GAMMA-RINGS: SOME INTERPRETATIONS USED IN THE STUDY OF THEIR RADICALS

Mariana DUMITRU [1]

În lucrare se studiază câteva concepte de gamma-inele şi inele ternare, inspirate din studiul matricelor dreptunghiulare peste câmpuri. Găsim unele proprietăţi ale acestora, necesare pentru definirea radicalilor acestor structuri algebrice şi obţinerea unor teoreme de structură. În final, se dezvoltă unele legături între idealele maximale ale unui gamma-inel şi idealele maximale ale inelelor asociate.

We study some concepts of gamma-rings and ternary rings, inspired by the study of rectangular matrices over fields. We point out some of their properties which are necessary for defining radicals for such algebraic structures and getting structure theorems. Some connections between maximal ideals in the gamma-rings and those in the associated rings.

Keywords: Gamma-rings; modules; radicals.
MSC 2000: 16Y99

1. Introduction

The concept of Γ-ring has a special place among generalizations of rings. This place is explained by the fact that the Γ-rings have appeared in connection with the Abelian additive groups of all linear mappings between two finite dimensional spaces over a field. As the ring of all square matrices over a field (a division ring or a ring) plays such an important role in classical ring theory, the question to improve the algebraic structure of the additive group of all rectangular matrices of the same type over a division ring is quite natural. But a binary multiplication on this set, although possible, for example, element wise, has no appropriate interpretations. In compensation, M.R. Hestenes [12] has noted that a ternary multiplication on the set $M_{m,n}(D)$ of all matrices of type $m\times n$ over the division ring D, namely $abc=a\cdot {}^{t}b\cdot c$, where ${}^{t}b$ denotes the transpose of the matrix b and "·" is the usual multiplication between two matrices, which is additive with respect to the three arguments, hence it is distributive over addition, and satisfies a kind of associativity law, $(abc)de=a(dcb)e=ab(cde)$. From this remark to the remark that one can define a ternary external multiplication of the type $A\times\Gamma\times A\to A$, where $A=M_{m,n}(D)$ and $\Gamma=M_{n,m}(D)$ and the ternary multiplication is

[1] Lecturer, Mathematics Department, University Maritime of Constanta, Romania

given by the usual multiplication of three matrices, was only a step and it was made by N. Nobusawa [36]. This multiplication is also three-distributive with respect to the addition and it satisfies a kind of associativity law $(a\alpha b)\beta c=a\alpha(b\beta c)$. Of course, one can endow Γ with a ternary multiplication $\Gamma\times A\times\Gamma\to\Gamma$ satisfying the same conditions.

Following one or another way of generalization, many authors (we give at the end of the papers an exhaustive bibliography) have developed a theory of ternary rings, many ideas of this development being borrowed from the so rich theory of rings, as it was expected.

One of the most visited ideas was that of building radicals in order to use them for studying the intimate structure of a Γ-ring. W.E. Barnes [4], J. Luh [30-32], W.G. Lister [28], W.E. Coppage and J. Luh [11], H.C. Myung [33, 34], S. Kyuno [16-20, 24, 25], A.G. Spera [47], T.S. Ravisankar and U.S. Shukla [44], R.A. Stephenson [48], L. Profera [41], G.L. Booth [6] have introduced and studied some notions of radical in ternary rings of Nobusawa, Lister, Hestenes or in triple systems. They had to define many notions as: prime ideal, nilpotency, strong nilpotency, regularity, quasi-regularity and so on.

However it is a direct way to consider some kinds of radicals in Γ-rings. One can study the same radicals in the associated rings to a Γ-ring, namely the ring of left and right operators over the Γ-ring, whose existence was observed still by Nobusawa [36, 37], and these operators rings were used to constructing of a Morita context associated to the given Γ-ring (see Nobusawa [37, 38], Parvathi and Rajendran [39], Parvathi and Adhikari [40]). It is interesting that there exists a correspondence between ideals of these operator rings and the ideals of the Γ-ring. In this paper, we study the possibility to obtain the radicals for a Γ-ring by using the radicals of the associated operator rings and also the radicals of other rings which can be defined in connection to the given Γ-ring. We can obtain in this way some information about the classes of Γ-rings having some properties with respect to radicals of a given type.

2. Definitions and notations

Definition 2.1. Let A and Γ be two Abelian additive groups. A is called a **Γ-ring**, if a ternary composition $A\times\Gamma\times A\to A$ is defined on A, $(x,\alpha,y)\to x\alpha y$, such that the following axioms are fulfilled:

$$(x+y)\alpha z=x\alpha z+y\alpha z,\quad x(\alpha+\beta)y=x\alpha y+x\beta y,\quad x\alpha(y+z)=x\alpha y+x\alpha z, \tag{1}$$

$$(x\alpha y)\beta z=x\alpha(y\beta z),\quad \text{for all } x,y,z\in A \text{ and } \alpha,\beta\in\Gamma. \tag{2}$$

We give first the necessary notions for such an algebraic system as ideals, homomorphisms, modules. In the end of the section, we shall discuss the other possible definitions for ternary rings and the connection among them.

Definition 2.2. An additive subgroup I of the Γ-ring A is called a **right** (**left**) **ideal** of A, if $x\alpha y\in I$ (respectively $y\alpha x\in I$) for all $x\in I$, $\alpha\in\Gamma$, $y\in A$. A left and right ideal of A is called an **ideal** of A.

Definition 2.3. A Γ-ring A is said to be **simple**, if $A\Gamma A\neq 0$ and the only ideals of A are 0 and A itself.

We can easily see that $M_{m,n}(D)$ is a simple $M_{n,m}(D)$-ring.

Definition 2.4. Let A be a Γ-ring, $(M, +)$ an Abelian group. If there exists a mapping

$$\varphi : M\times\Gamma\times A\to \mathrm{M},\ \ \varphi(m,\alpha,x):=m\alpha x,$$

such that the following conditions hold

$$(m+n)\alpha x=m\alpha x+n\alpha x,\ m\alpha(x+y)=m\alpha x+m\alpha y, \tag{3}$$

$$m\alpha(x\beta y)=(m\alpha x)\beta y, \tag{4}$$

for all $x,y\in A$, $m,n\in M$ and $\alpha,\beta\in\Gamma$, then M is called a **right ΓA-module**.

A Γ-ring A is obviously a right ΓA-module; D^m and $M_m(D)$ can be made into ΓA-modules in a natural way, where $\Gamma=M_{m,n}(D)$ and $A=M_{n,m}(D)$.

Let M be a right ΓA-module, A being a Γ-ring. Consider the subsets $M_1\subseteq M$, $\Gamma_1\subseteq\Gamma$, $A_1\subseteq A$. We denote by $M_1\Gamma_1A_1$ the additive subgroup generated by the elements of the form $m_1\alpha_1x_1$, with $m_1\in M_1$, $\alpha_1\in\Gamma_1$ and $x_1\in A_1$. As in the classical theory of modules over a ring, we can define the notions of submodules, submodule generated by a subset of M, quotient module with respect to a submodule, homomorphisms of ΓA-modules, direct sums of ΓA-modules.

So, if M and M' are two ΓA-modules, then a mapping $\varphi : M\to M'$ is called a **homomorphism of ΓA-modules**, if $\varphi(m+n) = \varphi(m)+\varphi(n)$ and $\varphi(x\alpha y) = \varphi(x)\alpha y$, for all $m,n\in M$, $\alpha\in\Gamma$ and $x, y\in A$.

The kernel of φ as a homomorphism of groups is also a submodule of M, as well as the image of φ, $\mathrm{Im}\varphi$, which is a submodule of M' and we have an isomorphism of modules from $\mathrm{Im}\varphi$ to $M/\ker(\varphi)$.

If A and A' are two Γ-rings, then we called a **homomorphism of Γ-rings** a homomorphism of additive groups $\varphi : A \to A'$ which satisfies the condition $\varphi(x\alpha y) = \varphi(x)\alpha\, \varphi(y)$, for all $x, y\in A$. Its kernel is an ideal of A, because for all $x\in \ker(\varphi)$, $\alpha\in\Gamma$ and $a\in A$, we have $\varphi(x\alpha a) = \varphi(x)\alpha\, \varphi(a) = 0\alpha\, \varphi(a) = 0$ and similarly $\varphi(a\alpha x) = 0$, therefore $x\alpha a$ and $a\alpha x$ belong to $\ker(\varphi)$.

Define the annihilator of a right ΓA-module M by

$$Ann_{\Gamma A}(M)=\{a\in A \mid M\Gamma a=0\};$$

it is easily seen that $Ann_A(M)$ is an ideal of the Γ-ring A. If we denote by $Ann_{\Gamma A}(m)$ the set of all elements $a\in A$ such that $m\Gamma a=0$, we show that this is a right ideal of A and $Ann_{\Gamma A}(M) = \bigcap_{m\in M} Ann_{\Gamma A}(m)$.

Denote $(A: I) = \{ a\in A \mid A\Gamma a\subseteq I\}$, for I a right ideal of the Γ-ring A. We verify by direct calculations that $(A: I)$ is a right ideal of A. Indeed, if $a,b\in(A: I)$, then for all $\alpha\in\Gamma$ and $x\in A$, we have: $a+b$, $x\alpha a\in (A: I)$, because of the equalities $y\alpha(a+b) = y\alpha a+y\alpha b\in I$ and $y\beta(x\alpha a)=(y\alpha x)\alpha a\in I$, for all $y\in A$ and $\beta\in\Gamma$.

The isomorphism theorems hold for ΓA-modules, but we prove only the following

Proposition 2.1. *Let N be a submodule of M, which is a ΓA-module, where A is a Γ-ring. Then there exists a one-to-one correspondence between the submodules of M containing N and the submodules of M/N.*

Proof. If T' is a submodule of the right ΓA-module M/N, then $T=\{m\in M \mid m+N\in T'\}$ is submodule of M, containing N, since for any $n\in M$, $n+N=M=\overline{0}\in T'$.

Moreover, if T is a submodule of M such that $N\subseteq T$, then $T' =\{t+N \mid t\in T\}$ is a submodule in M/N. This is the asked bijection.

Definition 2.5. A right ΓA-module M is called **irreducible** or **simple**, if $M\Gamma A\neq 0$ and it has only the improper submodules 0 and M.

Proposition 2.2. *For a right ΓA-module M, the following statements are equivalent:*

(*i*) *M is irreducible.*

(*ii*) *For each* $m\in M$, $m\neq 0$, $m\Gamma A=M$.

(*iii*) *For each* $m\in M$, $m\neq 0$, *there exists* $\alpha\in\Gamma$, $m\alpha A=M$.

Proof. $(i) \Rightarrow (ii)$: The set $\{m\in M \mid m\Gamma A=0\}$ is a submodule of M, as we can verify. If this is not 0, then $m\Gamma A=0$, for all $m\in M$, because it coincides with M. But this is not possible, hence the only $m\in M$ for which $m\Gamma A=0$ is $m=0$. Now for $m\in M$, $m\neq 0$, $m\Gamma A$ is a submodule of M which is not 0, hence it is M.

$(i) \Rightarrow (iii)$: Take $m\in M$, $m\neq 0$. Then $m\Gamma A=M$, hence there exists $\alpha\in\Gamma$ such that $m\alpha A\neq 0$. Moreover, for all $\alpha\in\Gamma$, $m\alpha A$ is a submodule of M, the image in M of the homomorphism of modules $\varphi_{m,\alpha}: A\to M$, $\varphi_{m,\alpha}(x)= m\alpha x$. As this is not 0 and M is an irreducible module, we have $m\alpha A=M$.

(iii) or $(ii) \Rightarrow (i)$: For any submodule $S\neq 0$ of M, taking an $s\in S$, $s\neq 0$, there exists $\alpha\in\Gamma$, such that $s\alpha A=M$, but $s\alpha A\subseteq S$, hence $S=M$.

Proposition 2.3. *The irreducible right ΓA-modules are exactly those right ΓA-modules for which there exists a maximal right ideal I of A such that M is isomorphic to A/I as right ΓA-modules.*

Proof. If I is a maximal right ideal of A, then, giving the correspondence between the modules of A and A/I as right A-modules, we obtain that A/I does not have any right submodules different from 0 and A/I, if such a submodule exists, then it has a corresponding ideal J in A such that $I\subseteq J\subseteq A$ and $I\neq J$, $J\neq A$, as it is not possible. Conversely, if M is an irreducible module over A, taking an $m\in M\backslash\{0\}$, and the corresponding $\alpha\in\Gamma$ from Proposition 1.2, (iii), we obtain a homomorphism of right ΓA-modules, $\varphi_{m,\alpha}: A\to M$, defined by $\varphi_{m,\alpha}(x)= m\alpha x$, for all $x\in A$.

This is surjective, since $\varphi_{m,\alpha}(\mathrm{A})= m\alpha A=M$, hence M is isomorphic to $A/\ker(\varphi_{m,\alpha})$ and $I=\ker(\varphi_{m,\alpha})$ is a right ideal of A (it is a submodule of the right ΓA-module A), which is maximal, since A/I is an irreducible right ΓA-module, by the first part of the proof.

If I is a maximal right ideal of the Γ-ring A, then A/I is an irreducible right ΓA-module. By applying Proposition 2.2, (iii), we obtain that for all $e+I$, $e\notin I$, hence $e+I\neq\bar{0}$, there exists $\alpha\in\Gamma$, such that $\bar{e}\,\alpha A=A/I$, where $\bar{e}= e+I$. But this means the equality $e\alpha A+I=A$. Therefore, for each $x\in A$, $x\text{-}e\alpha x\in I$.

We have proved the following statement:

Proposition 2.4. *If $I\neq A$ is a maximal right ideal of A, then there exists an element $e\in A$, such that, for some $\alpha\in\Gamma$, the elements of the form $x\text{-}e\alpha x\in I$, for all $x\in A$.*

This property gives the possibility to define the regularity in Γ-rings.

Definition 2.6. A right ideal of A is called **regular**, if there exist $e\in A$ and $\alpha\in\Gamma$, such that, for all $x\in A$, $x\text{-}e\alpha x\in I$.

The maximal right ideals of A are regular.

Definition 2.7. A right ΓA-module is called **faithful**, if its annihilator is the zero ideal of A. That means $m\Gamma A=0$ implies $m=0$, for $m\in M$, M a faithful ΓA-module.

Definition 2.8. Let A be a Γ-ring. If there exists a right ΓA-module M which is faithful and irreducible, then A is called **primitive**. An ideal I of A is called **primitive**, if the Γ-ring A/I is primitive.

If M is an irreducible right A-module, then $A/Ann_{\Gamma A}(M)$ is a primitive Γ-ring. (Ravisankar and Shukla [44], Proposition 1.4.)

Now we have the great part of notions which we need to define some radicals and to consider some of their properties.

We discuss about the definitions of ternary rings, in order to see the generality of our considerations here.

Definition 2.9. A Γ-ring is called a **Γ-ring in the sense of Nobusawa** (after Barnes [3]) , if Γ is also structured as an A-ring by a ternary composition:

$$\cdot : \Gamma\times A\times\Gamma\to\Gamma,\ (\alpha,a,\beta)\to\alpha a\beta,$$

such that the following two additional conditions hold:

$$(a\alpha b)\beta c=a(\alpha b\beta)c, \text{ for all } a,b,c\in A \text{ and } \alpha,\beta\in\Gamma; \qquad (5)$$

$$x\alpha y=0, \text{ for all } x,y\in A, \text{ implies } \alpha=0. \qquad (6)$$

We can note here the idea taken from the example formed by the set of all rectangular matrices of the same type over a field.

If we give up to the condition (6) which is quite restrictive, we obtain the **weak Γ_N-ring** of Ravisankar and Shukla [44]. (The notation of Γ_N-ring was used in the same paper for Γ-rings in the sense of Nobusawa).

Definition 2.10. An **associative triple system T of the first (second) type** is an Abelian group $(T,+)$ endowed with a trilinear composition:

$$\langle\cdot,\cdot,\cdot\rangle : T\times T\times T\to T,$$

such that it satisfies the conditions in the definition of a weak Γ_N-ring where $\Gamma=T$ (resp. the conditions (1),(2) and

$$<<a,b,c>,d,r> = <a,<d,c,b>,r>, \text{ for all } a,b,c,d,r\in T\text{)}. \qquad (5')$$

The associative triple systems of second type are **Hestenes ternary rings**, while those of first type are **Lister ternary rings**.

We have made some considerations on right ΓA-modules, just because we need them for our definition of Jacobson radical and for the translation of some properties of radicals from the same radicals considered in the rings of operators or other rings associated to Γ-rings to the radicals in Γ-rings.

3. Operator rings associated to a Γ-ring

Let A be a Γ-ring. Consider the mapping:

$$\psi : \Gamma\times A\to \mathrm{End}(A),\ \ \psi((\alpha,a)) = [\alpha,a],$$

where $[\alpha,a] : A\to A$, defined by $[\alpha,a](x)=x\alpha a$, for all $x\in A$, is an endomorphism of the additive group $(A,+)$, as one can easily verify. This mapping is additive in both arguments, hence there exists a homomorphism of groups $\theta : \Gamma \otimes_Z A\to End(A)$, such that $\theta\left(\sum_{i=1}^{n}(\alpha_i\otimes a_i)\right)=\sum_{i=1}^{n}[\alpha_i,a_i]$. The image of θ, denoted by R, can be endowed with a structure of associative ring by defining the binary multiplication:

$$\sum_i[\alpha_i,a_i]\cdot\sum_j\left[\beta_j,b_j\right]=\sum_{i,j}\left[\alpha_i,a_i\beta_j b_j\right].$$

This is the **right operator ring** associated to the Γ-ring A. Similarly, we can introduce **the left operator ring** L.

If the operator rings associated to the Γ-ring A have identities, then we say the **Γ-ring *A* has left and right unities**. This case, which has many similarities to rings, was very well analyzed by Kyuno [19], Nobusawa [36] and Parvathi and Rajendran [37].

The Γ-ring A can be considered as an L-R-bimodule, by defining the scalar multiplication as follows:

$$x\cdot\left(\sum_i[\alpha_i,a_i]\right)=\sum_i x\alpha_i a_i\ , \text{ for all } x\in A \text{ and } \sum_i[\alpha_i,a_i]\in R,$$

$$\left(\sum_i [a_i,\alpha_i]\right)\cdot x=\sum_i a_i\alpha_i x\,,\ \text{ for all } x\in A \text{ and } \sum_i [a_i,\alpha_i]\in L\,.$$

If L and R have identities, then we have

Proposition 3.1. (Parvathi and Rajendran [39], Theorem 1.1.). *Let A be a Γ-ring with left and right unities. Then:*
(*i*) ${}_LA_R$ *is a faithfully balanced bimodule.*
(*ii*) ${}_LA$ *and* A_R *are progenerators.*
(*iii*) *L and R are Morita equivalent rings.*

(For definitions in theory of rings and modules that we use in the paper, see Anderson and Fuller [1].)

In the general case, when L and R do not have identities, there exists also a correspondence between the sets of ideals of L or R and the set of ideals (on the right hand or on the left hand) in A.

Proposition 3.2. *Let R and L be Morita equivalent rings. Then there exists a one-to-one correspondence between preradicals for the category of R-modules and those for the category of L-modules, preserving the properties of preradicals like inclusion, intersections, sums of preradicals.*

Proof. Let F: R-mod→L-mod and G: L-mod→R-mod be the factors giving the Morita equivalence and $f: FG\rightarrow \mathbf{1}_{L\text{-mod}}$ and $g: GF\rightarrow \mathbf{1}_{R\text{-mod}}$ be the corresponding natural isomorphism. If r and s are preradicals for R-mod and L-mod respectively, taking an R-module M and an L-module N, we define the correspondences:

$${}_Lr(R) = f(F(r(G(N)))) \quad \text{and} \quad {}_Rs(M) = g(G(s(F(M)))).$$

The properties of Morita equivalence (see [1]) give immediately that ${}_Lr$ and ${}_Rs$ are preradicals in L-mod and R-mod respectively. We have $G{}_Lr=rG$ and $F{}_Rs=sF$, hence ${}_R({}_Lr)=r$ and ${}_L({}_Rs)$=s. The correspondences $r\rightarrow {}_Lr$ and $s\rightarrow {}_Rs$ preserve the inclusion. We show only that these correspondences preserve also the intersection of a family of preradicals:

$${}_L\left(\bigcap_i r_i\right)(N)=fF\left(\bigcap_i r_i\right)G(N)\subseteq\bigcap_i fFr_iG(N)=\left(\bigcap_i {}_Lr_i\right)(N)$$

and

$${}_R\left(\bigcap_i {}_Lr_i\right)\subseteq\bigcap_i {}_R({}_Lr_i)=\bigcap_i r_i\,,\quad \text{hence}\quad \bigcap_i {}_Lr_i={}_L\left({}_R\left(\bigcap_i {}_Lr_i\right)\right)\subseteq{}_L\left(\bigcap_i r_i\right).$$

The proof for preserving the sums of preradicals is analogous.

As an obvious consequence of this proposition we have

Corollary 3.1. *If R and L are Morita equivalent rings, then there exists a one-to-one correspondence between their ideals which define a radical of a given type.*

Hence the set of all prime ideals of R and the set of all prime ideals of L correspond one to another by a bijection, which give the correspondence between their prime radicals. The same statement holds for maximal ideals therefore for Jacobson radicals of the two rings.

Now we establish a correspondence between the set of left ideals of a Γ-ring A and the set of left ideals of the right operator ring R associated to A.

Proposition 3.3. (*i*) *If I is a left ideal of R, then* $I^*=\{a\in A \mid [\Gamma,a]\subseteq I\}$ *is a left ideal of A.*

(*ii*) *Let F be a family of left ideals of R. Then* $\bigcap_{I\in F} I^* = \left(\bigcap_{I\in F} I\right)^*$.

(*iii*) *If J is an ideal of A, then* $J_* = \left\{\sum_i [\alpha_i, a_i] \in R \mid A\sum_i [\alpha_i, a_i] \subseteq J\right\}$ *is an ideal of R (both ideals are left ideals).*

Proof. Indeed, for arbitrary $a,b\in I^*$ and $x\in A$, $\alpha\in\Gamma$, we have a-b, $x\alpha a\in I^*$, because of the implications: $[\Gamma, a], [\Gamma, b]\subseteq I \Rightarrow [\Gamma, a] - [\Gamma, b]\subseteq I$ and $[\Gamma, x\alpha a]$ is generated by the elements of the form $[\beta,x\alpha a]$ which are equal to $[\beta,x]\cdot[\alpha,a]$ which belong to I, as I is a left ideal of R. The other two statements can be proved in a similar way.

Proposition 3.4. *There exists a one-to-one correspondence between the set of all right R-modules and the set of right* Γ*A-modules.*

Proof. Indeed, if M is a right R-module, then M can be endowed with a structure of right ΓA-module by defining ternary composition $m\alpha x=m[\alpha,x]$. Conversely, if M is a right ΓA-module, taking the scalar multiplication on M over R

$$m\left(\sum_i [\alpha_i, a_i]\right) = \sum_i m\alpha_i a_i ,$$

we obtain a structure of right R-module on M.

Moreover we have a "transport" of the primitivity from R to A.

Proposition 3.5. A Γ*-ring A is primitive if and only if* $A\Gamma x=0$ *implies* $x=0$ *and the right operator R is a primitive ring.*

Proof is based on the correspondence between irreducible modules in the two rings and we do not insist on it.

We end this section by noting that the operator rings have a real significance in the context of one of the most important examples of Γ-rings. Let G and G' be two Abelian additive groups, $A\subseteq Hom(G,G')$ and $\Gamma\subseteq Hom(G',G)$ be two subgroups, such that $A\Gamma A\subseteq A$ and $\Gamma A\Gamma\subseteq\Gamma$, where the ternary operations come from the binary compositions of mappings.

We obtain a Γ_N-ring (weak) A, which is sometimes called a **Γ-ring of homomorphisms between** G **and** G'. For this Γ-ring, R contains all products $a_i\alpha_i$, $\alpha_i\in\Gamma$, $a_i\in A$, hence R is a subring of $End(G)$, while L is a subring of $End(G')$. They can have identities or do not have.

Using an idea suggested by this example, we can obtain a "generalized" Morita equivalence in the cases without identities in L and R, replacing this condition by the conditions $R^2 = R$ and $L^2 = L$. We have analyzed this case in another paper.

4. Prime and Jacobson radicals of a Γ-ring

First we recall some definitions and we get some properties.

Definition 4.1. (Barnes [2]) An ideal I of a Γ-ring A, $I\neq A$, is called **prime**, if, for any two ideals J and K of A, $J\Gamma K\subseteq I$ implies $J\subseteq I$ or $K\subseteq I$. The **prime radical of** A is then

$$P(A)=\bigcap\{P \mid P \text{ is a prime ideal of } A\}.$$

Proposition 4.1. *There is a bijection between the set of all prime ideals of the* Γ*-ring A and the set of all prime ideals of R (or L).*

Proof. We show easily that the bijection in Proposition 3.4. has a restriction to the sets in the statement of the above proposition.

It is obvious the following statement:

Corollary 4.1. *With the notations of Proposition 2.4., we have*

$$(P(R))^{*}=P(A),$$

where $P(R)$ *is the prime radical of the ring* R *.*

Definition 4.2. The **Jacobson radical** $J(A)$ of the Γ-ring A is the ideal equal to $\bigcap\{Ann_{\Gamma A}(M) \mid M$ is an irreducible right ΓA-module$\}$, if there exist irreducible right <-> ΓA-modules, and $J(A)=A$, if such right ΓA-modules do not exist.

The relation of this definition to the definitions given by Coppage and Luh [11] and Ravisankar and Shukla [44] is discussed in another place. We note here another property of $J(A)$ which is based on the property of irreducible ΓA-modules in Proposition 3.3.

Proposition 4.2. *For the Jacobson radical of a* Γ*-ring A, the following equality holds*

$$J(A)=\bigcap\{(A:I) \mid I \text{ is a maximal right ideal of } A\}.$$

Now we consider a new possibility of associating some rings to a Γ-ring A.

Proposition 4.3. *Let A be a* Γ*-ring and* $\gamma\in\Gamma$. *Then A can be endowed with a structure of associative ring by the multiplication* $x \cdot y = x\gamma y$, *for all* $x,y\in A$.

We denote this ring by A_γ. Each right ideal of A is also a right ideal of A and each right ΓA -module M can be considered as a right A_γ -module, by the following multiplication $m \cdot a = m\gamma a$, for all $m\in M$ and $a\in A$.

We omit the proof, since it is straightforward.

Proposition 4.4. *Every maximal right ideal of A is included into a maximal right ideal of* A_γ. *Moreover, denoting by* $J(A_\gamma)$ *the Jacobson radical of* A_γ, *we have the equality:*

$$J(A)=\bigcap_{\gamma\in\Gamma} J(A_\gamma).$$

Proof. Using Proposition 2.2., we can see that each irreducible right ΓA-module M becomes an irreducible right A_γ -module, by taking $\alpha=\gamma$, hence $J(A)\subseteq J(A_\gamma)$, for all $\gamma\in\Gamma$, therefore $J(A)\subseteq\bigcap_{\gamma\in\Gamma} J(A_\gamma)$.

For the converse inclusion, let us remark that:

$\bigcap_{\gamma\in\Gamma} J(A_\gamma) = \bigcap_{\gamma\in\Gamma}(\bigcap\{\ I_\gamma \mid I_\gamma$ is a maximal right ideal of $A\}) \subseteq \bigcap\{I \mid I$ is a maximal right ideal of $A\}$, just because some of I_γ are greater then the maximal right ideals of A which are regular, that is $i\gamma a \in I_\gamma$, for all $i \in I_\gamma$ and $a\in A$, but $i\Gamma a$ is not always included into I_γ.

We cannot deduce so directly a relationship between $P(A)$ and $J(A)$ but there are some possibilities to connect them. We intend to study it in a future paper, as well as definitions of other radicals of gamma-rings.

5. Conclusions

In this paper, we analysed some possibilities of defining ternary rings, obtaining the old definitions of N. Nobusawa, W.E. Barnes and W.G. Lister, M.R. Hestenes.

The necessity of studying gamma-rings appeared when we want to define ternary operations on groups of homomorphisms between two Abelian groups.

By establishing correspondences between ideals of the operator rings associated to a Γ-ring and the ideals of the considered Γ-ring, we got Morita equivalences for Γ-rings and we studied the radicals in Γ-rings.

REFERENCES

[1] *F.W. Anderson and K.R. Fuller*, Rings and Categories of Modules, Springer-Verlag, Berlin, 1973.
[2] *M. Ashci*, The commutativity in prime gamma-rings, Intern. Math. Forum, **2** (2007), No.3, pp.103-108.
[3] *L. Bader and A. Franchetta*, Anelli ternari di Hestenes quasiprimitivi, Rend. Matem. (7) **1** (1981), pp. 411-430.
[4] *W.E. Barnes*, On the Γ-rings of Nobusawa, Pacific J. Math., **18** (1966), pp. 411-422.
[5] *F. Bartolozzi and G. Panella*, Anelli ternari di Hestenes semplici, artiniani e privi di ideali bilateri effetivi, Ricerche Mat., **26** (1977), pp. 255-275.
[6] *G.L. Booth*, A Brown-McCoy radical for Γ-rings, preprint.
[7] *S.G. Botha, A. Buys*, Prime ideals in categories, Comm. in Algebra, **13** (1985), pp. 1171-1186.
[8] *A. Buys and G.K. Gerber*, Prime and k-prime ideals in Ω-groups, preprint.
[9] *A. Caggegi*, Anelli ternari di Hestenes con ideale destro minimale, Anali di Matem., (1981), pp. 169-191.
[10] *A. Caggegi*, Anelli ternari di Hestenes semplici, artiniani a destra e privi di ideali bilateri effetivi, Rend. Mat., Ser 8, **13** (1980), pp. 85-104.
[11] *W.E. Coppage and J. Luh*, Radicals of gamma rings, J. Math. Soc. Japan, **23** (1971), pp. 40-52.
[12] *M.R. Hestenes*, A ternary algebra with applications to matrices and linear transformations, Arch. Rational Mech. Anal, **11** (1962), pp. 138-194.
[13] *M.R. Hestenes*, On a ternary algebra, Scripta Math., **29** (1973), pp. 253-272.

[14] *D.F. Hsu*, On prime ideals and primary decompositions in gamma-rings, Math. Japonica, **21** (1976), pp. 455-460.
[15] *H. Kandamar*, The k-derivation of a gamma-ring, Turk. J. Math., **24** (2000), pp. 221-231.
[16] *S. Kuyno*, On the radicals of Γ-rings, Osaka J. Math., **12** (1975), pp. 639-645.
[17] *S. Kuyno*, On the semisimple gamma-rings, Tohoku Math. J., **29** (1977), pp. 217-225.
[18] *S. Kuyno*, On prime gamma-rings, Pacific J. Math., **75** (1978), pp. 185-190.
[19] *S. Kuyno*, A gamma-ring with the right and left unities, Math. Japonica, **24** (1979), pp. 191-193.
[20] *S. Kuyno*, Coincidence of the right Jacobson radical and the left Jacobson radical in a gamma-ring, Tsukuba J.Math., **3** (1979), pp. 31-35.
[21] *S. Kuyno*, Nobusawa's gamma-rings rings with the right and left unities, Math. Japonica, **25** (1980), pp. 179-180.
[22] *S. Kuyno*, A gamma-ring with minimum conditions, Tsukuba J. Math., **5** (1981), pp. 47-65.
[23] *S. Kuyno*, On a Nobusawa's gamma-ring $\begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix}$, Math. Japonica, **26** (1981), pp. 55-64.
[24] *S. Kuyno*, Notes on Jacobson radicals of gamma-rings, Math. Japonica, **27** (1982), pp. 107-111.
[25] *S. Kuyno*, Prime ideals in gamma-rings, Pacific J. Math., **98** (1982), pp. 375-379.
[26] *S. Kuyno*, Subdirect sums of Nobusawa gamma-rings, Math. Japonica, **28** (1983), pp. 31-36.
[27] *L.C.A. Leeuwen and R. Wiegandt*, Radicals, semisimple classes and torsion theories, Acta Math. Acad. Sci. Hungar., **36** (1980), pp. 37-47.
[28] *W.G. Lister*, Ternary rings, Trans. Amer. Math. Soc., **154** (1971), pp. 37-55.
[29] *O. Loos*, Assoziative Triplesysteme, Manuscripta Math., **7** (1972), pp. 103-112.
[30] *J. Luh*, On the theory of simple Γ-rings, Michigan Math. J., **16** (1969), pp. 65-75.
[31] *J. Luh*, On primitive Γ-rings with minimal one-sided ideals, Osaka J. Math., **5** (1968), pp. 165-173.
[32] *J. Luh*, The structure of primitive gamma-rings, Osaka J. Math., **7** (1970), pp. 267-274.
[33] *H.C. Myung*, Note on the Jacobson radical in associative triple systems of second type, J. Korean Math. Soc., **12** (1975), pp. 29-35.
[34] *H.C. Myung*, A characterization of the Jacobson radical in ternary algebras, Proc. Amer. Math. Soc., **38** (1973), pp. 228-234.
[35] *N. Nobusawa*, On a generalization of the ring theory, Osaka J. Math., **1** (1964), pp. 81-89.
[36] *N. Nobusawa*, Structure theorems of a module over a ring with bilinear mapping, Canad. Math. Bull., **10** (1967), pp. 649-652.
[37] *N. Nobusawa*, On duality in Γ-rings, Math. J. Okayama Univ., **25** (1983), pp. 69-73.
[38] *N. Nobusawa*, Γ-rings and Morita equivalence of rings, Math. J. Okayama Univ., **26** (1984), pp. 151-156.
[39] *M. Parvathi and P.A. Rajendran*, Gamma-rings and Morita equivalence, Comm. in Algebra, **12** (14)(1984), pp. 1781-1786.
[40] *M. Parvathi and P. R. Adhikari*, Γ-rings and Morita equivalence for rings with local units, Comm. in Algebra, **13** (1985), pp. 585-598.
[41] *L. Profera*, Anelli ternari di Hestenes semplici e artiniani, Atti Accad. Naz. Lincei Rend., Ser. 8, **62** (1977), pp. 292-300.
[42] *L. Profera*, Sullo zoccolo di un anello ternario di Lister, Boll.U.M.I., Ser. VI, **1** (1982), pp. 59-67.
[43] *L. Profera*, Anelli ternari di Lister semisemplici, Note di Matem., **2** (1982), pp. 1-56.
[44] *T.S. Ravisankar and U.S. Sukla*, Structure of Γ-rings, Pacific J. Math., **80** (1979), pp. 539-559.

[45] *M.F. Smiley*, An introduction to Hestenes rings, Amer. Math. Monthly, **76** (1969), pp. 245-248.

[46] *A.J.M. Snyders and S. Veldsman*, Ideal extensions of Γ-rings, J. Austral. Math. Soc., Ser. A, **54** (1993), pp. 368-392.

[47] *A.G. Spera*, Radicale di un anello di Hestenes, Atti Accad. Sci. Lett. Arti Palermo, I, Ser.IV, **35** (1975-76), pp. 283-296.

[48] *A.G. Spera*, Sugli anelli ternari di Hestenes, Rend. Circ. Mat. Palermo, Ser.II, **27** (1978), pp. 289-304.

[49] *R.A. Stephenson*, Jacobson structure theory for Hestenes ternary rings, Trans. Amer. Math. Soc., **177** (1973), pp. 91-98.

[50] *Z.K. Warsi*, On decomposition of primary ideals of Γ-rings, Indian J. Pure Appl. Math., **9** (1978), pp. 912-917.