GENERALIZATIONS OF BANACH, KANNAN AND CIRIC FIXED POINT THEOREMS IN $b_v(s)$ METRIC SPACES

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Generalizations of a metric space is one of the most important research areas in mathematics. In literature, there are several generalized metric spaces. The latest generalized metric space is $b_v(s)$ metric space which is introduced by Mitrovic and Radenovic in 2017. In this paper, we prove Kannan and Ciric fixed point theorems and generalize Banach fixed point theorem for weakly contractive mappings in $b_v(s)$ metric spaces. Our results extend and generalize some corresponding result.

Keywords: Kannan fixed point theorem, Banach fixed point theorem, Ciric fixed point theorem, weakly contractive mapping, $b_v(s)$ metric space

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1. Introduction and Preliminaries

Let (E, ρ) be a metric space and S a mapping on E. If there exists a point $u \in E$ such that $Su = u$, then point u is said to be a fixed point of S and the set of all fixed points of S is denoted by $F(S)$. Fixed point theory is one of the most important and famous theory in mathematics since it has applications to very different type of problems arise in different branches. So, uniqueness and existence problems of fixed points are also important. Two of the well known fixed point theorems are Banach and Kannan fixed point theorems. Banach fixed point theorem proved by Stefan Banach in 1920 guarantees that a contractive mapping (a mapping S is called contractive if there exists a $c \in [0, 1)$ such that $\rho(Su, Sw) \leq c\rho(u, w)$ for all $u, w \in E$) defined on a complete metric space has a unique fixed point, (see [1]). In 1968, Kannan proved another fixed point theorem for mapping satisfying

$$\rho(Su, Sw) \leq \gamma(\rho(u, Su) + \rho(w, Sw)) \tag{1}$$

for all $u, w \in E$ and $\gamma \in \left[0, \frac{1}{2}\right)$, (see [2]). Although contractivity condition implies the uniform continuity of S, Kannan type mappings (mappings which satisfy the inequality (1)) need not to be continious. Also both of these theorems characterize the completeness of metric spaces. Because of the importance of these theorems many authors generalized them for different type of contractions (see [18, 19, 20, 21, 22, 23, 24, 25]). In one of these studies, Rhoades [26] proved that a mapping which satisfies for all $u, w \in E$

$$\rho(Su, Sw) \leq \rho(u, w) - \varphi(\rho(u, w)) \tag{2}$$

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has a unique fixed point where (E,ρ) is a complete metric space and $\varphi:\mathbb{R}^{+}\cup\{0\}\to\mathbb{R}^{+}\cup\{0\}$ is a nondecreasing and continuous function such that $\varphi(t)=0$ iff $t=0$. Here, $\mathbb{R}$ is the set of all real numbers. A mapping which satisfies inequality (2) is called weakly contractive. It is clear that weakly contractive mappings can be reduced to contractive mappings by taking $\varphi(t)=ct$ for $c\in(0,1]$.

It is well known that metric spaces are very important tool for all branches of mathematics. So mathematicians have been tried to generalize this space and transform their studies to more generalized metric spaces.

As one of the most famous generalized metric spaces, in 1989, b-metric spaces was introduced by the following way.

Definition 1.1. [3] *Let E be a nonempty set and mapping $\rho:E\times E\to[0,\infty)$ a function. (E,ρ) is called b-metric space if there exists a real number $s\geq 1$ such that following conditions hold for all $u,w,z\in E$:*

(1) $\rho(u,w)=0$ *iff* $u=w$*;*
(2) $\rho(u,w)=\rho(w,u)$*;*
(3) $\rho(u,w)\leq s[\rho(u,z)+\rho(z,w)]$.

Clearly a b-metric space for $s=1$ is exactly a metric space. After this definition, many authors proved fixed point theorems for different type mappings in this space (see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]). Also they gave some generalization of Banach contraction principle, Reich and Kannan fixed point theorems. In this sense, Czerwik [4, 5] generalized Banach contraction principle to b-metric spaces. Following b-metric spaces, some generalized version of this space such as extended b-metric space, dislocated b-metric space, recrangular b-metric space, partial b-metric space, partial ordered b-metric space, etc. were introduced. The latest generalized metric space was introduced by Mitrovic and Radenovic [17] in 2017 by the following way.

Definition 1.2. [17] *Let E be a set, $\rho:E\times E\to[0,\infty)$ a function and $v\in\mathbb{N}$. Then (E,ρ) is said to be a $b_v(s)$ metric space if there exists a real number $s\geq 1$ such that following conditions hold for all $u,w\in E$ and for all distinct points $z_1,z_2,...,z_v\in E$, each of them different from u and w:*

(1) $\rho(u,w)=0$ *iff* $u=w$*;*
(2) $\rho(u,w)=\rho(w,u)$*;*
(3) $\rho(u,w)\leq s[\rho(u,z_1)+\rho(z_1,z_2)+\cdots+\rho\left(z_v,w\right)]$.

$b_v\left(s\right)$ metric space generalizes not only b-metric space but also rectangular metric space, v-generalized metric space and recrangular b-metric space by the following way. In the definition,

(1) If $v=s=1$, then we derive usual metric space.
(2) If $s=1$, then we derive v-generalized metric space.
(3) If $v=2$ and $s=1$, then we derive rectangular metric space.
(4) If $v=2$, then we derive b-rectangular metric space.
(5) If $v=1$, then we derive b-metric space.

Mitrovic and Radenovic proved Banach contraction principle and Reich fixed point theorem in this space. In the following, we give an example for $b_v\left(s\right)$ metric space.

Example 1.1. *Let $E = \mathbb{N}$. Define mapping $\rho : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ by*

$$\rho_\theta(u, w) = \begin{cases} 0, & \text{if } u = w \\ 1, & \text{if } u \text{ or } w \notin \{1, 2\} \text{ and } u \neq w \\ 10, & \text{if } u, w \in \{1, 2\} \text{ and } u \neq w \end{cases}$$

for all $u, w \in \mathbb{N}$. Then, it is obvious that (E, ρ) is a $b_v(s)$ metric space with $v = 8$ and $s = \frac{10}{9}$.

Motivated and ispired by the above studies, in this paper, we generalize Banach fixed point theorem for weakly contractive mappings and prove Kannan and Ciric fixed point theorems in $b_v(s)$ metric spaces.

2. Main Results

In this section we first give a generalization of Banach fixed point theorem for weakly contractive mappings in $b_v(s)$ metric space.

Theorem 2.1. *Let E be a complete $b_v(s)$ metric space and S a weakly contractive mapping on E. Then S has a unique fixed point.*

Proof. Let $u_0 \in E$ be an arbitrary initial point. Define sequence $\{u_n\}$ by $u_{n+1} = Su_n$. If $u_n = u_{n+1}$, then it is clear that u_n is a fixed point of S. So, let us assume that $u_n \neq u_{n+1}$ for all n. Moreover, for all different n and m, assumption $u_n \neq u_m$ can be proved in a similar way to other studies. Then, from definition of weakly contractive mapping we have

$$\begin{aligned} \rho(u_{n+1}, u_{n+p+1}) &= \rho(Su_n, Su_{n+p}) \\ &\leq \rho(u_n, u_{n+p}) - \varphi(\rho(u_n, u_{n+p})) \end{aligned}$$

for all $n, p \in \mathbb{N}$ where $\mathbb{N}$ is the set of pozitive integer. Set $\alpha_n = \rho(u_n, u_{n+p})$. Since α_n is nonnegative and φ is nondecreasing function, we can write

$$\alpha_{n+1} \leq \alpha_n - \varphi(\alpha_n) \leq \alpha_n. \tag{3}$$

This means that $\{\alpha_n\}$ is a nonincreasing sequence. Also, we know that $\{\alpha_n\}$ is bounded below. Hence it has a limit $\alpha \geq 0$. Assume that $\alpha > 0$. Since φ is nondecreasing function we have

$$\varphi(\alpha_n) \geq \varphi(\alpha) > 0.$$

Therefore, from (3) we obtain

$$\alpha_{n+1} \leq \alpha_n - \varphi(\alpha).$$

Thus, we get $\alpha_{N+m} \leq \alpha_m - N\varphi(\alpha)$ which is a contradiction for large enough N. So, our assumption is wrong, that is $\alpha = 0$. Since $\alpha_n = \rho(u_n, u_{n+p}) \to 0$ as $n \to \infty$, $\{u_n\}$ is a Cauchy sequence in E. Hence, $\{u_n\}$ converges to a point u in E because of the completeness of E. Now, we show that $\rho(u, Su) = 0$, i.e., u is a fixed point of S. Using weak contractivity of S and definition of $b_v(s)$ metric ρ, we have

$$\begin{aligned} \rho(u, Su) &\leq s\left[\rho(u, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \ldots + \rho(u_{n+v-1}, u_{n+v}) + \rho(u_{n+v}, Su)\right] \\ &= s\left[\rho(u, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \ldots + \rho(u_{n+v-1}, u_{n+v}) + \rho(Su_{n+v-1}, Su)\right] \\ &\leq s\left[\rho(u, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \ldots + \rho(u_{n+v-1}, u_{n+v})\right. \\ &\quad \left. + \rho(u_{n+v-1}, u) - \varphi(\rho(u_{n+v-1}, u))\right]. \end{aligned}$$

Since $\rho(u_n, u_{n+p}) \to 0$ and $u_n \to u$ as $n \to \infty$ and $\varphi(0) = 0$, we obtain $\rho(u, Su) = 0$, i.e., u is a fixed point of S.

Now, we need to show that u is a unique fixed point. Suppose to the contrary that there exist a different fixed point w. Since S is a weakly contractive mapping, we have

$$\rho(u,w)=\rho(Su,Sw)\leq\rho(u,w)-\varphi(\rho(u,w))<\rho(u,w).$$

This is a contradiction. So $u=w$, that is u is a unique fixed point of S. This completes the proof. □

Remark 2.1. *In Theorem 2.1,*

(1) *if we take $v=s=1$ and $\varphi(t)=ct$, then we derive Banach fixed point theorem.*
(2) *if we take $\varphi(t)=ct$ then we derive Theorem 2.1 of* [17]
(3) *if $v=1$ and $\varphi(t)=ct$, then we derive Theorem 2.1 of* [27].
(4) *if $v=2$ and $\varphi(t)=ct$, then we derive Theorem 2.1 of* [28] *and so main theorem of* [29].
(5) *if $v=s=1$, then we derive main theorem of* [26].

Now, we prove Kannan fixed point theorem in $b_v(s)$ metric spaces.

Theorem 2.2. *Let E be a complete $b_v(s)$ metric space and S a Kannan type mapping on E such that $s\gamma\leq 1$. Then S has a unique fixed point.*

Proof. Let $u_0\in E$ be an arbitrary initial point. Define sequence $\{u_n\}$ by $u_{n+1}=Su_n$. If $u_n=u_{n+1}$, then it is clear that u_n is a fixed point of S. So, let us assume that $u_n\neq u_{n+1}$ for all n. Moreover, we can assume that $x_n\neq x_m$ for all different n and m in a similar way to [29]. Since S is a Kannan type mapping we have

$$\begin{aligned}\rho(u_n,u_{n+1}) &= \rho(Su_{n-1},Su_n)\\ &\leq \gamma\left[\rho(u_{n-1},Su_{n-1})+\rho(u_n,Su_n)\right]\\ &= \gamma\left[\rho(u_{n-1},Su_{n-1})+\rho(u_n,u_{n+1})\right]\end{aligned}$$

and so,

$$\rho(u_n,u_{n+1})\leq\frac{\gamma}{1-\gamma}\rho(u_{n-1},Su_{n-1})\leq\left(\frac{\gamma}{1-\gamma}\right)^n\rho(u_0,Su_0).\tag{4}$$

This implies that

$$\lim_{n\to\infty}\rho(u_n,u_{n+1})=0.\tag{5}$$

Using inequality (4), we get

$$\begin{aligned}\rho(u_n,u_{n+p}) &= \rho\left(S^nu_0,S^{n+p}u_0\right)\\ &\leq \gamma\left[\rho\left(S^{n-1}u_0,S^nu_0\right)+\rho\left(S^{n+p-1}u_0,S^{n+p}u_0\right)\right]\\ &\leq \gamma\left[\left(\frac{\gamma}{1-\gamma}\right)^{n-1}\rho(u_0,Su_0)+\left(\frac{\gamma}{1-\gamma}\right)^{n+p-1}\rho(u_0,Su_0)\right].\end{aligned}$$

Since $\gamma\in\left[0,\frac{1}{2}\right)$, $\rho(u_n,u_{n+p})\to 0$ as $n\to\infty$. It means that $\{u_n\}$ is a Cauchy sequence in E. By completeness of E, there exists a point u in E such that $u_n\to u$. Now we show that u is a fixed point of S. For any $n\in\mathbb{N}$, we have

$$\begin{aligned}\rho(u,Su) &\leq s\left[\rho(u,u_{n+1})+\rho(u_{n+1},u_{n+2})+\ldots+\rho(u_{n+v-1},u_{n+v})+\rho(u_{n+v},Su)\right]\\ &= s\left[\rho(u,u_{n+1})+\rho(u_{n+1},u_{n+2})+\ldots+\rho(u_{n+v-1},u_{n+v})+\rho(Su_{n+v-1},Su)\right]\\ &\leq s\left[\rho(u,u_{n+1})+\rho(u_{n+1},u_{n+2})+\ldots+\rho(u_{n+v-1},u_{n+v})\right.\\ &\quad\left.+\gamma\left[\rho(u_{n+v-1},Su_{n+v-1})+\rho(u,Su)\right]\right],\end{aligned}$$

and so

$$(1 - s\gamma)\,\rho\,(u, Su) \leq s\,[\rho\,(u, u_{n+1}) + \rho\,(u_{n+1}, u_{n+2}) + \ldots + (1+\gamma)\,\rho\,(u_{n+v-1}, u_{n+v})]\,.$$

Using inequality (5), we have $\rho\,(u, Su) = 0$, i.e., $u \in F\,(S)$. Now we need to show that u is a unique fixed point. Suppose to the contrary that there exist a different fixed point w. Since S is a Kannan type mapping, we have

$$\rho\,(u, w) = \rho\,(Su, Sw) \leq \gamma\,[\rho\,(u, Su) + \rho\,(w, Sw)] = 0.$$

Therefore $u = w$ which is a contradiction. So, u is a unique fixed point of S. □

Remark 2.2. *In Theorem 2.2,*

(1) *if $v = s = 1$, then we obtain Kannan fixed point theorem* [2] *in complete metric spaces.*
(2) *if $v = 2$, then we derive Theorem 2.4 of* [29].
(3) *if $v = 2$ and $s = 1$, then we obtain main theorem of* [30] *without the assumption of orbitally completeness of the space and the main theorem of* [25].

Now, we give Ciric fixed point theorem in $b_v(s)$ metric space. But, to prove this theorem we need to prove following definition and lemma.

Definition 2.1. *Let E be a $b_v(s)$ metric space. A sequence $\{u_n\}$ is said to converge to u in the strong sense if and only if $\{u_n\}$ is Cauchy and converges to $u \in E$.*

Lemma 2.1. *Let E be a $b_v(s)$ metric space. Let $\{u_n\}$ and $\{w_n\}$ be sequences in E converging to u and w in the strong sense, respectively. Then*

$$\rho(u, w) \leq s \liminf_{n\to\infty} \rho(u_n, w_n)$$

holds.

Proof. From definition of $b_v(s)$ metric, we know that following inequality holds for all $u, w \in E$:

$$\begin{aligned}\rho(u, w) \quad \leq \quad & s\,[\rho(u, u_n) + \rho(u_n, w_n) + \rho(w_n, w_{n+1}) \\ & +\ldots + \rho(w_{n+v-3}, w_{n+v-2}) + \rho(w_{n+v-2}, w)]\,.\end{aligned}$$

So, by taking liminf from both side, we get

$$\begin{aligned}\rho(u, w) \quad \leq \quad & s \liminf_{n\to\infty}\,[\rho(u, u_n) + \rho(u_n, w_n) + \rho(w_n, w_{n+1}) \\ & +\ldots + \rho(w_{n+v-3}, w_{n+v-2}) + \rho(w_{n+v-2}, w)]\end{aligned}$$

Since $\lim_{n\to\infty} \rho(u, u_n) = 0$ and $\lim_{n\to\infty} \rho(w, w_n) = 0$, we obtain

$$\rho(u, w) \leq s \liminf_{n\to\infty} \rho(u_n, w_n).$$

for all $u, w \in E$. □

Theorem 2.3. *Let E be a complete $b_v(s)$ metric space and let S be a mapping on E such that there exists $r \in [0, 1)$ satisfying*

$$d(Su, Sw) \leq r \max\{\rho(u, w), \rho(u, Su), \rho(w, Sw), \rho(u, Sw), \rho(w, Su)\} \tag{6}$$

for any $u, w \in E$. If $rs < 1$, then S has a unique fixed point.

Proof. Fix $a \in E$ and set

$$O(a,m,n) = \left\{S^j a : j \in \mathbb{N} \cup \{0\}, m \leq j \leq n\right\}$$

and

$$O(a,m,\infty) = \left\{S^j a : j \in \mathbb{N} \cup \{0\}, m \leq j\right\}.$$

Let $dO(a,m,n)$ and $dO(a,m,\infty)$ denotes the diameters of $O(a,m,n)$ and $O(a,m,\infty)$, respectively for $m,n \in \mathbb{N} \cup \{0\}$ with $m \leq n$. By (6), we note

$$O(a,m,n) \leq rO(a,m-1,n). \tag{7}$$

We also note by (6)

$$\max\left\{\rho(a,S^j a) : 1 \leq j \leq n\right\} = O(a,0,n) \tag{8}$$

for $n \in \mathbb{N}$.

Let $\{S^n a\}$ be a sequence in E. For this sequence, there exist two cases. Either there exists $k,l \in \mathbb{N}$ such that $k < l$ and $S^k a = S^l a$ or $S^n a$ are all different for all n.

In the first case, we have

$$dO(a,k,l-1) = dO(a,k+1,l) \leq rdO(a,k,l) = rdO(a,k,l-1)$$

by (7). Hence

$$dO(a,k,\infty) = dO(a,k,l-1) = 0$$

which implies S^k is a fixed point of S. We next consider the second case. Fix $n \in \mathbb{N}$ with $n > v$. By (8), there exists $l \in \mathbb{N}$ with $l \leq n$ such that $\rho(a,S^l a) = dO(a,0,n)$. If $l > v$, then we get

$$\begin{aligned}
dO(a,0,n) &= \rho(a,S^l a) \\
&\leq s[\sum_{j=0}^{v-1} \rho(S^j a, S^{j+1} a) + \rho(S^v a, S^l a)] \\
&\leq s[vdO(a,0,v) + dO(a,v,l)] \\
&\leq s[vdO(a,0,v) + r^v dO(a,0,l)] \\
&\leq s[vdO(a,0,v) + r^v dO(a,0,n)]
\end{aligned}$$

and hence

$$dO(a,0,n) \leq \frac{sv}{(1-sr^v)} dO(a,0,v). \tag{9}$$

If $l \leq v$, then it is easy to see that (9) holds. Since $n \in \mathbb{N}$ is arbitrary, $\{dO(a,0,n)\}$ is bounded, so $dO(a,0,\infty) < \infty$. By (6), we note

$$dO(a,m,\infty) = rdO(a,m-1,\infty) \leq ... \leq r^m dO(a,0,\infty)$$

for $m \in \mathbb{N}$, which implies that $\{S^n a\}$ is a Cauchy sequence. From the completeness of E, $\{S^n a\}$ converges to some $z \in E$. It follows from Lemma 2.1 that

$$\begin{aligned}
\rho(z,Sz) &\leq s \liminf_{n\to\infty} \rho(S^n a, Sz) \\
&\leq s \liminf_{n\to\infty} r \max\left\{\rho(S^{n-1}a,z), \rho(S^{n-1}a,S^n a),\right. \\
&\qquad \left.\rho(z,Sz), \rho(S^{n-1}a,Sz), \rho(S^n a,z)\right\} \\
&= sr\rho(z,Sz).
\end{aligned}$$

From hypothesis, since we assume that $rs < 1$, then we have z is a fixed point of S. Now we prove that S has a unique fixed point. Let assume to the contrary that y and z are distinct fixed points of S. Then, we have

$$\begin{aligned} \rho(y,z) &= \rho(Sy,Sz) \\ &\leq r\max\{\rho(y,z),\rho(y,Sy),\rho(z,Sz),\rho(y,Sz),\rho(z,Sy)\} \\ &= r\rho(y,z). \end{aligned}$$

Since $r < 1$, then we get a contradiction. So, our assumption is wrong. Hence fixed point is unique. This completes the proof. □

3. Conclusion

As a result, we proved Banach, Kannan and Ciric fixed point theorems in complete $b_v\{s\}$ metric space. Our results generalize many theorems proved in different generalized metric spaces.

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