ABOUT PROBABILITIES ON LUKASIEWICZ-MOISIL ALGEBRAS (I)

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În această lucrare, prezentăm unele proprietăţi ale unui gen de probabilităţi pe algebre Lukasiewicz – Moisil.

In this paper, we present some properties of a kind of states on Lukasiewicz – Moisil algebras.

Keywords: Lukasiewicz – Moisil algebra, state, conditional ρ-state.
Mathematics Subject Classification 2000:06D30

1. Introduction

In classical probability theory the set of events has a structure of Boolean algebra, because we use the Boolean logic. If we consider another logic, the set of events has on algebraic structure defined by the associated Lindenbaum – Tarski algebra. If we consider the n – valued Moisil logic, the set of events is a Lukasiewicz – Moisil algebra (n-valued). Is more difficult to define the notion of probability (= state) in this case. In [3] the authors study a state on a Lukasiewicz – Moisil algebra similar to the states on a MV_n – algebra. In a Boolean algebra, the states we defined in terms of the biresiduum.

This remark can be extended to Lukasiewicz – Moisil algebra [4] considering three biresidual ρ_C, ρ_H, and ρ_W. The corresponding ρ – states are connected to their restriction to the Boolean center. In our paper we consider another biresiduum ρ_M and we define ρ_M – states. Their study is similar to ρ_H – states. We study also conditional ρ_M – states and continuous ρ_M – states.

2. Definitions. Preliminaries.

2.1. States on Boolean algebras

Consider a Boolean algebra $(B, V, \wedge, -, 0, 1)$.

Definiton 2.1.1. A function $m : B \to [0,1]$ is a state on *B*, if the following conditions holds:

(I) $m(x \vee y) = m(x) + m(y)$, if $x \wedge y = 0$.

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(II) $m(1) = 1$.

Proposition 2.1.2. If $m : B \to [0,1]$, the following are equivalent:

(I) m is a state on B.

(II) $1 + m(x \wedge y) = m(x \vee y) + m(S_B(x,y)), \quad x, y \in B,$
$m(0) = 0, \quad m(1) = 1.$

In this proposition the bresidunm $S_B : BxB \to B,$
Is defined by $S_B(x,y) = (x \to y) \wedge (y \to x),$ where $x \to y$ is the Boolean implication $x \to y = \overline{x} \vee y.$

We have the following well-known lemma.

Lemma 2.1.3. If m is a state on B, the following properties hold for any $x, y, z \in B$:

(I) $m(x \vee y) = m(x) + m(y) - m(x \wedge y)$.

(II) $m(\overline{x}) = 1 - m(x)$.

(III) $m(x \wedge \overline{y}) = m(x) - m(x \wedge y)$.

(IV) $x \le y \Rightarrow m(x) \le m(y)$.

(V) $m(x \to y) = 1 - m(x) + m(x \wedge y)$.

(VI) [4] $m(z) + m(x \wedge y \wedge z) = m((x \vee y) \wedge z) + m(S_B(x,y) \wedge z)$. This is the generalization of the condition 2.1.2 (II).

2.2. Lukasiewicz – Moisil algebras

The n-valued Lukasiewicz – Moisil algebras were introduced by Moisil [5]. They are algebraic models for Moisil logics [1]. An extensive study of Lukasiewicz – Moisil algebras is the monograph [1].

Definition 2.2.1. A n - valued Lukasiewicz – Moisil algebra (LM algebra), $n \ge 2$ is an algebra $(A, V, \wedge N, \{\rho_i\} i \in \overline{1, n-1}, 0, 1)$ such that:

(I) $(A, \vee, \wedge, N, 0, 1)$ is a De Morgan algebra.

(II) $\rho_i : A \to A$ are lattice endomorphisms. $(i \in \overline{1, n-1})$,

(III) $\rho_i x \vee N\rho_i x = 1, \rho_i x \wedge N\rho_i x = 0$, for any $i, x \in A$, (ρ_i are chrysippian endomorphisms).

(IV) $\rho_i \circ \rho_j = \rho_j, \quad \forall i, j.$

(V) $i \le j \Rightarrow \rho_i \le \rho'_j, \quad \forall i, j.$

(VI) $\rho_i \circ N = N \circ \rho_{n-i}, \quad \forall i.$

(VII) $(\forall i)(\rho_i x = \rho_i y) \Rightarrow x = y$ (Moisil's determination principle).

Define the center of a LM algebra A by $C|A| = \{x \in A | x$ is a chrysippian element$\}$. $C|A|$ is a Boolean algebra and we have:

Proposition 2.2.2. [1] If A is a LM algebras the following hold:

(I) $x \in C(A) \Leftrightarrow \quad \rho_i x = x, \quad \forall i \Leftrightarrow (\exists i)(\rho_i x = x) \Leftrightarrow$
$x \vee Nx = 1 \Leftrightarrow x \wedge Nx = 0.$

(II) If $\vee x_i$ exists for a family $(x_i) i \in I, i \sigma I$ then $\rho_j(\vee xi) = \underset{i \in I}{\vee} \rho_j xi$, for any $j = \overline{1, n-1}$, and if $\underset{iGI}{\wedge} x_i$, exists, then $\rho_j(\wedge x_i) = \underset{iGI}{\wedge} \rho_j x_i$, for any $j = \overline{1, n-1}$.

(III) If $\underset{iGI}{\vee} x_i$ exists for x family $(x_i)_{i \in I}$, then $N \underset{i \in I}{\vee} x_i = \underset{i \in I}{\wedge} Nx_i$ and its dual.

(IV) If $\underset{i \in I}{\vee} x_i$ exists for a family $(x_i) i \in I$, then $x \wedge \underset{i \in I}{\vee} \quad x_i = \underset{i \in I}{\vee}(x \wedge x_i)$, for any $x \in A$ and its dual.

(V) $\rho_1 x \leq x \leq \rho_{n-1} x, \forall x \in A$

Example 2.2.3. The n – element chain $0 < \frac{1}{n-1} < \ldots < \frac{n-2}{n-1} < 1$ is a LM algebra with operations

$$N\left(\frac{j}{n-1}\right) = 1 - \frac{j}{n-1} = \frac{n-1-j}{n-1}, \quad j = \overline{0, n-1},$$

$$\rho_i\left(\frac{j}{n-1}\right) = \begin{cases} 0, i + j < n \\ 1, i + j \geq n \end{cases} \quad j = \overline{0, n-1}, i = \overline{1, n-1}$$

We note this algebra by L_n.

Example 2.2.4. Let B be a Boolean algebra. The set $B^{[n-1]} = \{(x_1 \ldots x_{n-1}) \in B^{n-1} | x_1 \leq x_2 \leq \ldots \leq x_{n-1}\}$ is a LM algebra, if we put $N(x_1, \ldots, x_n) = (\overline{x_{n-1}}, \ldots, \overline{x_1})$, and $\rho_i(x_1, \ldots x_{n-1}) = (x_i, \ldots x_i)$, for $i = \overline{1, n-1}$, $C(B^{[n-1]}) = \{(x, \ldots, x) | x \in B\} \cong B$. If A is a LM algebra, the map $\rho : A \to (CA)^{[n-1]}$ defined by $\rho(x) = (\rho_1 n, \ldots, \rho_{n-1} x), \forall x \in A$ is a monomorphism and ρ becames an isomorphism **iff** A is a Post algebra.

3 ρ_M - states on LM algebras

In this section A is a LM algebra.

Definition 3.1. [3] A function $s : A \to [0,1]$ is a <u>state</u> on A, if the following conditions hold:

(I) $s'(x \vee y) = s(x) + s(y) - s(x \wedge y), \quad x, y \in A$
(II) $s(0) = 0, s(1) = 1$
(III) $s(x) = \dfrac{s(\rho_1, x) + \ldots + s(\rho_{n-1} x)}{n-1}, \quad x \in A$

Lemma 3.2. [3] If *s* as a state on *A*, $s|C(A)$ is a state on the Boolean algebra $C(A)$ and conversely of *m* is state on the Boolean algebra $C(A)$, there exists a unique state *s* on *A* such that $s|C(A) = m$.

On *A* we have many implications (residua). For instance $x \underset{H}{\rightarrow} y = y \vee \bigwedge_{i=1}^{n-1} (N\rho_i x \vee \rho_i y)$ (Heyting implication), $x \underset{C(A)}{\rightarrow} y = \bigwedge_{i=1}^{n-1} (\rho_i x \rightarrow \rho_i y)$, (Cignoli implication) $x \underset{M}{\rightarrow} N\rho_1 x \vee y$ (Monteiro implication)/ The corresponding biresidua are

$$\rho_H(x, y) = \left(x \underset{H}{\rightarrow} y\right) \wedge \left(y \underset{H}{\rightarrow} x\right),$$
$$\rho_C(x, y) = \bigwedge_{i=1}^{n-1} S_{C(A)}(\rho_i x, \rho_i y),$$
$$\rho_M(x, y) = \left(x \underset{M}{\rightarrow} y\right) \wedge \left(y \underset{M}{\rightarrow} x\right)$$

The next definition is suggested by **Proposition 2.1.2.**

Definition 3.3. If $\rho \in \{\rho_H, \rho_C, \rho_M\}$, a function $s : A \rightarrow [0,1]$ is a ρ - *state* on *A* if it satisfies the conditions:

$$1 + s(x \wedge y) = s(x \vee y) + s(\rho(x, y)), \quad x, y \in A, s(0) = 0, s(1) = 1.$$

ρ_H - states and ρ_C – states were studied in $[4]$.

In our paper we talk about ρ_M - states.

Proposition 3.4. The following properties hold:

(I) $x \leq y \Rightarrow x \underset{M}{\rightarrow} = 1$

(II) $x \leq y \Leftrightarrow \rho_i x \underset{M}{\rightarrow} \rho_i y = 1, \quad i = \overline{1, n-1}.$

(III) $x \underset{M}{\rightarrow} \rho_1 x = 1.$

(IV) $\rho_M |C(A) = S_{C(A)}.$

(V) $S_{C(A)}(\rho_1(x), \rho_1 y) = \rho_1 \rho_M(x, y)$

Proof. (I), (II), (III) are known. See for instance [2]. For (V) we have

$$S_{C(A)}(\rho_1(x), \rho_1(y)) = (N\rho_1 X \vee \rho_1 y) \wedge (N\rho_1 y \vee \rho_1 x) =$$
$$= \rho_1(N\rho_1 X \vee y) \wedge \rho_1(N\rho_1 y \vee x) = \rho_1((N\rho_1 X \vee y) \wedge (N\rho_1 y \vee x) = \rho_1 \rho_M(x, y)$$

Note $S_{C(A)} = S$ and $\rho_M = \rho$.

Remark 3.5. If s is a ρ- state on *A*, $s|C(A)$ is a state on the Boolean algebra *C(A)*.

Proposition 3.6. If s is - ρ state on *A*, then $s(x)=s(\rho_1 x)$, for any $x \in A$.

Proof. Take $y=\rho_1(x)$, in the first condition of Definition 3.3:

$$1+s(x \wedge \rho_1 x)=s(x \vee \rho_1(x))+s(\rho(x,\rho_1 x)) \text{ so}$$
$$1+s(\rho_1 x)=s(x)+s(\rho(x,\rho_1 x))$$

by Proposition 2.2.2. (V) and

$$s(\rho(x,\rho_1 x))=s((\rho_1 x \underset{M}{\to} x) \wedge (x \underset{M}{\to} \rho_1 x))=$$
$$=s(1 \wedge 1)=s(1)=1 \text{ by Proposition 3.4., so}$$
$$s(x)=s(\rho_1 x)$$

Proposition 3.7. If s is a ρ - state on *A*, the following properties hold for any $x,y,z \in A$:

(I) $s(x \vee y)=s(x)+s(y)-s(x \wedge y)$

(II) $s(N_x)=1-s(\rho_{n-1}x), s(N\rho_1 x)=1-s|x)$

(III) $x \le y \Rightarrow s(x) \le s(y)$

(IV) $s(x \wedge Ny)=s(x)-s(x \wedge \rho_{n-1}y)$

(V) $s(x \underset{M}{\to} y)=1-s(x)+s(x \wedge y)$

(VI) $s(z)+s(x \wedge y \wedge z)=s((x \vee y) \wedge z)+s(\rho(x \wedge y) \wedge z)$

Proof.

(I) $s(x \vee y)=s(\rho_1(x \vee y))=s(\rho_1 x \vee \rho_1 y)=$
$=s(\rho_1 x)+s(\rho_1 y)-s(\rho_1 n \wedge \rho_1 y)=$
$=s(x)+s(y)-s(x \wedge y)$

(II) $s(Nx)=s(\rho_1 Nx)=s(N\rho_{n-1}x)=1-s(\rho_{n-1}x)$ so
$s(N\rho_1 x)=1-s(\rho_{n-1}\rho_1 x)=1-s(x)$

(III) $x \le y \Rightarrow \rho_1 x \le \rho_1 y \Rightarrow s(\rho_1 x) \le s(\rho_1 y) \Rightarrow s(x) \le s(y)$

(IV) $s(x \wedge Ny)=s(\rho_1 x \vee N\rho_{n-1}y)=s(\rho_1 x)-s(\rho_1 x \wedge y_{n-1}y)=$
$=s(x)-s(x \wedge \rho_{n-1}y)$

(V) $s(x \underset{M}{\to} y)=s(N\rho_1 x \vee y)=s(N\rho_1 x \vee \rho_1 y)=$
$=1-s(\rho_1 x)+s(\rho_1 y)-s(N\rho_1 x \wedge \rho_1 y)=$

$$- s(\rho_1 y) + s(\rho_1 y \wedge \rho_1 x) = 1 - s(x) + s(x \wedge y)$$

(VI) $s(z) + s(x \wedge y \wedge z) = s(\rho_1 z) + s(\rho_1 x \wedge \rho_1 y \wedge \rho_1 z) =$

$$s((\rho_1 x \vee \rho_1 y) \wedge \rho_1 z) + s(S((\rho_1 x_1 \rho_1 y) \wedge \rho_1 z)) =$$

$$= s(\rho_1 (x \vee y) \wedge \rho_1 z)) + s(\rho_1 \rho(x, y) \wedge \rho_1 z) \quad \text{so}$$

$$s(z) + s(x \wedge y \wedge z) = s((x \vee y) \wedge z) + s(\rho(x, y) \wedge z)$$

Proposition 3.7 is a L M version of Proposition 2.1.3.

Proposition 3.8. If m is a state on Boolean algebra *C(A)*, then there exists a unique ρ - state s on A such that $s|C(A) = m$.

Proof. Consider the function $s : A \to [0,1]$ defined by $s = m \circ \rho_1$. By Proposition 2.1.2 we have:

$$1 + s(x \wedge y) = 1 + m(\rho_1 x \wedge \rho_1 y) = m(\rho_1 x \vee \rho_1 y) + m(S(\rho_1 x, \rho_1 y)) =$$

$$= m(\rho_1 (x \vee y)) + m(\rho_1 (\rho(x, y))) = s(x \vee y) + s(\rho(x, y))$$

If $x \in C(A), s(x) = m(\rho_1 x) = m(x)$. The unicity of s is trivial.

Corollary 3.9.

(I) s is a ρ - state on A iff $s(x) = s(\rho_1 x)$ and $s|C(A)$ is a state on the Boolean algebra *C(A)*, where s is a function $s : A \to [0,1]$.

(II) There is a bijection between the set of ρ - states on A and the set of states on the Boolean algebra *C(A)*.

Example 3.10. Let $L_3 = \left\{0, \frac{1}{2}, 1\right\}$ be the there valued Lukasiewicz – Moisil algebra. (Example 2.2.3). The unique state, on L_3 is $s\left(\frac{1}{2}\right) = \frac{s\left(\rho_1 \frac{1}{2}\right) + s\left(\rho_2 \frac{1}{2}\right)}{2} = \frac{1}{2}$, the unique ρ_H – state on L_3 is $s_1\left(\frac{1}{2}\right) = s_1\left(\rho_2 \frac{1}{2}\right) = 1$, the unique ρ_M – state on L_3 is $s_2\left(\frac{1}{2}\right) = s_2\left(\rho_1 \frac{1}{2}\right) = 0$. They are distinct. In this algebra we have not ρ_C – states.

Example 3.11. Consider the three valued Lukasiewicz – Moisil algebra $L_2 \times L_3$. In [4] is proved that the function $s(0,0) = s\left(0, \frac{1}{2}\right) = s(0,1) = 0,\ s(1,0) = s\left(1, \frac{1}{2}\right) = s(1,1) = 1$ is the unique ρ_C – state on $L_2 \times L_3$. But $s(x) = s(\rho_1 x)$ and $s|C(A)$ is a state on the Boolean algebra *C(A)*. By Corollary 3.9 (I) s is a ρ_M – state. Analogously s is a ρ_H – state and a state. If we put $m(0,1)=\frac{1}{3}$ and $m(1,0)=\frac{2}{3}$, $m(0,0)=0, m(1,1)=1$, m is a state on *C(A)* and it defines by Proposition 3.8, Proposition 9 [4] and Proposition 3.4 [3], the $\rho_M -$ state $s_1\left(0,\frac{1}{2}\right)=0, s_1\left(1,\frac{1}{2}\right)=\frac{2}{3}$, the $\rho_H -$ state $s_2\left(0,\frac{1}{2}\right)=\frac{1}{3}, s_2\left(1,\frac{1}{2}\right)=1$, the state $s_3\left(0,\frac{1}{2}\right)=\frac{1}{6}, s_3\left(1,\frac{1}{2}\right)=\frac{5}{6}$.

In this case the four sets of „states" are distinct, but not disjoint.

Remark 3.12. If m is a state on the Boolean algebra *C(A)*, then the $\rho_M -$ state $s_1 = m\circ\rho_1$, the state $s=\frac{m\circ\rho_1+m\circ\rho_2+...+m\circ\rho_{n-1}}{n-1}$, and the $\rho_H -$ state $s_2 = m\circ\rho_{n-1}$ satisfy the inequalities: $s_1\le s\le s_2$. By the previous examples they can be distinct. But we have:

Proposition 3.13. If s is a $\rho_M -$ state on A, the following conditions are equivalent:

(I) s a state on A

(II) $s(\rho_{n-1}x)=s(x), x\in A$

(III) s is a $\rho_H -$ state on A

Proof. (I)$\Rightarrow$(II): we have $s(x)=s(\rho_1 x)$ and by Definition 3.1 (III) it follows that $(n-2)s(x)=\sum_{i=2}^{n-1}s(\rho_i x)$. But $s(x)\le s(\rho_2 x)\le s(\rho_3 x)\le ...\le s(\rho_{n-1}x)$. This implies $s(x)=s(\rho_{n-1}x)$.

(II) $\Rightarrow$(III): we must verify the condition of Definition 3.3:

$$1+s(x\wedge y)=1+s(\rho_{n-1}(x\wedge y))=1+s(\rho_{n-1}n\wedge\rho_{n-1}y)=$$
$$=s(\rho_{n-1}x\vee\rho_{n-1}y)+s(S(\rho_{n-1}x,\rho_{n-1}y))=s(\rho_{n-1}(x\vee y))+$$
$$+s(\rho_{n-1}\rho_H(x,y))=s(x\vee y)+s(\rho_H(x,y))$$

(III) $\Rightarrow$(I): $s(x)=s(\rho_1 x)=s(\rho_2 x)=...=s(\rho_{n-1}x)$ and by Proposition 8 [4], $s(x\vee y)=s(x)+s(y)-s(x\wedge y)$.

Proposition 3.14.

a) If m is a state on a Boolean algebra B, then there is a unique ρ- state s on $B^{[n-1]}$ such that $s(x,\cdots x)=m(x)$, for every $x\in B$.

b) If A is a LM algebra and m is a ρ- states on A, there is a unique ρ-state s on $C(A)^{[n-1]}$ such that $s(\rho(x))=m(x)$, for any $x\in A$.

Proof. a) The application $x \mapsto (x,...x)$ defines an isomorphism between the Boolean algebra B and $C\left(B^{[n-1]}\right)$, so $m'(x,...x) = m(x)$ is a state on the Boolean algebra $C\left(B^{[n-1]}\right)$ and we apply Proposition 3.8.
b) The application $\rho : A \to C(A)^{[n-1]}$ induces an isomorphism between centers.

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