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SOME APPLICATIONS OF THE HILBERT TRANSFORM

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În acest articol, se dau unele proprietăţi ale transformării Hilbert, cu interpretări în teoria matematică a semnalelor aleatoare. Este studiată reprezentarea semnalelor analitice ca o generalizare a formei complexe a semnalelor sinusoidale, un rol special fiind acordat proprietăţilor spectrale, legate de conceptul de densitate de putere. Este de asemenea analizat un exemplu practic din domeniul circuitelor electronice.

In this paper we recall some properties of the Hilbert transform interpreted in terms of the mathematical theory of the random signals. We study the representation of the analitical signals as a generalization of the complex form of the sinusoidal signals, a special part being focused on the spectral properties, in connection with the concept of power density. It is also presented an example taken from practice of the electronic circuits.

Keywords: Fourier transform, Hilbert transform, analytical representation.

1. Introduction

By making use of the Hilbert transform, one can describe some constructions used in the signal theory in mathematical rigorous terms. In this paper, one recalls the main facts regarding the Hilbert transform and one analyzes some concepts like that of analytic associated signal, convolution filters, spectral densities of stationary random signal.

2. On the Hilbert transform

Let $h : \mathbb{R} \longrightarrow \mathbb{C}$ be a function from $L^2 = L^2(\mathbb{R})$. For any $f \in L^1 \cap L^2$, one can consider the convolution $H_f = h * f$, which belongs to L^2. If the Fourier image $\hat{h} = \mathcal{F}h$ is bounded, then the linear operator $H: L^2 \to L^2$ is well defined and bounded; indeed, if $f \in L^1 \cap L^2$, then $\widehat{H_f} = \hat{h}.\hat{f}$ and if $\|\hat{h}\| \leq M$ ($M > 0$ fixed), then by the Parseval relation, $\|H_f\| \leq M.\|f\|$ and since $L^1 \cap L^2$ is dense in L^2, this inequality extends to L^2.

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Take now $h(t) = \frac{1}{\pi t}\,(t \neq 0)$. This function is not integrable on $\mathbb{R}$; instead, its primitive $\frac{1}{\pi} ln|t|$ is and one can take its associated distribution and then, the distribution $VP\frac{1}{t}$, defined by

$$VP\frac{1}{t}(\varphi) = p.v.\int_{-\infty}^{\infty} \frac{\varphi(t)}{t} dt \quad \text{("principal value")}$$

for any test-function φ. One knows that $t.VP\frac{1}{t} = 1$, so h can be identified with the distribution $\frac{1}{\pi}VP\frac{1}{t}$ [2]. For any function $s: \mathbb{R} \to \mathbb{C}$ from L^2, one defines its Hilbert image

$$H_s = \frac{1}{\pi}VP\frac{1}{t} * s. \tag{1}$$

Explicitly, for any $t \in \mathbb{R}$,

$$H_s(t) = p.v.\int_{-\infty}^{\infty} h(t-u)s(u)du = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{s(u)}{t-u} du.$$

Since $\mathcal{F}\left\{VP\frac{1}{t}\right\} = -i\pi.sgn\omega$, from (1) one gets:

$$\widehat{H_s}\,(\omega) = -i.\hat{s}(\omega).sgn\omega. \tag{2}$$

Because $\hat{s} \in L^2$, it follows that H_s is a function which belongs to L^2.

PROPOSITION 1. Suppose that $s: \mathbb{R} \to \mathbb{R}$ belongs to L^2 and $s(t) = 0$ for any $t < 0$. If $\hat{s}(\omega) = A(\omega) + iB(\omega)$, $\omega \in \mathbb{R}$, then $A(\omega)$ is even and $B(\omega)$ is odd; moreover,

$$A = \frac{1}{\pi}VP\frac{1}{\omega} * B \;\; and \;\; B = -\frac{1}{\pi}VP\frac{1}{\omega} * A. \tag{3}$$

Proof. Indeed, since $Supp\, s \subset [0,\infty)$, then $s(t) = s(t) * sgnt$, for any $t \in \mathbb{R}$; hence $\hat{s}(\omega) = (\hat{s} * \widehat{sgn})(\omega)$. But $\widehat{sgn}(\omega) = \frac{1}{i\pi}VP\frac{1}{\omega}$, therefore $\hat{s}(\omega) = \hat{s}(\omega) * \frac{1}{i\pi}VP\frac{1}{\omega}$. Thus, $A(\omega) + iB(\omega) = \frac{1}{i\pi}VP\frac{1}{\omega} * \big(A(\omega) + iB(\omega)\big)$ and it remains to separate the real and imaginary parts.

One knows that the Hilbert operator

$$H: L^2 \to L^2, \qquad s \mapsto \frac{1}{\pi}VP\frac{1}{t} * s = H_s$$

is linear and continuous. Moreover, for any $s \in L^2$, $H(H_s) = -s$, that is $H^2 = -id$. Particularly, H is bijective and $H^{-1} = -H$. From the relation (2), one obtains another definition of the Hilbert transform: $H_s = -i\mathcal{F}^{-1}(\hat{s}.sgn)$. One also knows the following table:

$s(t)$	$\sin t$	$\cos t$	$\delta(t)$
$H_s(t)$	$-\cos t$	$\sin\ t$	$\frac{1}{\pi t}$

According to prop. 1, one gets: $A = H_B$ and $B = -H_A$. Such a pair of Hilbert images can also be obtained in another way. Namely, let $f: \mathbb{R} \to \mathbb{R}$ be a function from $L^1 \cap L^2$ and put

$$a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt\,,\quad b(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt.$$

By the Fourier representation formula,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(u)\cos\omega(t-u)\,du$$

$$= \int_{0}^{\infty} (a(\omega)\cos\omega t + b(\omega)\sin\omega t)d\omega,$$

for any real t. Consider now the conjugate-function g, defined by

$$g(t) = \int_{0}^{\infty} (-a(\omega)\sin\omega t + b(\omega)\cos\omega t)d\omega.$$

Then

$g(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} f(u)\sin\omega(u-t)\,du =$
$\frac{1}{\pi}\lim_{A\to\infty}\int_{0}^{A} d\omega \int_{-\infty}^{\infty} f(u)\sin\omega(u-t)\,du =$
$\frac{1}{\pi}\lim_{A\to\infty}\int_{-\infty}^{\infty} f(u)\frac{1-\cos A(u-t)}{u-t}du = \frac{1}{\pi}\lim_{A\to\infty}\int_{-\infty}^{\infty}\frac{1-\cos Ax}{x}f(t+x)dx =$
$\lim_{A\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-\cos Ax}{x}[f(t+x) - f(t-x)]dx = \frac{1}{\pi}\int_{0}^{\infty}\frac{f(t+x)-f(t-x)}{x}dx =$
$\frac{1}{\pi}p.v.\int_{-\infty}^{\infty}\frac{f(x)}{x-t}dx$. Thus, $g = -H_f$ and similarly, $f = H_g$.

NOTE. One can also define the discrete Hilbert transform. Namely, consider the sequence $h = (h_n)$, $n \in \mathbb{Z}$, defined by $h_n = \frac{2}{\pi n}$ if n is odd and $h_n = 0$ for n even. For any sequence $s = (s_n)$, $n \in \mathbb{Z}$, such that $s_n = 0$ for $n < 0$ and $\sum_n |s_n| < \infty$, its discrete Hilbert image is $H_s = (z_n)$, where $z_n = \sum_p h_{n-p}\, s_p$ ($H_s = h * s$, in analogy with (1)). If $s_n = p_n + iq_n$ and put $p = (p_n)$, $q = (q_n)$, $q = H_p$ and $p = -H_q$ and it founds again (3); [5]

3. . Some representations of the signals

It is useful to introduce some terminology from Signal Processing.

Recall that any function $s: \mathbb{R} \to \mathbb{C}$ from L^2 is also called a signal with finite energy; its norm $\|s\|_2 = \left(\int_{-\infty}^{\infty} |s(t)|^2\right)^{\frac{1}{2}}$ is the energy of s. Any linear continuous operator $T: L^2 \to L^2$ is also called a continual filter. If there is a function $p \in L^1 \cap L^2$ (called weight) such that $Tx = p * x$ for any x, then one says that T is a convolution filter, with the transfer function $\hat{p}(\omega) = \mathcal{F}\{p(t)\}$.

By definition, if $s: \mathbb{R} \to \mathbb{R}$ belongs to L^2, one can consider a new signal $\dot{s} \in L^2$, defined by

$$\dot{s}(t) = s(t) + iH_s(t),\ for\ t \in \mathbb{R}, \tag{4}$$

called the analytic signal associated to s [1], [4]. Mathematically, this terminology is justified by the fact that under some conditions, $\dot{s}$ is the restriction of a complex holomorphic (≡ analytic) function.

Example. For $s(t) = \cos t$, one gets $\dot{s}(t) = \cos t + i \sin t$ and for $s(t) = \sin t$, $\dot{s}(t) = \sin t - i \cos t$. By extension, one defines the distribution $\dot{\delta}(t) = \delta(t) + \frac{i}{\pi} VP \frac{1}{t}$.

PROPOSITION 2. The filter $T : L^2 \to L^2$, $s(t) \mapsto \dot{s}(t)$ is a convolution filter, with the transfer function

$$H(\omega) = \begin{cases} 2, & \omega > 0 \\ 0, & \omega < 0 \end{cases} \tag{5}$$

Proof. For any $s \in L^2$, we have

$$\dot{s}(t) =^{by(4)} s(t) + iH_s(t) =^{by(1)} s(t) + i\left(\frac{1}{\pi} VP \frac{1}{t} * s(t)\right)$$
$$= \left(\delta(t) + \frac{i}{\pi} VP \frac{1}{t}\right) * s(t) = \dot{\delta}(t) * s(t),$$

hence $Ts = \dot{\delta} * s$. Thus, T is a convolution filter. Its transfer function is

$$\mathcal{F}\{\dot{\delta}(t)\} = \mathcal{F}\{\delta(t)\} + \frac{i}{\pi}\mathcal{F}\left\{VP \frac{1}{t}\right\} = 1 + \frac{i}{\pi}(-i\pi.sgn\omega) = 1 + sgn\omega = H(\omega),$$

for any real $\omega \neq 0$.

COROLLARY. For any $s \in L^2$ and any ω real, $\mathcal{F}\{\dot{s}(t)\}(\omega) = H(\omega).\hat{s}(\omega)$, i.e. $\hat{\dot{s}} = H.\hat{s}$.

NOTE. The passage from s to $\dot{s}$ is a generalization of the complex form of the sinusoidal signals. By (4), the imaginary part of $\dot{s}$ is just the Hilbert image of s and conversely, if we know $\dot{s}$, then the signal can be directly recovered, since $s = Re\,\dot{s}$.

Fix $\omega_0 > 0$. For any signal $s: \mathbb{R} \to \mathbb{R}$ from L^2, one can consider $\tilde{s}$, s_r, s_i, defined by

$$\tilde{s}(t) = \dot{s}(t).e^{-i\omega_0 t};\ \ s_r(t) = Re\,\tilde{s}(t)\ and\ \ s_i(t) = Im\,\tilde{s}(t),$$

for ant real t.

NOTE. If $T: L^2 \to L^2$, $x(t) \longmapsto y(t)$ is a convolution filter with the transfer function $H(\omega)$, then by proposition 2, the filter $x(t) \longmapsto \tilde{y}(t) = \dot{y}(t).e^{-i\omega_0 t}$ ($\omega_0 > 0$ fixed) will have the transfer function

$$H_1(\omega) = \begin{cases} 2H(\omega + \omega_0), & \omega > 0 \\ 0, & \omega < 0 \end{cases}.$$

Thus, if $H(\omega)$ is symmetrical around ω_0 (i.e. null outside an interval centered at ω_0), then H_1 is even; the filter $x(t) \longmapsto y(t)$ is said band-pass and $x(t) \longmapsto \tilde{y}(t)$ low-pass. Therefore, by considering the Hilbert transform, the band-pass filters can be replaced by low-pass ones.

From (4), it follows that $s(t) = Re\,\dot{s}(t) = Re\left(\tilde{s}(t).e^{(i\omega_0 t)}\right) = Re\left(s_r(t) + i\,s_i(t)\right)e^{i\omega_0 t} = s_r(t)\cos\omega_0 t - s_i(t)\sin\omega_0 t$ Hence, from s_r and s_i, one can recover s; conversely, $s_r(t) = Re\,\tilde{s}(t) =^{by(4)} Re\left(\dot{s}(t).e^{-i\omega_0 t}\right) = s(t)\cos\omega_0 t + H_s(t).\sin\omega_0 t$ and $s_i(t) = -s(t)\sin\omega_0 t + H_s(t).\cos\omega_0 t$.

These formulas are different ways of signal representations, which use the Hilbert transforms. In what follows, we will estimate their spectral densities and mean powers.

4. Spectral densities estimations

Let (Ω, K, P) be a probability field and $s = (s_\tau)$, $\tau \in \mathbb{R}$ be a stationary random signal (= random process) relative to this field; for any $\tau \in \mathbb{R}$, s_τ is supposed to be a random variable from $L^2(\Omega)$, hence having means and variances. Then one can define its autocorrelative function R

$$R(t) = M(s_{t+\tau}\,.\,\overline{s_\tau}),\ \text{for any real } t; \tag{6}$$

this is independent in τ, one to the stationarity hypothesis. Obviously, $R(0) = M|s_0|^2$ and R is even. The Fourier image of R, $P_s(\omega) = \int_{-\infty}^{\infty} R(t)e^{-i\omega t}dt$ is called the power spectral density of the random signal s. By the Fourier inversion formula,

$$R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_s(\omega)e^{it\omega}d\omega. \tag{7}$$

Since R is even, $P_s(\omega) = P_s(-\omega) = 2\int_0^{\infty} R(t)\cos\omega t\,dt$ and for any ω real, $P_s(\omega)$ is real.

PROPOSITION 3. For any stationary random signal s, the power spectral density of the analytic signal $\dot{s}$ is

$$P_{\dot{s}}(\omega) = \begin{cases} 4P_s(\omega), & \omega > 0 \\ 0, & \omega < 0 \end{cases}. \tag{8}$$

Proof. Generally, if $T: L^2 \to L^2$, $x(t) \mapsto y(t)$, is a convolution filter with the weight G hence $y = G * x$, then it holds the following relation for the autocorrelations of inputs/outputs: $R_y(t) = G(-t) * G(t) * R_x(t)$ and for spectral densities, $P_y(\omega) = |H(\omega)|^2 P_x(\omega)$, for any real ω; here $H(\omega) = \mathcal{F}\{G(t)\}$ is the transfer function of T [1]. Applying this for the filter $s(t) \mapsto \dot{s}(t)$ from prop.2, it follows that $P_{\dot{s}}(\omega) = |H(\omega)|^2 . P_s(\omega)$ and apply the proposition 2.

In analogy to proposition 3, we will give the power spectral densities for $\tilde{s}$, s_r and s_i.

PROPOSITION 4. Fix $\omega_0 > 0$ and a stationary random signal s in $L^2(\Omega)$. Then

a) $P_{\tilde{s}}(\omega) = P_{\dot{s}}(\omega + \omega_0)$;

b) $P_{s_r}(\omega) = P_{s_i}(\omega)$ and if $P_s(\omega)$ is symmetrical around ω_0, i.e. null outside an interval centered in ω_0, this equals $\frac{1}{2} P_{\tilde{s}}(\omega)$.

Proof. a)

$R_{\tilde{s}}(t) =^{by (6)} M\left(\tilde{s}(t+\tau).\overline{\tilde{s}(\tau)}\right) = M\left(\dot{s}(t+\tau).e^{-i\omega_0(t+\tau)}.\overline{\dot{s}(\tau)}.e^{i\omega_0\tau}\right) =$
$M\left(\dot{s}(t+\tau).\overline{\dot{s}(\tau)}.e^{-i\omega_0 t}\right) = R_{\dot{s}}(t).e^{-i\omega_0 t}$.

Then $P_{\tilde{s}}(\omega) = \mathcal{F}\{R_{\tilde{s}(t)}\} = \int_{-\infty}^{\infty} R_{\dot{s}}(t).e^{-i(\omega+\omega_0)t}dt = P_{\dot{s}}(\omega + \omega_0)$.

b) On the other hand, $R_{\tilde{s}}(t) = M\left(\left(s_r(t+\tau) + is_i(t+\tau)\right).\overline{\left(s_r(\tau) + \imath s_\imath(\tau)\right)}\right) =$
$M\left[s_r(t+\tau).\overline{s_r(\tau)} + s_i(t+\tau).\overline{s_\imath(\tau)} + i\left(s_i(t+\tau).s_r(\tau) - s_r(t+\tau).s_i(\tau)\right)\right] =$
$R_{s_r}(t) + R_{s_i}(t) + i[M\left(s_i(t+\tau).s_r(\tau) - M\left(s_r(t+\tau).s_i(\tau)\right)\right]$By direct computation, $R_{s_r}(t) = R_{s_i}(t)$, hence $P_{s_r}(\omega) = P_{s_i}(\omega)$ and $M\left(s_r(t+\tau).s_i(\tau)\right) = -M\left(s_r(\tau).s_i(t+\tau)\right)$, therefore, $R_{\tilde{s}}(t) = 2R_{s_r}(t) + 2iM\left(s_i(t+\tau).s_r(\tau)\right)$. If $P_s(\omega)$ is symmetrical around ω_0, then by a), $P_{\tilde{s}}(\omega)$ is symmetrical around $\omega = 0$, hence an even function. Then $R_{\tilde{s}}$ has only real values and consequently, $M\left(s_i(t+\tau).s_r(\tau)\right) = 0$ for any t and $R_{\tilde{s}} = 2R_{s_r}$.

Example. Fix $\omega_0 > 0$, $B > 0$ and suppose that $s = (s_\tau)$, $\tau \in \mathbb{R}$ is an ideal random signal with the power spectral density

$$P_s(\omega) = \begin{cases} N_0, & \omega \in [\omega_0 - B, \omega_0 + B] \\ 0, & otherwise \end{cases}$$

In this case, one says that s is a white noise with the frequency band width equal to 2B. By prop. 4, we can explicit the mean powers for s, $\tilde{s}$, s_r and s_i. Namely,

$$M|s_0|^2 =^{by(6)} R(0) =^{by(7)} \frac{1}{2\pi}\int_{-\infty}^{\infty} P_s(\omega)d\omega = \frac{1}{2\pi}\int_{\omega_0-B}^{\omega_0+B} N_0 d\omega = \frac{1}{\pi}B.N_0.$$

Similarly,
$M\left|\tilde{s_0}^2\right| = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{\tilde{s}}\,(\omega)d\omega =^{prop.3} \frac{2}{\pi}\int_0^{\infty} P_s\,(\omega+\omega_0)d\omega = \frac{2}{\pi}\int_{\omega_0}^{\infty} P_s(\omega)d\omega = \frac{2}{\pi}\int_{\omega_0}^{\omega_0+B} N_0 d\omega = \frac{2}{\pi}B.N_0$ and also, $M|s_r|^2 = M|s_i|^2 = \frac{1}{2}M|\tilde{s}|^2 = \frac{1}{\pi}B.N_0$.

An application

Consider an electronic circuit like that from figure 1, where the input signal s is the tension, which can be regarded as a stationary random signal with the power spectral density $P_s(\omega)$. Denote by $y(t)$ the tension at the output. By Kirchhoff law, $s(t) = Ri(t) + y(t)$, where the current intensity is $i(t) = C.y'(t)$. Thus $RCy'(t) + y(t) = s(t)$ for any real t. By applying the Fourier transform, it follows that $RC(i\omega).Y(\omega) + Y(\omega) = S(\omega)$, hence the transfer function of the circuit, regarded as a filter $x(t) \longmapsto y(t)$, will be

$$H(\omega) = \frac{Y(\omega)}{S(\omega)} = \frac{1}{1 + RC\omega i}.$$

If $G(t) = \mathcal{F}^{-1}\{H(\omega)\} = \begin{cases} ke^{-kt}, t \geq 0 \\ 0, t < 0 \end{cases}$, where $k = \frac{1}{RC}$, it follows that $y(t) = G(t) * s(t)$. Moreover, $|H(\omega)|^2 = \frac{1}{1+R^2C^2\omega^2} = \frac{1}{1+\left(\frac{\omega}{k}\right)^2}$, hence $P_y(\omega) = P_s(\omega)\frac{1}{1+\left(\frac{\omega}{k}\right)^2}$. For $|\omega| \gg k$, $H(\omega)$ is neglectable and so will be $P_y(\omega)$. The previous circuit is an example of low-pass filter.

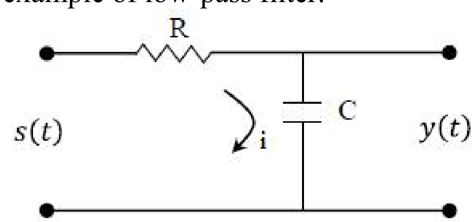

Fig. 1.

5. Conclusions

In 1 we have recalled some properties of the Hilbert transform. In 2 we analyzed the notion of analytic signal $\dot{s}$ associated to a given signal $s \in L^2(\mathbb{R})$ and we introduced a modulation $\tilde{s}(t) = \dot{s}(t)e^{-i\omega_0 t}$ around a fixed frequency ω_0. This assures a correspondence between band-pass and low-pass filters. The filter $s(t) \longmapsto \dot{s}(t)$ plays a special part (prop. 2, 3). Another concept which is analyzed in 3 is that of power spectral density; this permits to estimate the mean powers of some signals and to study the behavior of some filters (prop. 4). In this work, we presented some concepts taken from Electrical Engineering and we gave some significant examples.

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