CURVATURES FOR AN EIGENVALUE OF A PERIODIC STURM-LIOUVILLE PROBLEM

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We define two curvature functions for an eigenvalue $\lambda \in \mathbb{R}$ of a periodic Sturm-Liouville problem having a two-dimensional space of eigenfunctions. For the classical example of the Fourier analysis (which corresponds to the unit circle) these functions are different. The circles centered in the origin arise in several ways.

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1. From Euclidean plane curves to linear differential equations and Riccati equations

Fix a finite real interval $I = [a, b]$ and the smooth regular plane curve $r : I \subseteq \mathbb{R} \to \mathbb{E}^2 = (\mathbb{R}^2, < \cdot, \cdot >_{can})$ having the Wronskian $W(r) > 0$ (hence r is not a line through the origin O of $\mathbb{R}^2$); if $W(r) < 0$ we change the orientation of the curve. Expressing the given curve as $r(\cdot) = (x(\cdot), y(\cdot))$ its components functions x, y are solutions of the Wronskian linear differential equation:

$$\begin{cases} W(x, y, u = u(\cdot)) := \begin{vmatrix} x & y & u \\ x' & y' & u' \\ x'' & y'' & u'' \end{vmatrix} = 0 \to \mathcal{E}^2 : u''(t) + p(t)u'(t) + q(t)u(t) = 0, \\ p := -\frac{[W(r)]'}{W(r)}, \quad q := \frac{W(r')}{W(r)}, \quad \mathcal{E}^2 : \frac{d}{dt}\left(\frac{u'}{W(r)}\right) + \frac{W(r')u}{(W(r))^2} = 0. \end{cases}$$

It is well-known that the general solution of $\mathcal{E}^2$ is provided by two real constants C_1, C_2 through the formula:

$$u(t) = C_1 x(t) + C_2 y(t), \quad C_1 = \frac{W(u, y)}{W(r)}, \quad C_2 = \frac{W(x, u)}{W(r)}$$

and this means that the real linear space of all solutions is two-dimensional. Let $k = k(t)$ be the curvature function of r; we suppose that r has no inflexion points, so $k > 0$ or $k < 0$ on I. A main hypothesis of this short note is that t is a arc-lenght parameter for r; then $I = [0, L(r)]$ with $L(r) > 0$ the length of r and then the value $W(r)$ is the Euclidean distance form the origin O to the tangent line of the curve. The well-known expression of the curvature ([1, p. 37]) gives:

$$k = x'y'' - y'x'' = W(r') \neq 0 \to q \neq 0.$$

Example 1.1. *The most simple example is the circle $\mathcal{C}(O, R > 0)$ with the arc-length parametrization:*

$$r(t) = R\left(\cos\frac{t}{R}, \sin\frac{t}{R}\right), \quad t \in I = [0, 2\pi R].$$

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We have $W(r) = constant = R > 0$ *and* $W(r') = constant = \frac{1}{R} = k$ *and then the self-adjoint form of* $\mathcal{E}^2$ *is the well-known:*

$$\mathcal{E}^2 : \left(\frac{u'}{R}\right)' + \frac{u}{R^3} = 0 \to u'' + \frac{u}{R^2} = 0.$$

More generally, for the ellipse $E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *we have the non-arc-length parametrization* $E : r(t) = (a \cos t, b \sin t)$, $t \in [0, 2\pi]$ *with* $W(r) = W(r') = ab = constant$ *and then* $\mathcal{E}^2 : u'' + u = 0$. $\square$

Remark 1.1. *i) The circle example suggests the expression of the derivative curves* r' *and* r'' *for the general framework:*

$$r'(t) = (-\sin K(t), \cos K(t)), \quad K(t) = \int_0^t k(s)ds \to r''(t) = -k(t)exp(iK(t)).$$

For the last relation we use the famous trigonometric Euler formula. With one step further we have:

$$r'''(t) = -k'(t)exp(iK(t)) - k^2(r)r'(t)$$

and then it follows that the arc-length defined functions $x(\cdot)$, $y(\cdot)$ *are also solutions of the Wylczynski equation:*

$$\mathcal{E}^3 : kU''' - k'U'' + k^3U' = 0$$

which we have studied in [7]*. The general Wylczynski equation of a curve on a* 2*-dimensional manifold* (M^2, g) *is the equation* (1.5) *from* [6].

ii) It is well-known that the set of homogeneous linear second order differential equations $\mathcal{E}^2 : u''(t) + p(t)u'(t) + q(t)u(t) = 0$ *is in one-to-one correspondence with the set of Riccati equations:*

$$\mathcal{R} : z'(t) = A(t)z^2(t) + B(t)z(t) + C(t).$$

Indeed, starting with $\mathcal{E}^2$ *we consider* $u'(t) = z(t)u(t)$ *and from* $u''(t) = (z'(t) + z^2(t))u(t)$ *it results* $\mathcal{R}$ *with:*

$$A = -1, \quad B = -p, \quad C = -q.$$

Hence, to the given plane curve we associate the Riccati equation:

$$\mathcal{R}(r) : z'(t) = -z^2(t) + \frac{[W(r)]'}{W(r)}(t)z(t) - \frac{W(r')}{W(r)}(t).$$

The circle $\mathcal{C}(O, R)$ *parametrized as in the previous example has the Riccati equation:*

$$\mathcal{R}(\mathcal{C}) : z'(t) = -z^2(t) - \frac{1}{R^2}.$$

$\square$

2. Curvatures for an eigenvalue of a periodic Sturm-Liouville problem

In the following we travel in the reverse way: from a differential equation to an associated geometry and hence curvature. Fix the functions $a, b, c \in C^1(\mathbb{R})$ satisfying the following hypothesis:

H1) $a > 0$ and $c > 0$ on $\mathbb{R}$;

H2) all three functions are periodic with the period $L > 0$.

Hence, we can consider the *periodic Sturm-Liouville* problem on $I = [0, L]$:

$$\mathcal{SL} : [(a(t)u'(t))' + b(t)u(t)] + \lambda c(t)u(t) = 0, \quad u(0) = u(L), \quad u'(0) = u'(L).$$

It is well-known ([2, p. 31]) that in contrast to *the regular* Sturm-Liouville problems a fixed eigenvalue λ may have two linearly independent eigenfunctions (x, y). Therefore, we assume from now that this is our setting and then we call *the geometrical curvature* of λ the curvature function k^g_λ of an arc-length parametrization for the curve $C(\mathcal{SL}) : t \in [0, L] \to \tilde{r}(t) = (x(t), y(t))$.

An example when the curve is precisely determined by the above conditions is provided by:

Proposition 2.1. *Suppose that both a and $b + \lambda c$ are non-zero constants. Then the arc-length parametrized curve $C(\mathcal{SL})$ is a circle centered in the origin $O(0, 0)$.*

Proof. By deriving the arc-length hypothesis we have $x'x'' + y'y'' = 0$. By multiplying this equation with the constant $a > 0$ and replacing x'', y'' from $\mathcal{SL}$ it follows $xx' + yy' = 0$ which gives the claimed conclusion. We point out that the periodicity hypothesis is not used. □

Moreover, we can associate a second curvature to our eigenvalue λ through the identification of the second form of $\mathcal{E}^2$ and $\mathcal{SL}$. Let us call *equational* this resulting curvature and hence:

$$W(r) = \frac{1}{a} > 0, \quad k^e_\lambda \cdot a^2 = b + \lambda c.$$

It follows the curvature function $k^e_\lambda : \mathbb{R} \to \mathbb{R}$:

$$k^e_\lambda(t) := \frac{b(t) + \lambda c(t)}{a^2(t)},$$

which is also periodic with the same period L. We remark that a somehow dual problem is studied in [3], namely cycloids in a normed plane whose radius of curvature and support function satisfy a differential equation of Sturm-Liouville type.

Example 2.1. *Dual to the example 1.1 we have the Fourier $\mathcal{FSL}$ provided by $L = 2\pi$ and the coefficient functions $a = c = 1$ and $b = 0$. Its 2-dimensional eigenvalues are well-known (in fact are resulting from the example 1.1): $\lambda_n = n^2$ with the corresponding basis of eigenfunctions $(\cos(nt), \sin(nt))$ for $n \in \mathbb{N}^*$. It results the curvature functions of λ_n as being indexed by the natural number n:*

$$k^g_n = 1 > 0, \quad k^e_n = \lambda_n = n^2 > 0.$$

It is worth to point out that for $n \geq 2$ we have $k^g_n < k^e_n$ and hence our second route (from second order ODE to plane curve) is not the reverse of the initial one. □

Proposition 2.2. *If $W(r) > 0$ and $k^g_\lambda = k^e_\lambda$ then the curve $\mathcal{C}(SL)$ is a line not containing O.*

Proof. Suppose $k^g_\lambda = k^e_\lambda$. The expression of the curvature implies the relation:

$$(b + \lambda c)W(r) = 0$$

which gives $\lambda = -\frac{b}{c} \in \mathbb{R}$ and this value reduces k^e_λ to 0 and then the curve is a line. Also $\mathcal{SL}$ reduces to the equation $au' = \mathcal{C}_1$ with the solution:

$$u(t) = \mathcal{C}_1 \int_0^t \frac{ds}{a(s)} + \mathcal{C}_2.$$

□

Example 2.2. *Although this example does not fit into the setting of periodic Sturm-Liouville problems we consider it interesting due to to similarities with the circles already studied in the previous examples. Let $(x = cn(\cdot, \rho), y = sn(\cdot, \rho), z = dn(\cdot, \rho))$ be the Jacobi elliptic functions as solutions of the homogeneous ordinary differential system (*[4, p. 30]*):*

$$\begin{cases} \frac{dx}{dt} = -zy, \quad x(0) = 1, \\ \frac{dy}{dt} = zx, \quad y(0) = 0, \\ \frac{dz}{dt} = -\rho^2 xy, \quad z(0) = 1, \end{cases}$$

with the modulus $0 \leq \rho^2 < 1$. *We compute now the curvature of the plane curve* $r_\rho(t) = (x(t), y(t))$. *From:*

$$\begin{cases} r'_\rho(t) = z(t)(-y(t), x(t)), \\ r''_\rho(t) = (-z^2(t)x(t) + \rho^2 x(t)y^2(t), -z^2(t)y(t) - \rho^2 x^2(t)y(t)) \end{cases}$$

it results:

$$k_\rho(t) = \frac{1}{[x^2(t) + y^2(t)]^{\frac{1}{2}}} > 0$$

and we remark that this function does not depends on z. *In fact, the functions* $\mathcal{F}_1, \mathcal{F}_2 : \mathbb{R}^3 \to \mathbb{R}$*:*

$$\mathcal{F}_1(x, y, z) := x^2 + y^2, \quad \mathcal{F}_2(x, y, z) := \rho^2 y^2 + z^2$$

are first integrals of the differential system above ([8, p. 130]*) and hence* $x^2(t) + y^2(t) = constant = x^2(0) + y^2(0) = 1$*; therefore* $k_\rho = constant = 1$. *In conclusion, our plane curve is exactly the unit circle* S^1 *with a new periodic parametrization* r_ρ *since both functions* $x(\cdot)$ *and* $y(\cdot)$ *are periodic with* $L = 4\tilde{L}$ *for (*[8, p. 131]*):*

$$\tilde{L} = \tilde{L}(\rho) := \int_0^1 \frac{ds}{\sqrt{(1 - s^2)(1 - \rho^2 s^2)}}.$$

In particular:

$$\tilde{L}(0) = \arcsin s \Big|_0^1 = \frac{\pi}{2}$$

for the trigonometrical functions $cn(\cdot, 0) = \cos(\cdot)$ *and* $sn(\cdot, 0) = \sin(\cdot)$.

We point out that a remarkable example is provided by $\rho = \frac{1}{\sqrt{2}}$ *which is the eccentricity of a remarkable class of ellipses, called self-adjoints, which are studied in* [5]. □

3. Conclusions

The current work provides a geometric approach to the theory of periodic Sturm-Liouville equations. This way to study differential equations is inspired by the classical theory of plane curves.

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