued and non-negative function in $L^1(\mathbb{R}^2)$. Finally, an inequality regarding the trace class norm of the power n of a product of two localization operators is given.

Keywords: α-windowed Fourier transform, localization operators, Schatten-von Neumann classes

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1. Introduction

The Fourier transform give a representation of a signal using global, periodic functions. However, to achieve localized descriptions, it's necessary to concentrate Fourier analysis on specific segments of the signal. This can be realised by applying a window function, $\phi(x)$, which isolates a portion of the signal for analysis. The window can then be shifted across different time intervals to cover the entire time domain of interest. This method is referred to as the windowed Fourier transform (WFT) or short-time Fourier transform (STFT), a concept introduced by Gabor in [10].

Daubechies in the paper [5] introduced a category of bounded linear operators known as time-frequency localization operators and studied them in the context of signal analysis. These linear operators were subsequently referred to as Daubechies operators in references [8] and [9].

There are a lot of time-frequency transforms such as short-time Fourier transfom (STFT), wavelet transform, Stockwel transform, linear canonical transform, ridgelet transform, curvelet transform and many others which constitute very important tools used in time-frequency analysis. For more details concerning the time-frequency transforms see for example [1], [5]-[7], [10]-[12] and [14].

The aim of this paper is to introduce and study localization operators related to α-WFT.

The paper is organized as follows. Section 2 is dedicated to providing preliminary results related to α-window Fourier transform (α-WFT), including the orthogonality relation and the inversion formula. The localization operators associated to α-WFT are introduced

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in Section 3. Within this section, we present some results regarding the L^2-boundedness of localization operators. In the last section, Section 4, we focus on the compactness and Schatten-von Neumann properties for this class of linear operators. We also give a trace formula for the trace class localization operators. Further, an inequality regarding the trace class norm of the power n of a product of two localization operators is given.

2. Preliminary results

In this section, we give a brief presentation of the α-windowed Fourier transform. For more details concerning this notion, see [4].

Definition 2.1 (WFT)**.** *For a fixed function $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ (called the window function), its window daughter function or its windowed Fourier kernel is denoted by $\phi_{\omega,u}$ and is defined by*

$$\phi_{\omega,u}(x) = \phi(x-u)\exp\{i\omega x\},$$

for all $x \in \mathbb{R}$. The WFT of $f \in L^2(\mathbb{R})$ with respect to the window function $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ is defined by

$$G_\phi f(\omega,u) = \int_{\mathbb{R}} f(x)\overline{\phi_{\omega,u}(x)}dx,$$

for all $(\omega,u) \in \mathbb{R}^2$.

Definition 2.2. [4] *For a window function $\phi \in L^2(\mathbb{R})\setminus\{0\}$ together with a fixed real number $\alpha \neq n\pi, n \in \mathbb{Z}$ (called the fractional parameter), a family of functions $\mathcal{F}_\phi^\alpha(\omega,u)$ is defined by*

$$\mathcal{F}_\phi^\alpha(\omega,u) = \left\{\phi_{\omega,u}^\alpha(x) := \phi(x-u)\exp\left\{i\omega x - \frac{i(x^2-u^2)\cot\alpha}{2}\right\}; \omega, u, x \in \mathbb{R}\right\}.$$

Lemma 2.1. *For $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ it follows that $\phi_{\omega,u}^\alpha \in L^2(\mathbb{R})$ and*

$$\left\|\phi_{\omega,u}^\alpha\right\|_{L^2(\mathbb{R})} = \|\phi\|_{L^2(\mathbb{R})}.$$

The proof of this lemma is straightforward.

Definition 2.3 (α-WFT,[4])**.** *Let $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ be a window function and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, the α-WFT of $f \in L^2(\mathbb{R})$ with respect to ϕ and α is defined by*

$$\begin{aligned} G_\phi^\alpha f(\omega,u) &= \left(f, \phi_{\omega,u}^\alpha\right)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\overline{\phi_{\omega,u}^\alpha(x)}dx \\ &= \int_{\mathbb{R}} f(x)\overline{\phi(x-u)\exp\left\{i\omega x - \frac{i(x^2-u^2)\cot\alpha}{2}\right\}}dx \\ &= \int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\exp\left\{-i\omega x + \frac{i(x^2-u^2)\cot\alpha}{2}\right\}dx, \end{aligned}$$

for all $(\omega,u) \in \mathbb{R}^2$.

Now, we recall some fundamental properties for the α-WFT that will be used in this paper. These results were obtained in the paper [4].

Proposition 2.1 (Orthogonality relation)**.** *Let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter. Then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} G_\phi^\alpha f(\omega,u)\overline{G_\psi^\alpha g(\omega,u)}d\omega du = 2\pi\,(\psi,\phi)_{L^2(\mathbb{R})}\,(f,g)_{L^2(\mathbb{R})},$$

for all $f, g \in L^2(\mathbb{R})$. Moreover, if $f = g$ and $\phi = \psi$, then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left|G_\phi^\alpha f(\omega,u)\right|^2 d\omega du = 2\pi \left\|\phi\right\|^2_{L^2(\mathbb{R})} \left\|f\right\|^2_{L^2(\mathbb{R})},$$

or equivalently

$$\left\|G_\phi^\alpha f\right\|_{L^2(\mathbb{R})} = (2\pi)^{\frac{1}{2}} \left\|\phi\right\|_{L^2(\mathbb{R})} \left\|f\right\|_{L^2(\mathbb{R})}.$$

Proposition 2.2 (Inversion formula)**.** *Let $\phi,\psi \in L^2(\mathbb{R})\setminus\{0\}$ be two window functions such that $(\psi,\phi)_{L^2(\mathbb{R})} \neq 0$ and let α be a fractional parameter. Then, any function $f \in L^2(\mathbb{R})$ can be reconstructed as follows*

$$f(x) = \frac{1}{2\pi\left(\psi,\phi\right)_{L^2(\mathbb{R})}} \int_{\mathbb{R}}\int_{\mathbb{R}} G_\phi^\alpha f(\omega,u)\psi^\alpha_{\omega,u}(x) d\omega du,$$

for all $x \in \mathbb{R}$. Moreover, if $\phi = \psi$, we obtain

$$f(x) = \frac{1}{2\pi\left\|\phi\right\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}}\int_{\mathbb{R}} G_\phi^\alpha f(\omega,u)\phi^\alpha_{\omega,u}(x) d\omega du$$

for all $x \in \mathbb{R}$.

3. Localization operators related to α-WFT

In this section we introduce the localization operator $L^\alpha_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ related to α-WFT, where $\phi,\psi \in L^2(\mathbb{R})\setminus\{0\}$ are two window functions and $\sigma : \mathbb{R}^2 \to \mathbb{C}$ is a fixed function (called the symbol). To this end, we use the inversion formula (or the reconstruction formula) from Proposition 2.3 by inserting a symbol. As soon as we have a reconstruction formula for some time-frequency transform, we are interested to study localization operators as in [2]-[5] and [15]. The idea of a localization operator is to pick out different areas of interests by inserting a weight function or a symbol into a reconstruction formula (or in a resolution of the identity formula). Throughout this paper, $\mathcal{B}(X)$ is the set of all bounded linear operators from the Hilbert space X to itself.

Definition 3.1. *Let $\phi,\psi \in L^2(\mathbb{R})\setminus\{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, the operator $L^\alpha_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ weakly defined by*

$$\begin{aligned}\left(L^\alpha_{\sigma,\phi,\psi}f,g\right)_{L^2(\mathbb{R})} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u) G_\phi^\alpha f(\omega,u)\overline{G_\psi^\alpha g(\omega,u)} d\omega du \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u)\left(f,\phi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})}\left(\psi^\alpha_{\omega,u},g\right)_{L^2(\mathbb{R})} d\omega du,\end{aligned} \tag{1}$$

for all $f,g \in L^2(\mathbb{R})$ or strongly defined by

$$L^\alpha_{\sigma,\phi,\psi}f = \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u)\left(f,\phi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})} \psi^\alpha_{\omega,u} d\omega du,$$

for all $f \in L^2(\mathbb{R})$, is called the localization operator related to α-WFT with respect to the symbol $\sigma \in L^1(\mathbb{R}^2)\cup L^\infty(\mathbb{R}^2)$.

In the sequel, we will give some results concerning the L^2-boundedness of localization operator $L^\alpha_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ when $\sigma \in L^p(\mathbb{R}^2), 1 \leq p \leq \infty$.

Proposition 3.1. *Let $\sigma \in L^1(\mathbb{R}^2)$ be a symbol, let $\phi,\psi \in L^2(\mathbb{R})\setminus\{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then the localization operator $L^\alpha_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a well defined bounded linear operator and*

$$\left\|L^\alpha_{\sigma,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq \left\|\phi\right\|_{L^2(\mathbb{R})} \left\|\psi\right\|_{L^2(\mathbb{R})} \left\|\sigma\right\|_{L^1(\mathbb{R}^2)}.$$

Proof. Let $f, g \in L^2(\mathbb{R})$. Then, using Cauchy–Schwarz inequality for the Hilbert space and Lemma 2.1 we have

$$\begin{aligned}\left|\left(L^\alpha_{\sigma,\phi,\psi} f, g\right)_{L^2(\mathbb{R})}\right| &= \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u)\left(f,\phi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})}\left(\psi^\alpha_{\omega,u}, g\right)_{L^2(\mathbb{R})} d\omega du\right| \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)|\left|\left(f,\phi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})}\right|\left|\left(\psi^\alpha_{\omega,u}, g\right)_{L^2(\mathbb{R})}\right| d\omega du \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)| \|f\|_{L^2(\mathbb{R})}\left\|\phi^\alpha_{\omega,u}\right\|_{L^2(\mathbb{R})}\left\|\psi^\alpha_{\omega,u}\right\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})} d\omega du \\ &\le \|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}\|\phi\|_{L^2(\mathbb{R})}\|\psi\|_{L^2(\mathbb{R})}\int_{\mathbb{R}}\int_{\mathbb{R}}|\sigma(\omega,u)|\, d\omega du \\ &\le \|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}\|\phi\|_{L^2(\mathbb{R})}\|\psi\|_{L^2(\mathbb{R})}\|\sigma\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

Thus, the proof is complete. ☐

Proposition 3.2. *Let $\sigma \in L^\infty(\mathbb{R}^2)$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then the localization operator $L^\alpha_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a well defined bounded linear operator and*

$$\left\|L^\alpha_{\sigma,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \le 2\pi \|\phi\|_{L^2(\mathbb{R})}\|\psi\|_{L^2(\mathbb{R})}\|\sigma\|_{L^\infty(\mathbb{R}^2)}.$$

Proof. Let $f, g \in L^2(\mathbb{R})$. Then, using the Cauchy-Schwarz inequality and the orthogonality relation (see Proposition 2.1) we have

$$\begin{aligned}&\left|\left(L^\alpha_{\sigma,\phi,\psi} f, g\right)_{L^2(\mathbb{R})}\right| = \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u)\left(f,\phi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})}\left(\psi^\alpha_{\omega,u}, g\right)_{L^2(\mathbb{R})} d\omega du\right| \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)|\left|\left(f,\phi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})}\right|\left|\left(\psi^\alpha_{\omega,u}, g\right)_{L^2(\mathbb{R})}\right| d\omega du \\ &\le \|\sigma\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\left(f,\phi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})}\right|\left|\left(\psi^\alpha_{\omega,u}, g\right)_{L^2(\mathbb{R})}\right| d\omega du \\ &\le \|\sigma\|_{L^\infty(\mathbb{R}^2)}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\left(f,\phi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})}\right|^2 d\omega du\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\left(\psi^\alpha_{\omega,u}, g\right)_{L^2(\mathbb{R})}\right|^2 d\omega du\right)^{\frac{1}{2}} \\ &= \|\sigma\|_{L^\infty(\mathbb{R}^2)}\left\|G^\alpha_\phi f\right\|_{L^2(\mathbb{R})}\left\|G^\alpha_\psi g\right\|_{L^2(\mathbb{R})} \\ &= 2\pi\|\sigma\|_{L^\infty(\mathbb{R}^2)}\|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}\|\phi\|_{L^2(\mathbb{R})}\|\psi\|_{L^2(\mathbb{R})}.\end{aligned}$$

Thus, the proof is complete. ☐

Theorem 3.1. *Let $\sigma \in L^p(\mathbb{R}^2), 1 \le p \le \infty$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then there exists a unique bounded linear operator $L^\alpha_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that*

$$\left\|L^\alpha_{\sigma,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \le (2\pi)^{\frac{1}{p'}} \|\phi\|_{L^2(\mathbb{R})}\|\psi\|_{L^2(\mathbb{R})}\|\sigma\|_{L^p(\mathbb{R}^2)},$$

where p and p' are conjugate indices to each other $\left(i.e.\ \frac{1}{p} + \frac{1}{p'} = 1\right)$ and $L^\alpha_{\sigma,\phi,\psi}$ is given by (1) for all $f, g \in L^2(\mathbb{R})$ and all simple functions (the finite linear combinations with complex coefficients of characteristic functions of measurable sets) σ on $\mathbb{R}^2$ for which $\mu\{(\omega,u) \in \mathbb{R}^2 : \sigma(\omega,u) \neq 0\} < \infty$.

Proof. (i). *Existence:* Let $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a unitary operator. Let $\sigma \in L^1(\mathbb{R}^2)$. Then, by Proposition 3.1, the linear operator $\widetilde{L}^\alpha_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $\widetilde{L}^\alpha_{\sigma,\phi,\psi} = SL^\alpha_{\sigma,\phi,\psi}S^{-1}$ is a bounded operator and

$$\left\|\widetilde{L}^\alpha_{\sigma,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \le \|\phi\|_{L^2(\mathbb{R})}\|\psi\|_{L^2(\mathbb{R})}\|\sigma\|_{L^1(\mathbb{R}^2)}. \tag{2}$$

Let $\sigma \in L^\infty(\mathbb{R}^2)$. Then, by Proposition 3.2, the linear operator $\widetilde{L}^\alpha_{\sigma,\phi,\psi}$ is also a bounded operator and

$$\left\|\widetilde{L}^\alpha_{\sigma,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq 2\pi \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^\infty(\mathbb{R}^2)}. \tag{3}$$

Let $\mathcal{D}$ be the set of all simple functions σ on $\mathbb{R}^2$ such that $\mu\{(\omega,u) \in \mathbb{R}^2 : \sigma(\omega,u) \neq 0\} < \infty$. Let $f \in L^2(\mathbb{R})$ and T be the linear transformation from $\mathcal{D}$ to the set of all functions in $L^2(\mathbb{R})$ defined by

$$T\sigma = \widetilde{L}^\alpha_{\sigma,\phi,\psi} f, \quad \forall \sigma \in \mathcal{D}.$$

Then, from (2), it follows that:

$$\begin{aligned}\|T\sigma\|_{L^2(\mathbb{R})} = \left\|\widetilde{L}^\alpha_{\sigma,\phi,\psi} f\right\|_{L^2(\mathbb{R})} &\leq \left\|\widetilde{L}^\alpha_{\sigma,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \|f\|_{L^2(\mathbb{R})} \\ &\leq \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)} \|f\|_{L^2(\mathbb{R})}\end{aligned}$$

and, from (3), it follows that:

$$\begin{aligned}\|T\sigma\|_{L^2(\mathbb{R})} = \left\|\widetilde{L}^\alpha_{\sigma,\phi,\psi} f\right\|_{L^2(\mathbb{R})} &\leq \left\|\widetilde{L}^\alpha_{\sigma,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \|f\|_{L^2(\mathbb{R})} \\ &\leq 2\pi \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^\infty(\mathbb{R}^2)} \|f\|_{L^2(\mathbb{R})}\end{aligned}$$

for all $\sigma \in \mathcal{D}$.

If we take $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha = \frac{1}{p}$, $\beta_1 = \beta_2 = \beta = \frac{1}{2}$, $M_1 = \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}$ and $M_2 = 2\pi \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}$ in the Riesz-Thorin interpolation theorem (see Theorem 12.4 in [15]), we get

$$\|T\sigma\|_{L^2(\mathbb{R})} = \left\|\widetilde{L}^\alpha_{\sigma,\phi,\psi} f\right\|_{L^2(\mathbb{R})} \leq (2\pi)^{\frac{1}{p'}} \|\sigma\|_{L^p(\mathbb{R}^2)} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}$$

for all $\sigma \in \mathcal{D}$, where p' is the conjugate index of p.

Therefore,

$$\left\|\widetilde{L}^\alpha_{\sigma,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq (2\pi)^{\frac{1}{p'}} \|\sigma\|_{L^p(\mathbb{R}^2)} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})}$$

for all $\sigma \in \mathcal{D}$.

Let $\sigma \in L^p(\mathbb{R}^2), 1 < p < \infty$. Then there exists a sequence $\{\sigma_j\}_{j\geq 1}$ of functions in $\mathcal{D}$ such that $\sigma_j \to \sigma$ in $L^p(\mathbb{R}^2)$ as $j \to \infty$. Then

$$\left\|\widetilde{L}^\alpha_{\sigma_i,\phi,\psi} - \widetilde{L}^\alpha_{\sigma_j,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq (2\pi)^{\frac{1}{p'}} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma_i - \sigma_j\|_{L^p(\mathbb{R}^2)} \to 0$$

as $i, j \to \infty$. Therefore, $\left\{\widetilde{L}^\alpha_{\sigma_j,\phi,\psi}\right\}_{j\geq 1}$ is a Cauchy sequence in $\mathcal{B}\left(L^2(\mathbb{R})\right)$.

Using the completeness of $\mathcal{B}\left(L^2(\mathbb{R})\right)$, we can find a bounded linear operator $\widetilde{L}^\alpha_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that $\widetilde{L}^\alpha_{\sigma_j,\phi,\psi} \to \widetilde{L}^\alpha_{\sigma,\phi,\psi}$ in $\mathcal{B}\left(L^2(\mathbb{R})\right)$ as $j \to \infty$. So,

$$\begin{aligned}\left\|\widetilde{L}^\alpha_{\sigma,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} &\leq \left\|\widetilde{L}^\alpha_{\sigma,\phi,\psi} - \widetilde{L}^\alpha_{\sigma_j,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} + \left\|\widetilde{L}^\alpha_{\sigma_j,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \\ &\leq \left\|\widetilde{L}^\alpha_{\sigma,\phi,\psi} - \widetilde{L}^\alpha_{\sigma_j,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} + (2\pi)^{\frac{1}{p'}} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma_j\|_{L^p(\mathbb{R}^2)} \\ &\to (2\pi)^{\frac{1}{p'}} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^p(\mathbb{R}^2)}\end{aligned}$$

as $j \to \infty$. Thus the operator $L^\alpha_{\sigma,\phi,\psi} = S^{-1}\widetilde{L}^\alpha_{\sigma,\phi,\psi} S$ belongs to $\mathcal{B}(L^2(\mathbb{R}))$ and satisfies the conclusion of the theorem if $\sigma \in L^p(\mathbb{R}^2), 1 < p < \infty$.

(ii). *Uniqueness:* Let $\sigma \in L^p(\mathbb{R}^2), 1 < p < \infty$ and suppose that $P_\sigma : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is another bounded linear operator satisfying the conclusion of the theorem. Let $Q : L^p(\mathbb{R}^2) \to$

$\mathcal{B}(L^2(\mathbb{R}))$ be the linear operator defined by $Q\sigma = L^{\alpha}_{\sigma,\phi,\psi} - P_\sigma$, for all $\sigma \in L^p(\mathbb{R}^2)$. In this case,

$$\|Q\sigma\|_{\mathcal{B}(L^2(\mathbb{R}))} = \left\|L^{\alpha}_{\sigma,\phi,\psi} - P_\sigma\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \le 2^{\frac{1+p'}{p'}} \pi^{\frac{1}{p'}} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^p(\mathbb{R}^2)}$$

for all $\sigma \in L^p(\mathbb{R}^2)$.

Since $L^{\alpha}_{\sigma,\phi,\psi}$ and P_σ are bounded linear operators satisfying the conclusions of the theorem, the operator $Q\sigma = L^{\alpha}_{\sigma,\phi,\psi} - P_\sigma$ is equal to the zero operator on $L^2(\mathbb{R})$ for all $\sigma \in \mathcal{D}$. Thus, $Q : L^p(\mathbb{R}^2) \to \mathcal{B}(L^2(\mathbb{R}))$ is a bounded linear operator that is equal to zero on the dense subspace $\mathcal{D}$ of $L^p(\mathbb{R}^2)$. Therefore $P_\sigma = L^{\alpha}_{\sigma,\phi,\psi}$ for all functions $\sigma \in L^p(\mathbb{R}^2)$. □

4. S_p norm inequalities, $1 \le p \le \infty$

In this section, we prove that the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ related to α-WFT is in the Schatten-von Neumann class $S_p, 1 \le p \le \infty$.

We begin by recalling the definition of Schatten-von Neumann classes. To this end, we first remind a well-known result concerning the canonical form for a compact operator. Let X be a separable and complex Hilbert space in which the inner product and the norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively.

Theorem 4.1 (see [15])**.** *Let $T : X \to X$ be a compact operator. Then we can find an orthonormal basis $\{u_n\}_{n\ge1}$ for $N(T)^\perp$ (the orthogonal complement of the null space $N(T)$ of T) consisting of eigenvectors of $|T| = (T^*T)^{\frac{1}{2}} : X \to X$ and an orthonormal set $\{v_n\}_{n\ge1}$ in X such that*

$$T = \sum_{n=1}^{\infty} s_n(T)(\cdot, u_n)v_n,$$

where $s_n(T), n \ge 1$ are the positive singular values of $T : X \to X$ (i.e. $s_n(T)$ is the eigenvalues of $|T| : X \to X$ corresponding to the eigenvectors $\{u_n\}_{n\ge1}$) and the series converges to T strongly.

Definition 4.1. *A compact operator $T : X \to X$ is said to be in the Schatten-von Neumann class $S_p, 1 \le p < \infty$, if*

$$\sum_{n=1}^{\infty} (s_n(T))^p < \infty.$$

Thus, $S_p, 1 \le p < \infty$, is a complex Banach space in which the norm $\|\cdot\|_{S_p}$ is given by

$$\|T\|_{S_p} = \left(\sum_{n=1}^{\infty} (s_n(T))^p\right)^{\frac{1}{p}}, \qquad T \in S_p.$$

We let S_∞ be the C^*-algebra $\mathcal{B}(X)$ of all bounded linear operators on X. Thus, $\|\cdot\|_{S_\infty} = \|\cdot\|_{\mathcal{B}(X)}$, where $\|\cdot\|_{\mathcal{B}(X)}$ denotes the norm in $\mathcal{B}(X)$. Let us remark that $S_p \subseteq S_q, 1 \le p \le q \le \infty$. Usually, S_1 is called the trace class and S_2 is the Hilbert-Schmidt class.

Definition 4.2. *If $T : X \to X$ is a bounded linear operator in the trace class S_1, then we can define the trace $tr(T)$ by*

$$tr(T) = \sum_{n=1}^{\infty} (T\varphi_n, \varphi_n),$$

where $\{\varphi_n\}_{n\ge1}$ is any orthonormal basis for X (the fact that the trace is independent of the specific orthonormal basis follows from Proposition 2.6 in [15]).

It can be proven that if $T : X \to X$ is a positive operator in the trace class S_1, then

$$\|T\|_{S_1} = tr(T),$$

see Proposition 2.7 in [15]. For more details concerning the Schatten-von Neumann classes, see [15]. Let us recall the following theorem.

Theorem 4.2 (Theorem 1.4.8 in [16])**.** *Suppose that T is a compact operator from X into X. Then T is in $S_p, 1 \leq q < \infty$ if and only if*

$$\sum_{n=1}^{\infty} |(T\varphi_n, \xi_n)|^p < \infty,$$

for all orthonormal sequences $\{\varphi_n\}_{n\geq 1}$ and $\{\xi_n\}_{n\geq 1}$ in X. Moreover,

$$\|T\|_{S_p} = \sup\left\{\sum_{n=1}^{\infty} |(T\varphi_n, \xi_n)|^p \,; \{\varphi_n\}_{n\geq 1}, \{\xi_n\}_{n\geq 1} \text{ orthonormal sequences of } X\right\}.$$

Proposition 4.1. *Let $\sigma \in L^1(\mathbb{R}^2)$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then the localization operator $L^\alpha_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in the Hilbert-Schmidt class S_2 and*

$$\left\|L^\alpha_{\sigma,\phi,\psi}\right\|^2_{S_2} = \sum_{n=1}^{\infty} \left\|L^\alpha_{\sigma,\phi,\psi}\xi_n\right\|^2_{L^2(\mathbb{R})},$$

where $\{\xi_n\}_{n\geq 1}$ is any orthonormal basis for $L^2(\mathbb{R})$.

Proof. Let $\{\xi_n\}_{n\geq 1}$ be an orthonormal basis for $L^2(\mathbb{R})$. Using Fubini's Theorem, Parseval's identity in a Hilbert space, Cauchy–Schwarz inequality and Lemma 2.1, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left\|L^\alpha_{\sigma,\phi,\psi}\xi_n\right\|^2_{L^2(\mathbb{R})} = \left|\sum_{n=1}^{\infty} \left(L^\alpha_{\sigma,\phi,\psi}\xi_n, L^\alpha_{\sigma,\phi,\psi}\xi_n\right)_{L^2(\mathbb{R})}\right| \\
&= \left|\sum_{n=1}^{\infty} \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u) \left(\xi_n, \phi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})} \left(\psi^\alpha_{\omega,u}, L^\alpha_{\sigma,\phi,\psi}\xi_n\right)_{L^2(\mathbb{R})} d\omega du\right| \\
&= \left|\sum_{n=1}^{\infty} \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u) \left(\xi_n, \phi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})} \left(\left(L^\alpha_{\sigma,\phi,\psi}\right)^* \psi^\alpha_{\omega,u}, \xi_n\right)_{L^2(\mathbb{R})} d\omega du\right| \\
&= \left|\int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u) \left(\left(L^\alpha_{\sigma,\phi,\psi}\right)^* \psi^\alpha_{\omega,u}, \phi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})} d\omega du\right| \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)| \left\|\left(L^\alpha_{\sigma,\phi,\psi}\right)^*\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \left\|\psi^\alpha_{\omega,u}\right\|_{L^2(\mathbb{R})} \left\|\phi^\alpha_{\omega,u}\right\|_{L^2(\mathbb{R})} d\omega du \\
&= \left\|\left(L^\alpha_{\sigma,\phi,\psi}\right)^*\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \|\psi\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)} < \infty,
\end{aligned}$$

where $\left(L^\alpha_{\sigma,\phi,\psi}\right)^* : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the adjoint operator of $L^\alpha_{\sigma,\phi,\psi}$. So, according to the previous inequality and using Proposition 2.8 in [15], the linear operator $L^\alpha_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in the Hilbert–Schmidt class S_2 and hence compact. □

In the following we give some results concerning Schatten-von Neumann properties of the localization operator $L^\alpha_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, when its symbol $\sigma \in L^p(\mathbb{R}^2), 1 \leq p \leq \infty$.

Proposition 4.2. *Let $\sigma \in L^1(\mathbb{R}^2)$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then the localization operator $L^\alpha_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in the trace class S_1 and*

$$\left\|L^\alpha_{\sigma,\phi,\psi}\right\|_{S_1} \leq \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)}.$$

Proof. By Proposition 4.1 it follows that the localization operator $L^{\alpha}_{\sigma,\phi,\psi}$ is in S_2 and thus it is compact. Let $\{\varphi_n\}_{n\geq 1}$ and $\{\xi_n\}_{n\geq 1}$ be two orthonormal sequences for $L^2(\mathbb{R})$. Then, by Cauchy-Schwarz inequality we get

$$\left|\left(L^{\alpha}_{\sigma,\phi,\psi}\varphi_n,\xi_n\right)_{L^2(\mathbb{R})}\right| \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)| \left|\left(\varphi_n,\phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})}\right| \left|\left(\psi^{\alpha}_{\omega,u},\xi_n\right)_{L^2(\mathbb{R})}\right| d\omega du$$

$$\leq \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)| \left|\left(\varphi_n,\phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})}\right|^2 d\omega du\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)| \left|\left(\psi^{\alpha}_{\omega,u},\xi_n\right)_{L^2(\mathbb{R})}\right|^2 d\omega du\right)^{\frac{1}{2}},$$

for all $n \in \mathbb{N}$.

Summing over $n \in \mathbb{N}$ and using once again Cauchy-Schwarz inequality, we obtain

$$\sum_{n=1}^{\infty}\left|\left(L^{\alpha}_{\sigma,\phi,\psi}\varphi_n,\xi_n\right)_{L^2(\mathbb{R})}\right| \leq \left(\sum_{n=1}^{\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)| \left|\left(\varphi_n,\phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})}\right|^2 d\omega du\right)^{\frac{1}{2}}$$

$$\times \left(\sum_{n=1}^{\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)| \left|\left(\psi^{\alpha}_{\omega,u},\xi_n\right)_{L^2(\mathbb{R})}\right|^2 d\omega du\right)^{\frac{1}{2}}.$$

Using Fubini's Theorem, Bessel's inequality, Lemma 2.1 and the assumptions stated in the theorem we get

$$\begin{aligned}\sum_{n=1}^{\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)| \left|\left(\varphi_n,\phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})}\right|^2 d\omega du &= \int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)| \sum_{n=1}^{\infty}\left|\left(\varphi_n,\phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})}\right|^2 d\omega du \\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)| \left\|\phi^{\alpha}_{\omega,u}\right\|^2_{L^2(\mathbb{R})} d\omega du \\ &= \|\phi\|^2_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)},\end{aligned}$$

and similarly we can write

$$\sum_{n=1}^{\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(\omega,u)| \left|\left(\psi^{\alpha}_{\omega,u},\xi_n\right)_{L^2(\mathbb{R})}\right|^2 d\omega du \leq \|\psi\|^2_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)}.$$

Thus

$$\sum_{n=1}^{\infty}\left|\left(L^{\alpha}_{\sigma,\phi,\psi}\varphi_n,\xi_n\right)_{L^2(\mathbb{R})}\right| \leq \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)}.$$

Then, by Theorem 4.2 it follows that the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in the trace class S_1 and

$$\left\|L^{\alpha}_{\sigma,\phi,\psi}\right\|_{S_1} \leq \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)}.$$

This completes the proof. □

The following proposition gives a compactness result regarding the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, with its symbol $\sigma \in L^p(\mathbb{R}^2), 1 \leq p < \infty$.

Proposition 4.3. *Let $\sigma \in L^p(\mathbb{R}^2), 1 \leq p < \infty$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is compact.*

Proof. We denote by $\mathcal{D}$ the set of all simple functions σ on $\mathbb{R}^2$ such that $\mu\{(\omega,u) \in \mathbb{R}^2 : \sigma(\omega,u) \neq 0\} < \infty$. Let $\{\sigma_k\}_{k\geq 1}$ be a sequence of functions in $\mathcal{D}$ such that $\sigma_k \to \sigma$ in $L^p(\mathbb{R}^2)$ as $k \to \infty$. Then by Theorem 3.1, we get

$$\left\|L^{\alpha}_{\sigma_k,\phi,\psi} - L^{\alpha}_{\sigma,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq (2\pi)^{\frac{1}{p'}} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma_k - \sigma\|_{L^p(\mathbb{R}^2)} \to 0,$$

as $k \to \infty$, when p' is the conjugate index of p $\left(\text{i.e. } \frac{1}{p} + \frac{1}{p'} = 1\right)$. So $L^{\alpha}_{\sigma_k,\phi,\psi} \to L^{\alpha}_{\sigma,\phi,\psi}$ in $\mathcal{B}(L^2(\mathbb{R}))$ as $k \to \infty$. Since, by Proposition 4.2, $L^{\alpha}_{\sigma_k,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in the trace class S_1 and hence compact for all $k \in \mathbb{N}$, it follows that $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is compact. $\square$

Theorem 4.3. *Let $\sigma \in L^p(\mathbb{R}^2), 1 \le p \le \infty$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in the Schatten-von Neumann class S_p and*

$$\left\|L^{\alpha}_{\sigma,\phi,\psi}\right\|_{S_p} \le (2\pi)^{\frac{1}{p'}} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^p(\mathbb{R}^2)},$$

where p and p' are conjugate indices to each other.

Proof. By Proposition 4.2, we have

$$\left\|L^{\alpha}_{\sigma,\phi,\psi}\right\|_{S_1} \le \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)} \tag{4}$$

for all $\sigma \in L^1(\mathbb{R})$. By Proposition 3.2, we have

$$\left\|L^{\alpha}_{\sigma,\phi,\psi}\right\|_{S_\infty} = \left\|L^{\alpha}_{\sigma,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \le 2\pi \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^\infty(\mathbb{R}^2)}. \tag{5}$$

for all $\sigma \in L^\infty(\mathbb{R}^2)$.

So, by (4), (5) and the interpolation Theorems 2.10 and 2.11 in [15], the proof is complete. $\square$

In the following, a two sided estimate of the trace class norm of the localization operators $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, when $\sigma \in L^1(\mathbb{R}^2)$, is given.

Theorem 4.4. *Let $\sigma \in L^1(\mathbb{R}^2)$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi$, $n \in \mathbb{Z}$. Then, we have*

$$\frac{1}{\pi\left(\|\phi\|^2_{L^2(\mathbb{R})} + \|\psi\|^2_{L^2(\mathbb{R})}\right)} \|\sigma_{\phi,\psi}\|_{L^1(\mathbb{R}^2)} \le \left\|L^{\alpha}_{\sigma,\phi,\psi}\right\|_{S_1} \le \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)}, \tag{6}$$

where $\sigma_{\phi,\psi} : \mathbb{R}^2 \to \mathbb{C}$ is given by

$$\sigma_{\phi,\psi}(\omega,u) = \left(L^{\alpha}_{\sigma,\phi,\psi}\phi^{\alpha}_{\omega,u}, \psi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})}.$$

Proof. By Proposition 4.2 it follows that the localization operator $L^{\alpha}_{\sigma,\phi,\psi}$ is in the trace class S_1 and its trace class norm satisfies the estimate in the right-hand side of the relation (6). Now, we want to prove that the estimate in the left-hand side of the relation (6) is also valid.

To this end, we firstly prove that $\sigma_{\phi,\psi}$ is in the trace class S_1. Using Theorem 4.1, Fubini's Theorem and the orthogonality relation, we get

$$\begin{aligned}
&\|\sigma_{\phi,\psi}\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma_{\phi,\psi}(\omega,u)|\, d\omega du = \int_{\mathbb{R}}\int_{\mathbb{R}} \left|\left(L^{\alpha}_{\sigma,\phi,\psi}\phi^{\alpha}_{\omega,u}, \psi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})}\right| d\omega du \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left|\sum_{n=1}^{\infty} s_n\left(L^{\alpha}_{\sigma,\phi,\psi}\right)\left(\phi^{\alpha}_{\omega,u}, u_n\right)_{L^2(\mathbb{R})}\left(v_n, \psi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})}\right| d\omega du \\
&\le \frac{1}{2}\sum_{n=1}^{\infty} s_n\left(L^{\alpha}_{\sigma,\phi,\psi}\right)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\left(\phi^{\alpha}_{\omega,u}, u_n\right)_{L^2(\mathbb{R})}\right|^2 d\omega du + \int_{\mathbb{R}}\int_{\mathbb{R}} \left|\left(v_n, \psi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})}\right|^2 d\omega du\right) \\
&= \frac{1}{2}\sum_{n=1}^{\infty} s_n\left(L^{\alpha}_{\sigma,\phi,\psi}\right)\left(\left\|G^{\alpha}_{\phi}u_n\right\|^2_{L^2(\mathbb{R})} + \left\|G^{\alpha}_{\psi}v_n\right\|^2_{L^2(\mathbb{R})}\right) \le \pi\left(\|\phi\|^2_{L^2(\mathbb{R})} + \|\psi\|^2_{L^2(\mathbb{R})}\right)\left\|L^{\alpha}_{\sigma,\phi,\psi}\right\|_{S_1}
\end{aligned}$$

Therefore,

$$\frac{1}{\pi\left(\|\phi\|^2_{L^2(\mathbb{R})} + \|\psi\|^2_{L^2(\mathbb{R})}\right)} \|\sigma_{\phi,\psi}\|_{L^1(\mathbb{R}^2)} \le \left\|L^{\alpha}_{\sigma,\phi,\psi}\right\|_{S_1}$$

and this completes the proof of the theorem. □

Now, we give a formula for the trace $tr\left(L^{\alpha}_{\sigma,\phi,\psi}\right)$ of the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ when $\sigma \in L^1(\mathbb{R}^2)$.

Proposition 4.4. *Under the assumptions of Theorem 4.3, the trace $tr\left(L^{\alpha}_{\sigma,\phi,\psi}\right)$ of the localization operator $L^{\alpha}_{\sigma,\phi,\psi}$ is given by*

$$tr\left(L^{\alpha}_{\sigma,\phi,\psi}\right) = \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u)\left(\psi^{\alpha}_{\omega,u},\phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})} d\omega du.$$

Proof. By Proposition 4.2, the localization operator $L^{\alpha}_{\sigma,\phi,\psi}$ belongs to the trace class S_1. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R})$. Then, by Definition 4.2, Fubini's Theorem and Parseval's identity, we get

$$\begin{aligned} tr\left(L^{\alpha}_{\sigma,\phi,\psi}\right) &= \sum_{n=1}^{\infty}\left(L^{\alpha}_{\sigma,\phi,\psi}\xi_n,\xi_n\right) = \sum_{n=1}^{\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u)\left(\phi^{\alpha}_{\omega,u},\xi_n\right)_{L^2(\mathbb{R})}\left(\xi_n,\psi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})} d\omega du \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u)\sum_{n=1}^{\infty}\left(\phi^{\alpha}_{\omega,u},\xi_n\right)_{L^2(\mathbb{R})}\left(\xi_n,\psi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})} d\omega du \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u)\left(\psi^{\alpha}_{\omega,u},\phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})} d\omega du. \end{aligned}$$

and the proof is complete. □

Remark 4.1. *If σ is a real-valued and non-negative function in $L^1(\mathbb{R}^2)$ and $\phi = \psi$, then the estimates of the relation (6) are sharp.*

Proof. Firstly, using Proposition 2.7 in [15] and Proposition 4.4, we get

$$\left\|L^{\alpha}_{\sigma,\phi,\phi}\right\|_{S_1} = tr(L^{\alpha}_{\sigma,\phi,\phi}) = \|\phi\|_{L^2(\mathbb{R})}\|\sigma\|_{L^1(\mathbb{R}^2)}.$$

Therefore, the right-hand side estimate of the relation (6) is sharp. Using Fubini's Theorem, the orthogonality relation and Lemma 2.1, we get

$$\begin{aligned} \|\sigma_{\phi,\phi}\|_{L^1(\mathbb{R}^2)} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma_{\phi,\phi}(\omega,u) d\omega du = \int_{\mathbb{R}}\int_{\mathbb{R}}\left(L^{\alpha}_{\sigma,\phi,\phi}\phi^{\alpha}_{\omega,u},\phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})} d\omega du \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\sigma(\omega',u')\left(\phi^{\alpha}_{\omega,u},\phi^{\alpha}_{\omega',u'}\right)_{L^2(\mathbb{R})}\left(\phi^{\alpha}_{\omega',u'},\phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})} d\omega' du'\right) d\omega du \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\sigma(\omega',u')\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\left(\phi^{\alpha}_{\omega,u},\phi^{\alpha}_{\omega',u'}\right)_{L^2(\mathbb{R})}\right|^2 d\omega du\right) d\omega' du' \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\sigma(\omega',u')\left\|G^{\alpha}_{\phi}\phi^{\alpha}_{\omega',u'}\right\|^2_{L^2(\mathbb{R})} d\omega' du' \\ &= 2\pi\int_{\mathbb{R}}\int_{\mathbb{R}}\sigma(\omega',u')\left\|\phi^{\alpha}_{\omega',u'}\right\|^2_{L^2(\mathbb{R})}\|\phi\|^2_{L^2(\mathbb{R})} d\omega' du' = 2\pi\|\phi\|^4_{L^2(\mathbb{R})}\|\sigma\|_{L^1(\mathbb{R}^2)}. \end{aligned}$$

Thus, the left-hand side estimate of the realation (6) is also sharp. □

Now we state a result concerning the trace class norm of the power n of a product of two localization operators.

Proposition 4.5. *Let σ_1,σ_2 be two real-valued and non-negative functions in $L^1(\mathbb{R}^2)$ and let ϕ be a window function in $L^2(\mathbb{R})\setminus\{0\}$. Suppose that $L^{\alpha}_{\sigma_1,\phi,\phi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $L^{\alpha}_{\sigma_2,\phi,\phi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ commute with each other and the operator $L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a positive one. Then, the linear operators $L^{\alpha}_{\sigma_1,\phi,\phi}$, $L^{\alpha}_{\sigma_2,\phi,\phi}$ and $L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi}$ are positive and in the trace class S_1. Moreover,*

$$\left\|\left(L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi}\right)^n\right\|_{S_1} \leq \left\|L^{\alpha}_{\sigma_1,\phi,\phi}\right\|^n_{S_1}\left\|L^{\alpha}_{\sigma_2,\phi,\phi}\right\|^n_{S_1}, \forall n \in \mathbb{N}.$$

Proof. By Proposition 4.1, the linear operators $L^{\alpha}_{\sigma_1,\phi,\phi}, L^{\alpha}_{\sigma_2,\phi,\phi}$ are in the Hilbert-Schmidt class S_2 and by Proposition 4.2, these are in the trace class S_1. Using Lemma 1.4.13 in [16] we can state that the operator $L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in trace class S_1. So, by Proposition 2.7 in [15] it follows that

$$\left\|L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi}\right\|_{S_1} = tr\left(L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi}\right).$$

Now, we recall Theorem 1 in [13] which states that if A and B (defined from the Hilbert space $\mathcal{H}$ to itself) are positive operators in the trace class S_1, then

$$tr(AB)^n \leq (tr(A))^n\,(tr(B))^n,$$

for all $n \in \mathbb{N}$. But from the assumptions of the theorem it follows that the operators $L^{\alpha}_{\sigma_1,\phi,\phi}$, $L^{\alpha}_{\sigma_2,\phi,\phi}$, $L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi}$ are positive operators in the trace class S_1. So, using Theorem 1 in [13], we get

$$\left\|L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi}\right\|_{S_1} = tr\left(L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi}\right) \leq tr\left(L^{\alpha}_{\sigma_1,\phi,\phi}\right)tr\left(L^{\alpha}_{\sigma_2,\phi,\phi}\right) = \left\|L^{\alpha}_{\sigma_1,\phi,\phi}\right\|_{S_1}\left\|L^{\alpha}_{\sigma_2,\phi,\phi}\right\|_{S_1}.$$

Using the fact that $S_p \subseteq S_q, 1 \leq p \leq q \leq \infty$, Lemma 1.4.13 in [16] and mathematical induction we can prove that $\left(L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi}\right)^n$ is in the trace class S_1, for all $n \in \mathbb{N}$. We also can prove that the operator $\left(L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi}\right)^n$ is a positive one, for all $n \in \mathbb{N}$, using the mathematical induction, the hypotheses that $L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi}$ is a positive operator and that the operators $L^{\alpha}_{\sigma_1,\phi,\phi}$, $L^{\alpha}_{\sigma_2,\phi,\phi}$ commute with each other. By Proposition 2.7 in [15] and Theorem 1 in [13], we get

$$\begin{aligned}\left\|\left(L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi}\right)^n\right\|_{S_1} &= tr\left(L^{\alpha}_{\sigma_1,\phi,\phi}L^{\alpha}_{\sigma_2,\phi,\phi}\right)^n \leq \left(tr\left(L^{\alpha}_{\sigma_1,\phi,\phi}\right)\right)^n\left(tr\left(L^{\alpha}_{\sigma_2,\phi,\phi}\right)\right)^n \\ &= \left\|L^{\alpha}_{\sigma_1,\phi,\phi}\right\|^n_{S_1}\left\|L^{\alpha}_{\sigma_2,\phi,\phi}\right\|^n_{S_1},\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, the proof is complete. □

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