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HARMONIC WAVELET ANALYSIS - CONNECTION COEFFICIENTS FOR NONLINEAR PDE

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Undinele armonice au fost aplicate recent pentru rezolvarea problemelor de evoluţie şi, mai general, pentru a descrie operatori diferenţiali. În această lucrare sunt studiate proprităţile diferenţiale ale acestora şi este prezentată metoda de calcul a coeficienţilor de conexiune pentru probleme neliniare.

Harmonic wavelets were recently applied to the solution of evolution problems and, more generally, to describe operators. In this paper are studied their differential properties and, is analyzed the problem of the computation of connection coefficients for nonlinear problems.

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1. Introduction

The investigation of wavelet solution of differential problems (linear and nonlinear) [2, 3, 4, 5, 7] is been one of the most interesting applications of wavelet theory [1, 8, 10, 9]. In the mathematics , the harmonic wavelet theory was introduced by Newland [1], in 1993. Harmonic wavelets [1, 2, 3, 4, 5, 6, 9, 10] are complex functions and band-limited in the Fourier domain, so that they can be used to study frequency changes as well as oscillations in a small range time interval.

Besides the many advantages of using them (such as e.g., the localization and compression), harmonic wavelets have a main property, i.e., they form orthonormal bases (in suitable functional spaces). Thus, they easily fulfill one of the basic requirements of the Petrov-Galerkin method. If we restrict, in particular, to the Petrov-Galerkin method, wavelet harmonic bases are efficiently used to define the solution of PDE equations, integral equations, and more general integro-differential equations and operators (see e.g., [2, 3, 4, 5]). Thus, the solution of the differential (or integro-differential) problem is searched as a series of harmonic wavelets [1, 2, 3, 4, 5] and it determined (up to a given

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approximation) when its wavelet coefficients are computed (from an equivalent ordinary differential problem).

The present work takes a closer look at the problem of computation of connection coefficients for nonlinear problems, more specifically, for quadratic nonlinearities.

2. **Harmonic wavelets**

By definition [1], the harmonic scaling $\varphi(x)$ (the father wavelet) and wavelet $\psi(x)$ (the mother wavelet) are complex functions of the form

$$\varphi(x) \overset{def}{=} \frac{e^{i2\pi x}-1}{2\pi i x} \tag{1}$$

respectively ,

$$\psi(x) \overset{def}{=} \frac{e^{i4\pi x}-e^{i2\pi x}}{2\pi i x}\,. \tag{2}$$

Also, in corresponding multiresolution analysis, it observes that

$$\psi(x) = e^{i2\pi x}\varphi(x) = 2\varphi(2x)-\varphi(x)\,. \tag{3}$$

If we assume as Fourier transform of $f(x)$ the function $\widehat{f}(\omega)$

$$\widehat{f}(\omega) \overset{def}{=} \mathcal{F}\left[f(x)\right] \overset{def}{=} \int_{\mathbb{R}} f(x)e^{-i\omega x}\,dx \tag{4}$$

we can easily get Fourier transform for (1) and (2), i.e.,

$$\widehat{\varphi}(\omega) = \aleph\left(\omega+2\pi\right) \tag{5}$$

$$\widehat{\psi}(\omega) = \aleph(\omega) \tag{6}$$

where $\aleph(\omega)$ is the characteristic function

$$\aleph\left(\omega\right) = \begin{cases} 1, & 2\pi \le \omega \le 4\pi \\ 0, & \text{elsewhere.} \end{cases} \tag{7}$$

The dilated and translated instances of (1-2) are

$$\varphi_k^n(x) = 2^{\frac{n}{2}}\varphi(2^n x-k) \tag{8}$$

$$\psi_k^n(x) = 2^{\frac{n}{2}}\psi(2^n x-k) \tag{9}$$

for $\forall\, n,\, k \in \mathbb{Z}$ two parameters: n is the scale (refinement, compresion, or dilataion) parameter and k is the localization (translation) parameter. The first parameter shrinks or squeeze on the basic instance while the second can be used to shift the basic instance up to any point.

By taking into account the property of the Fourier transform, i.e.,

$$\mathcal{F}\left[f\left(ax\pm b\right)\right] = \frac{1}{|a|}e^{\pm\frac{i\omega b}{a}}\widehat{f}\left(\frac{\omega}{a}\right) \tag{10}$$

it is easy to prove that

$$\widehat{\varphi}_k^n(\omega) = 2^{-\frac{n}{2}} e^{-i\frac{\omega k}{2^n}} \aleph\left(\frac{\omega}{2^n} + 2\pi\right) \tag{11}$$

$$\widehat{\psi}_k^n(\omega) = 2^{-\frac{n}{2}} e^{-i\frac{\omega k}{2^n}} \aleph\left(\frac{\omega}{2^n}\right) \tag{12}$$

with $\forall\, n,\, k \in \mathbb{Z}$.

Moreover, concerning the application of the Petrov-Galerkin method to PDE, it is assumed that a certain unknown function (with its derivatives) can be expressed in terms of a basis (and its derivatives).

For this reason, in the Fourier domain, according to

$$\mathcal{F}\left[\frac{d^\ell}{dx^\ell} f(x)\right] = (i\omega)^\ell \widehat{f}(\omega) \tag{13}$$

and (11-12), the ℓ-th order derivatives of the harmonic scaling and wavelet basis have the explicit forms

$$\widehat{\frac{d^\ell}{dx^\ell}\varphi_k^n(x)} = \frac{2^{-\frac{n}{2}}}{2\pi} (i\omega)^\ell e^{-i\frac{\omega k}{2^n}} \aleph\left(\frac{\omega}{2^n} + 2\pi\right) \tag{14}$$

i.e.,

$$\widehat{\frac{d^\ell}{dx^\ell}\psi_k^n(x)} = \frac{2^{-\frac{n}{2}}}{2\pi} (i\omega)^\ell e^{-i\frac{\omega k}{2^n}} \aleph\left(\frac{\omega}{2^n}\right) . \tag{15}$$

3. Connection coefficients for nonlinear problems

In order to solve the nonlinear problems with Petrov-Galerkin, the nonlinear terms of PDE give rise to some more general connection coefficients [2, 3] and this leads to serious difficulties of numerical computation. Therefore, if we will restrict the calculus to quadratic terms, it follows that

$$\Phi_{pkh}^{snr} \stackrel{def}{=} \left\langle \psi_p^s(x)\, \psi_k^n(x)\, ,\, \psi_h^r(x) \right\rangle \tag{16}$$

$$\Gamma^{(\ell)snr}_{\;\;pkh} \stackrel{def}{=} \left\langle \psi_p^s(x)\, \frac{d^\ell}{dx^\ell}\left(\psi_k^n(x)\right)\, ,\, \psi_h^r(x) \right\rangle \tag{17}$$

$$\Delta^{(\ell,\, r)snr}_{\;\;pkh} \stackrel{def}{=} \left\langle \frac{d^\ell}{dx^\ell}\left(\psi_p^s(x)\right)\, \frac{d^j}{dx^j}\left(\psi_k^n(x)\right)\, ,\, \psi_h^r(x) \right\rangle \tag{18}$$

The inner product of two functions $f(x)$, $g(x)$, in Hilbert space $L^2(\mathbb{R})$, fulfills the Parceval equality and is defined

$$\langle f(x)\, ,\, g(x) \rangle \equiv \int_{\mathbb{R}} f(x)\, \overline{g(x)} dx = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega)\, \overline{\widehat{g(\omega)}} d\omega = \frac{1}{2\pi} \left\langle \widehat{f}(\omega)\, ,\, \widehat{g}(\omega) \right\rangle \tag{19}$$

Then, the group of coefficients can be rewritten as

$$\Phi^{snr}_{pkh} = \frac{1}{2\pi}\left\langle \widehat{\psi^s_p(x)\,\psi^n_k}(x)\,,\,\widehat{\psi^r_h(x)}\right\rangle \tag{20}$$

$$\Gamma^{(\ell)snr}_{\quad pkh} = \frac{1}{2\pi}\left\langle \widehat{\psi^s_p(x)\,\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}}\left(\psi^n_k(x)\right)\,,\,\widehat{\psi^r_h(x)}\right\rangle \tag{21}$$

$$\Delta^{(\ell,\,r)snj}_{\quad pkh} = \frac{1}{2\pi}\left\langle \widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi^s_p(x)\right)\,\frac{\mathrm{d}^j}{\mathrm{d}x^j}}\left(\psi^n_k(x)\right)\,,\,\widehat{\psi^r_h(x)}\right\rangle\,. \tag{22}$$

Thus, we need to find the coefficients (16-18). Therefore, we'll compute, previously, the formulas

$$\widehat{\psi^s_p(x)\,\psi^n_k}(x) \tag{23}$$

$$\widehat{\psi^s_p(x)\,\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}}\left(\psi^n_k(x)\right) \tag{24}$$

$$\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi^s_p(x)\right)\,\frac{\mathrm{d}^j}{\mathrm{d}x^j}}\left(\psi^n_k(x)\right)\,. \tag{25}$$

Taking into account the Fourier transform properties, i.e., the convolution in the frequency, we have that

$$\begin{aligned}\widehat{f(x)\,g(x)} &= \frac{1}{2\pi}\widehat{f}(\omega)*\widehat{g}(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(\omega-\tau)\,\widehat{g}(\tau)\mathrm{d}\tau \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(\tau)\,\widehat{g}(\omega-\tau)\mathrm{d}\tau = \widehat{g(x)\,f(x)}\,. \end{aligned}\tag{26}$$

In our case, according to (26), the expressions (23-25) are given by the following results.

Theorem 3.1. *For given* $s,\,n,\,k,\,p\in\mathbb{Z}$, *it is*

$$\begin{aligned}&\widehat{\psi^{\mathrm{m}}_\varepsilon(x)\,\psi^{\mathrm{M}}_\theta}(x) = \widehat{\psi^{\mathrm{M}}_\theta(x)\,\psi^{\mathrm{m}}_\varepsilon}(x) = \\ &= \frac{2^{-\frac{\mathrm{m}+\mathrm{M}}{2}}}{2\pi}\mathrm{e}^{-i\frac{\omega\varepsilon}{2^{\mathrm{m}}}}\left\{\aleph\left(\frac{\omega-2^{\mathrm{M}+1}\pi}{2^{\mathrm{m}}}\right)\int_{2^{\mathrm{M}+1}\pi}^{\omega-2^{\mathrm{m}+1}\pi}\mathrm{e}^{i\tau\left(\frac{\varepsilon}{2^{\mathrm{m}}}-\frac{\theta}{2^{\mathrm{M}}}\right)}\mathrm{d}\tau+\right.\\ &+\aleph\left(\frac{\omega-2^{\mathrm{M}+2}\pi}{2^{\mathrm{m}}}\right)\int_{\omega-2^{\mathrm{m}+2}\pi}^{2^{\mathrm{M}+2}\pi}\mathrm{e}^{i\tau\left(\frac{\varepsilon}{2^{\mathrm{m}}}-\frac{\theta}{2^{\mathrm{M}}}\right)}\mathrm{d}\tau+\left[\aleph\left(\frac{\omega}{2^{\mathrm{m}}+2^{\mathrm{M}}}\right)-\right.\\ &\left.\left.-\aleph\left(\frac{\omega-2^{\mathrm{M}+1}\pi}{2^{\mathrm{m}}}\right)-\aleph\left(\frac{\omega-2^{\mathrm{M}+2}\pi}{2^{\mathrm{m}}}\right)\right]\int_{\omega-2^{\mathrm{m}+2}\pi}^{\omega-2^{\mathrm{m}+1}\pi}\mathrm{e}^{i\tau\left(\frac{\varepsilon}{2^{\mathrm{m}}}-\frac{\theta}{2^{\mathrm{M}}}\right)}\mathrm{d}\tau\right\}\end{aligned}\tag{27}$$

where $\mathrm{m}=\frac{s+n-|s-n|}{2}$, $\mathrm{M}=\frac{s+n+|s-n|}{2}$ *and* $\varepsilon,\,\theta\in\{k,\,p\}$.

Proof. Taking account by (26), the relation (23) can be written as

$$\widehat{\psi^s_p(x)\,\psi^n_k}(x) = \frac{1}{2\pi}\widehat{\psi}^s_p(\omega)*\widehat{\psi}^n_k(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\psi}^s_p(\omega-\tau)\,\widehat{\psi}^n_k(\tau)\mathrm{d}\tau$$

$$\overset{\tau\to\omega-\xi}{=} \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\psi}^n_k(\omega-\tau)\,\widehat{\psi}^s_p(\tau)\mathrm{d}\tau = \widehat{\psi^n_k(x)\,\psi^s_p}(x)\,. \tag{28}$$

On the other hand, according to (28) and (12), it follows that

$$\begin{aligned}\widehat{\psi^s_p(x)\,\psi^n_k}(x) &= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\psi}^s_p(\omega-\tau)\,\widehat{\psi}^n_k(\tau)\mathrm{d}\tau\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\left(2^{-\frac{s}{2}}\mathrm{e}^{-i\frac{(\omega-\tau)p}{2^s}}\aleph\left(\frac{\omega-\tau}{2^s}\right)\right)\cdot\left(2^{-\frac{n}{2}}\mathrm{e}^{-i\frac{\tau k}{2^n}}\aleph\left(\frac{\tau}{2^n}\right)\right)\mathrm{d}\tau\\ &= \frac{2^{-\frac{s+n}{2}}}{2\pi}\mathrm{e}^{-i\frac{\omega p}{2^s}}\int_{\mathbb{R}}\mathrm{e}^{i\tau\left(\frac{p}{2^s}-\frac{k}{2^n}\right)}\aleph\left(\frac{\omega-\tau}{2^s}\right)\aleph\left(\frac{\tau}{2^n}\right)\mathrm{d}\tau\end{aligned} \tag{29}$$

respectively,

$$\widehat{\psi^n_k(x)\,\psi^s_p}(x) = \frac{2^{-\frac{s+n}{2}}}{2\pi}\mathrm{e}^{-i\frac{\omega k}{2^n}}\int_{\mathbb{R}}\mathrm{e}^{i\tau\left(\frac{k}{2^n}-\frac{p}{2^s}\right)}\aleph\left(\frac{\omega-\tau}{2^n}\right)\aleph\left(\frac{\tau}{2^s}\right)\mathrm{d}\tau\quad. \tag{30}$$

When $s\neq n$, let's say $s<n$, we have

$$\begin{aligned}&\widehat{\psi^s_p(x)\,\psi^n_k}(x) =\\ &\quad= \frac{2^{-\frac{s+n}{2}}}{2\pi}\mathrm{e}^{-i\frac{\omega p}{2^s}}\left\{\aleph\left(\frac{\omega-2^{n+1}\pi}{2^s}\right)\int_{2^{n+1}\pi}^{\omega-2^{s+1}\pi}\mathrm{e}^{i\tau\left(\frac{p}{2^s}-\frac{k}{2^n}\right)}\mathrm{d}\tau+\right.\\ &\quad+\aleph\left(\frac{\omega-2^{n+2}\pi}{2^s}\right)\int_{\omega-2^{s+2}\pi}^{2^{n+2}\pi}\mathrm{e}^{i\tau\left(\frac{p}{2^s}-\frac{k}{2^n}\right)}\mathrm{d}\tau++\left[\aleph\left(\frac{\omega}{2^s+2^n}\right)-\right.\\ &\quad\left.\left.-\aleph\left(\frac{\omega-2^{n+1}\pi}{2^s}\right)-\aleph\left(\frac{\omega-2^{n+2}\pi}{2^s}\right)\right]\int_{\omega-2^{s+2}\pi}^{\omega-2^{s+1}\pi}\mathrm{e}^{i\tau\left(\frac{p}{2^s}-\frac{k}{2^n}\right)}\mathrm{d}\tau\right\}.\end{aligned} \tag{31}$$

When $s>n$, we obtain

$$\begin{aligned}&\widehat{\psi^s_p(x)\,\psi^n_k}(x) =\\ &\quad= \frac{2^{-\frac{s+n}{2}}}{2\pi}\mathrm{e}^{-i\frac{\omega k}{2^n}}\left\{\aleph\left(\frac{\omega-2^{s+1}\pi}{2^n}\right)\int_{2^{s+1}\pi}^{\omega-2^{n+1}\pi}\mathrm{e}^{i\tau\left(\frac{k}{2^n}-\frac{p}{2^s}\right)}\mathrm{d}\tau+\right.\\ &\quad+\aleph\left(\frac{\omega-2^{s+2}\pi}{2^n}\right)\int_{\omega-2^{n+2}\pi}^{2^{s+2}\pi}\mathrm{e}^{i\tau\left(\frac{k}{2^n}-\frac{p}{2^s}\right)}\mathrm{d}\tau++\left[\aleph\left(\frac{\omega}{2^n+2^s}\right)-\right.\\ &\quad\left.\left.-\aleph\left(\frac{\omega-2^{s+1}\pi}{2^n}\right)-\aleph\left(\frac{\omega-2^{s+2}\pi}{2^n}\right)\right]\int_{\omega-2^{n+2}\pi}^{\omega-2^{n+1}\pi}\mathrm{e}^{i\tau\left(\frac{k}{2^n}-\frac{p}{2^s}\right)}\mathrm{d}\tau\right\}.\end{aligned} \tag{32}$$

For $s=n$ the last part of proof is imediately. Also, it easily follows that

$$\begin{aligned}\psi_k^n\widehat{(x)\,\psi_p^n}(x) &= \frac{2^{-n}}{2\pi}\mathrm{e}^{-i\frac{\omega k}{2^n}}\left[\aleph\left(\frac{\omega}{2^n}-2\pi\right)\int_{2^{n+1}\pi}^{\omega-2^{n+1}\pi}\mathrm{e}^{i\tau\frac{k-p}{2^n}}\mathrm{d}\tau+\right.\\ &+ \left.\aleph\left(\frac{\omega}{2^n}-4\pi\right)\int_{\omega-2^{n+2}\pi}^{2^{n+2}\pi}\mathrm{e}^{i\tau\frac{k-p}{2^n}}\mathrm{d}\tau\right]. \qquad (33)\end{aligned}$$

and, by the change of variable $\tau\to\omega-\xi$, it is

$$\begin{aligned}\psi_k^n\widehat{(x)\,\psi_p^n}(x) &= \frac{2^{-n}}{2\pi}\mathrm{e}^{-i\frac{\omega p}{2^n}}\left[\aleph\left(\frac{\omega}{2^n}-2\pi\right)\int_{2^{n+1}\pi}^{\omega-2^{n+1}\pi}\mathrm{e}^{i\xi\frac{p-k}{2^n}}\mathrm{d}\xi+\right.\\ &+ \left.\aleph\left(\frac{\omega}{2^n}-4\pi\right)\int_{\omega-2^{n+2}\pi}^{2^{n+2}\pi}\mathrm{e}^{i\xi\frac{k-p}{2^n}}\mathrm{d}\xi\right]. \qquad (34)\end{aligned}$$

□

In particular, according to formula (27) and taking into account that $k=p=0$, we have the following corollary.

Corollary 3.1. *For $k=p=0$, it is*

$$\begin{aligned}&\psi_0^s\widehat{(x)\,\psi_0^n}(x)=\psi_0^n\widehat{(x)\,\psi_0^s}(x) =\\ &\quad=\frac{2^{-\frac{\mathrm{m}+\mathrm{M}}{2}}}{2\pi}\left\{\aleph\left(\frac{\omega-2^{\mathrm{M}+1}\pi}{2^{\mathrm{m}}}\right)\left[\omega-\left(2^{\mathrm{m}+2}+2^{\mathrm{M}+1}\right)\pi\right]+\right.\\ &\quad+\aleph\left(\frac{\omega-2^{\mathrm{M}+2}\pi}{2^{\mathrm{m}}}\right)\left[\left(2^{\mathrm{m}+1}+2^{\mathrm{M}+2}\right)\pi-\omega\right]\\ &\quad+\left.2^{\mathrm{m}+1}\pi\left[\aleph\left(\frac{\omega}{2^{\mathrm{m}}+2^{\mathrm{M}}}\right)-\aleph\left(\frac{\omega-2^{\mathrm{M}+1}\pi}{2^{\mathrm{m}}}\right)-\aleph\left(\frac{\omega-2^{\mathrm{M}+2}\pi}{2^{\mathrm{m}}}\right)\right]\right\}. \qquad (35)\end{aligned}$$

Furthermore, the following corollary holds.

Corollary 3.2. *For $n=s$ and $k=p$, it is*

$$\begin{aligned}&\psi_k^n\widehat{(x)\,\psi_k^n}(x) =\\ &=\frac{2^{-n}}{2\pi}\mathrm{e}^{-i\frac{\omega k}{2^n}}\left[\aleph\left(\frac{\omega}{2^n}-2\pi\right)\left(\omega-2^{n+2}\pi\right)+\aleph\left(\frac{\omega}{2^n}-4\pi\right)\left(2^{n+3}\pi-\omega\right)\right]. \qquad (36)\end{aligned}$$

Theorem 3.2. *For given s, n, k, $p \in \mathbb{Z}$, and $\ell \in \mathbb{N}$, it is*

$$
\begin{aligned}
&\psi^{\mathrm{m}}_{\varepsilon}(x)\,\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi^{\mathrm{M}}_{\theta}(x)\right)} = \psi^{\mathrm{M}}_{\theta}(x)\,\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi^{\mathrm{m}}_{\varepsilon}(x)\right)} = \\
&= \frac{2^{-\frac{\mathrm{m}+\mathrm{M}}{2}}}{2\pi}\mathrm{e}^{-i\frac{\omega\varepsilon}{2^{\mathrm{m}}}}(i)^\ell \left\{ \aleph\left(\frac{\omega-2^{\mathrm{M}+1}\pi}{2^{\mathrm{m}}}\right)\int_{2^{\mathrm{M}+1}\pi}^{\omega-2^{\mathrm{m}+1}\pi}\tau^\ell \mathrm{e}^{i\tau\left(\frac{\varepsilon}{2^{\mathrm{m}}}-\frac{\theta}{2^{\mathrm{M}}}\right)}\mathrm{d}\tau + \right.\\
&+\aleph\left(\frac{\omega-2^{\mathrm{M}+2}\pi}{2^{\mathrm{m}}}\right)\int_{\omega-2^{\mathrm{m}+2}\pi}^{2^{\mathrm{M}+2}\pi}\tau^\ell \mathrm{e}^{i\tau\left(\frac{\varepsilon}{2^{\mathrm{m}}}-\frac{\theta}{2^{\mathrm{M}}}\right)}\mathrm{d}\tau + \left[\aleph\left(\frac{\omega}{2^{\mathrm{m}}+2^{\mathrm{M}}}\right) - \right.\\
&\left.\left. -\aleph\left(\frac{\omega-2^{\mathrm{M}+1}\pi}{2^{\mathrm{m}}}\right)-\aleph\left(\frac{\omega-2^{\mathrm{M}+2}\pi}{2^{\mathrm{m}}}\right)\right]\int_{\omega-2^{\mathrm{m}+2}\pi}^{\omega-2^{\mathrm{m}+1}\pi}\tau^\ell \mathrm{e}^{i\tau\left(\frac{\varepsilon}{2^{\mathrm{m}}}-\frac{\theta}{2^{\mathrm{M}}}\right)}\mathrm{d}\tau \right\} \qquad (37)
\end{aligned}
$$

where $\mathrm{m} = \frac{s+n-|s-n|}{2}$, $\mathrm{M} = \frac{s+n+|s-n|}{2}$ *and* ε, $\theta \in \{k, p\}$.

Proof. Taking account by (24), (26), (15), we have

$$
\psi^s_p(x)\,\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi^n_k(x)\right)} = \frac{2^{-\frac{n+s}{2}}}{2\pi} i^\ell \mathrm{e}^{-i\frac{\omega p}{2^s}}\int_{\mathbb{R}}\tau^\ell \mathrm{e}^{i\tau\left(\frac{p}{2^s}-\frac{k}{2^n}\right)}\aleph\left(\frac{\omega-\tau}{2^s}\right)\aleph\left(\frac{\tau}{2^n}\right)\mathrm{d}\tau \quad (38)
$$

$$
\begin{aligned}
&\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi^n_k(x)\right)\,\psi^s_p(x)} = \\
&\qquad = \frac{2^{-\frac{n+s}{2}}}{2\pi} i^\ell \mathrm{e}^{-i\frac{\omega k}{2^n}}\int_{\mathbb{R}}(\omega-\xi)^\ell \mathrm{e}^{i\xi\left(\frac{k}{2^n}-\frac{p}{2^s}\right)}\aleph\left(\frac{\xi}{2^n}\right)\aleph\left(\frac{\omega-\xi}{2^s}\right)\mathrm{d}\xi \\
&\qquad\qquad (39)
\end{aligned}
$$

from where, with the change of variable $\tau \to \omega - \xi$, we easily obtain that formulas (38) and (39) are equivalent. By a direct computation and according to Theorem 3.1, and (38-39), the formula (37) is proven. When $\ell = 0$, (23) trivially follows. Moreover, for $s = n$ we obtain that

$$
\begin{aligned}
\psi^n_p(x)\,\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi^n_k(x)\right)} &= \widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi^n_k(x)\right)\,\psi^n_p(x)} && (40)\\
\psi^n_p(x)\,\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi^n_k(x)\right)} &= \frac{2^{-n}}{2\pi}\mathrm{e}^{-i\frac{\omega k}{2^n}}(i)^\ell\left[\aleph\left(\frac{\omega}{2^n}-2\pi\right)\int_{2^{n+1}\pi}^{\omega-2^{n+1}\pi}\tau^\ell \mathrm{e}^{i\tau\frac{k-p}{2^n}}\mathrm{d}\tau + \right.\\
&\left. + \aleph\left(\frac{\omega}{2^n}-4\pi\right)\int_{\omega-2^{n+2}\pi}^{2^{n+2}\pi}\tau^\ell \mathrm{e}^{i\tau\frac{k-p}{2^n}}\mathrm{d}\tau\right] && (41)\\
\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi^n_k(x)\right)\,\psi^n_p(x)} &= \frac{2^{-n}}{2\pi}\mathrm{e}^{-i\frac{\omega p}{2^n}}(i)^\ell\left[\aleph\left(\frac{\omega}{2^n}-2\pi\right)\int_{2^{n+1}\pi}^{\omega-2^{n+1}\pi}(\omega-\xi)^\ell \mathrm{e}^{i\xi\frac{p-k}{2^n}}\mathrm{d}\xi + \right.\\
&\left. + \aleph\left(\frac{\omega}{2^n}-4\pi\right)\int_{\omega-2^{n+2}\pi}^{2^{n+2}\pi}(\omega-\xi)^\ell \mathrm{e}^{i\xi\frac{p-k}{2^n}}\mathrm{d}\xi\right]. && (42)
\end{aligned}
$$

□

Theorem 3.3. *For given $s,\, n,\, k,\, p\, \in \mathbb{Z}$ and $\ell,\, j\, \in \mathbb{N}$, it is*

$$\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi_\varepsilon^{\mathrm{m}}(x)\right)\frac{\mathrm{d}^j}{\mathrm{d}x^j}}\left(\psi_\theta^{\mathrm{M}}(x)\right)=\widehat{\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left(\psi_\theta^{\mathrm{M}}(x)\right)\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}}\left(\psi_\varepsilon^{\mathrm{m}}(x)\right)=$$
$$=\frac{2^{-\frac{\mathrm{m}+\mathrm{M}}{2}}}{2\pi}\mathrm{e}^{-i\frac{\omega\varepsilon}{2^{\mathrm{m}}}}i^{\ell+j}\left\{\aleph\left(\frac{\omega-2^{\mathrm{M}+1}\pi}{2^{\mathrm{m}}}\right)\int_{2^{\mathrm{M}+1}\pi}^{\omega-2^{\mathrm{m}+1}\pi}(\omega-\tau)^\ell\,\tau^j\mathrm{e}^{i\tau\left(\frac{\varepsilon}{2^{\mathrm{m}}}-\frac{\theta}{2^{\mathrm{M}}}\right)}\mathrm{d}\tau+\right.$$
$$+\aleph\left(\frac{\omega-2^{\mathrm{M}+2}\pi}{2^{\mathrm{m}}}\right)\int_{\omega-2^{\mathrm{m}+2}\pi}^{2^{\mathrm{M}+2}\pi}(\omega-\tau)^\ell\,\tau^j\mathrm{e}^{i\tau\left(\frac{\varepsilon}{2^{\mathrm{m}}}-\frac{\theta}{2^{\mathrm{M}}}\right)}\mathrm{d}\tau+\left[\aleph\left(\frac{\omega}{2^{\mathrm{m}}+2^{\mathrm{M}}}\right)-\right.$$
$$\left.\left.-\aleph\left(\frac{\omega-2^{\mathrm{M}+1}\pi}{2^{\mathrm{m}}}\right)-\aleph\left(\frac{\omega-2^{\mathrm{M}+2}\pi}{2^{\mathrm{m}}}\right)\right]\int_{\omega-2^{\mathrm{m}+2}\pi}^{\omega-2^{\mathrm{m}+1}\pi}(\omega-\tau)^\ell\,\tau^j\mathrm{e}^{i\tau\left(\frac{\varepsilon}{2^{\mathrm{m}}}-\frac{\theta}{2^{\mathrm{M}}}\right)}\mathrm{d}\tau\right\}. \tag{43}$$

where $\mathrm{m}=\frac{s+n-|s-n|}{2}$, $\mathrm{M}=\frac{s+n+|s-n|}{2}$ *and* $\varepsilon,\,\theta\,\in\{k,\,p\}$.

Proof. It follows imediately according to Theorem 3.1 and Theorem 3.2. Moreover, for $s=n$, we obtain the following

$$\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi_p^n(x)\right)\frac{\mathrm{d}^j}{\mathrm{d}x^j}}\left(\psi_k^n(x)\right)=$$
$$=\frac{2^{-n}}{2\pi}\mathrm{e}^{-i\frac{\omega k}{2^n}}i^{\ell+j}\left[\aleph\left(\frac{\omega}{2^n}-2\pi\right)\int_{2^{n+1}\pi}^{\omega-2^{n+1}\pi}(\omega-\tau)^\ell\,\tau^j\mathrm{e}^{i\tau\frac{k-p}{2^n}}\mathrm{d}\tau+\right.$$
$$\left.+\aleph\left(\frac{\omega}{2^n}-4\pi\right)\int_{\omega-2^{n+2}\pi}^{2^{n+2}\pi}(\omega-\tau)^\ell\,\tau^j\mathrm{e}^{i\tau\frac{k-p}{2^n}}\mathrm{d}\tau\right] \tag{44}$$

respectively, by the change of variable $\tau\to\omega-\xi$

$$\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi_p^n(x)\right)\frac{\mathrm{d}^j}{\mathrm{d}x^j}}\left(\psi_k^n(x)\right)=$$
$$=\frac{2^{-n}}{2\pi}\mathrm{e}^{-i\frac{\omega p}{2^n}}i^{\ell+j}\left[\aleph\left(\frac{\omega}{2^n}-2\pi\right)\int_{2^{n+1}\pi}^{\omega-2^{n+1}\pi}\xi^\ell\,(\omega-\xi)^j\,\mathrm{e}^{i\xi\frac{p-k}{2^n}}\mathrm{d}\xi+\right.$$
$$\left.+\aleph\left(\frac{\omega}{2^n}-4\pi\right)\int_{\omega-2^{n+2}\pi}^{2^{n+2}\pi}\xi^\ell\,(\omega-\xi)^j\,\mathrm{e}^{i\xi\frac{p-k}{2^n}}\mathrm{d}\xi\right] \tag{45}$$

□

Thus, taking account by (20), (21, (22)

$$\Phi^{snr}_{pkh} = \frac{1}{2\pi}\left\langle \widehat{\psi^s_p(x)\,\psi^n_k}(x)\,,\, 2^{-\frac{r}{2}}\mathrm{e}^{-i\frac{\omega h}{2^r}}\aleph\left(\frac{\omega}{2^r}\right)\right\rangle \tag{46}$$

$$\Gamma^{(\ell)snr}_{\quad pkh} = \frac{1}{2\pi}\left\langle \widehat{\psi^s_p(x)\,\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi^n_k(x)\right)}\,,\, 2^{-\frac{r}{2}}\mathrm{e}^{-i\frac{\omega h}{2^r}}\aleph\left(\frac{\omega}{2^r}\right)\right\rangle \tag{47}$$

$$\Delta^{(\ell,\,r)snj}_{\quad pkh} = \frac{1}{2\pi}\left\langle \widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(\psi^s_p(x)\right)\,\frac{\mathrm{d}^j}{\mathrm{d}x^j}\left(\psi^n_k(x)\right)}\,,\, 2^{-\frac{r}{2}}\mathrm{e}^{-i\frac{\omega h}{2^r}}\aleph\left(\frac{\omega}{2^r}\right)\right\rangle \tag{48}$$

from where, according to the Theorems 3.1-3.3, we finally get the coefficients (16), (17) and (18), respectively.

However, in particular, we can easily get simple expressions for the connection coefficients. With a direct (numerical) computation we have, e.g.,

$$\Phi^{000}_{000} = \frac{1}{4\pi^2}\left\langle \aleph\left(\omega-2\pi\right)\left(\omega-4\pi\right)+\aleph\left(\omega-4\pi\right)\left(8\pi-\omega\right),\,\aleph\left(\omega\right)\right\rangle = 0$$

$$\Gamma^{(1)000}_{\quad 000} = \frac{i}{4\pi^2}\left\langle \aleph\left(\omega-2\pi\right)\int_{2\pi}^{\omega-2\pi}\tau\mathrm{d}\tau+\aleph\left(\omega-4\pi\right)\int_{\omega-4\pi}^{4\pi}\tau\mathrm{d}\tau,\,\aleph\left(\omega\right)\right\rangle = 0$$

$$\begin{aligned}\Delta^{(1,\,1)000}_{\quad 000} = -\frac{1}{4\pi^2}\Big\langle \aleph\left(\omega-2\pi\right)\int_{2\pi}^{\omega-2\pi}\left(\omega-\tau\right)\tau\mathrm{d}\tau+ \\ +\aleph\left(\omega-4\pi\right)\int_{\omega-4\pi}^{4\pi}\left(\omega-\tau\right)\tau\mathrm{d}\tau,\,\aleph\left(\omega\right)\Big\rangle = 0\,.\end{aligned}$$

4. Conclusions

In general, the computation of the connection coefficients is not always an accessible way. Therefore, the application of the Petrov-Galerkin method for the wavelet approximation of solution of PDE depends on the connection coefficients. It should be noticed that it is not possible to be given a simple formula for their explicit form, because it involves many difficult computations. However, in particular, we can easily get simple expressions for the connection coefficients.

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