A NEW THREE-STEP ITERATIVE ALGORITHM FOR SOLVING THE SPLIT FEASIBILITY PROBLEM

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In this paper, we propose a new three-step iterative algorithm for solving the split feasibility problem in Hilbert space. Under proper assumptions, the sequence generated by the new iterative algorithm converges strongly to a solution of the SFP. Convergence rate of our algorithm is faster than previously existing iterative algorithms. To illustrate the effectiveness of our algorithm, we provide some numerical results.

Keywords: Three-step iterative; Strong convergence; Hilbert space

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces, C and Q be closed, convex, and nonempty subsets of H_1 and H_2, respectively. And let $A : H_1 \to H_2$ be a bounded and linear operator. The split feasibility problem (abbreviate SFP) can be mathematically described by finding a point x in C such that

$$x \in C, \ Ax \in Q. \tag{1.1}$$

The SFP was first proposed by Censor and Elfving [5] for solving a class of inverse problems. Recently, since the SFP is widely applied in medical image reconstruction [9, 10], the intensity-modulated radiation therapy [6, 7] and signal processing [3], it has gained extremely attention.

There are various algorithms to solve the SFP, see [3, 4, 6, 13, 18, 19] and the references therein. Particularly, Byrne [4] presented a CQ-algorithm, for which the iterative step x_k is formulated as follows:

$$x_{k+1} = P_C[I - \gamma A^*(I - P_Q)A]x_k, \ k \geq 0, \tag{1.2}$$

where $0 < \gamma < \frac{2}{\|A\|^2}$, P_C and P_Q denote the projections onto sets C and Q, respectively, and $A^* : H_2^* \to H_1^*$ is the adjoint of A. Due to the CQ-algorithm's own virtues—simple calculation, it has become a practical tool to solve the SFP, and various versions of the CQ-algorithm have been applied in many literature, such as [13, 18], etc.

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The three-step iterative was first introduced by Noor to consider the approximate solutions of variation inclusions in Hilbert space [11]. It is more valid than one-step and two-step iterative methods for solving the problems of pure and applied sciences [2]. Recently, three-step iterative has been used to solve the SFP, and it has gained great efficiency, such as Dang's [8]. He introduced the following three-step iterative algorithm:

$$\begin{cases} \omega_n = (1-\alpha_n)x_n + \alpha_n P_C[(1-\lambda_n)U]x_n, \\ \upsilon_n = (1-\beta_n)x_n + \beta_n P_C[(1-\lambda_n)U]\omega_n, \\ x_{n+1} = (1-\gamma_n)x_n + \gamma_n P_C[(1-\lambda_n)U]\upsilon_n, \end{cases} \tag{1.3}$$

where $U = I - \gamma A^*(I - P_Q)A$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ are real sequences in (0,1). In addition, Mihai [15] introduced a new three-step iterative method for finding fixed points of nonexpansive mapping.

Inspired by the above works, we combine the idea of Mihai's three-step iterative with Byrne's CQ-algorithm for solving the SFP (1.1). This is the core part of this paper. The structure of the present paper is as follows. In Section 2, we provide some concepts and lemmas that will be very useful for our convergence analysis. In Section 3, we propose the three-step iterative method and prove its convergent results. In Section 4, we illustrate that our algorithm is effective by some numerical results. In the last part, we summarize this paper.

2. **Preliminaries**

For the sake of convenience, we present some notations used in this paper. Let H be a real Hilbert space, its inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. I denotes the identity operator in H. $F(T)$ denotes the fixed points of T, i.e., $F(T) = \{x \in H : Tx = x\}$. $x_n \rightharpoonup x$ and $x_n \to x$ denote sequence $\{x_n\}$ converges weakly and strongly to x, respectively. In this paper, we assume that the solution set Ω of the SFP (1.1) is nonempty, let

$$\Omega = \{x \in C : Ax \in Q\} = C \cap A^{-1}Q,$$

then, Ω is closed, convex, and nonempty set.

In addition, let C be a closed, convex, and nonempty subset of Hilbert space H, for $x \in H$, P_C and $d(x,C)$ denote the orthogonal projection from x onto C and metric distance from x onto C, respectively, which are defined by

$$P_C(x) := \arg\min_{y\in C} \|x-y\| \ \textit{and}\ d(x,C) := \inf\{\|x-y\| : y \in C\}.$$

The following lemma presents some important properties of the orthogonal projection operator, in which (i) is taken from [1, theorem 3.14]; (ii) and (iii) from [1, proposition 4.8].

Lemma 2.1. ([1]) *Let C be a closed, convex, and nonempty subset of H, then for any $x, y \in H$ and $z \in C$,*

(i) $\langle x - P_Cx, z - P_Cx\rangle \le 0$;

(ii) $\|P_Cx - P_Cy\|^2 \le \langle P_Cx - P_Cy, x - y\rangle$;

(iii) $\|P_Cx - z\|^2 \le \|x - z\|^2 - \|P_Cx - x\|^2$.

Definition 2.1. Let $T : H \to H$ be an operator, then

(i) T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall \ x, y \in H;$$

(ii) T is averaged if

$$T = (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$, and $S : H \to H$ is nonexpansive;

(iii) T is ν-inverse strongly monotone(ν-ism), with $\nu > 0$, if

$$\langle x - y, Tx - Ty\rangle \geq \nu\|Tx - Ty\|^2, \ \forall \ x, y \in H;$$

(iv) T is λ-Lipschitz continuous, with $\lambda > 0$, if

$$\|Tx - Ty\| \leq \lambda\|x - y\|, \ \forall \ x, y \in H;$$

(v) T is firmly nonexpensive, if

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \ \forall \ x, y \in H.$$

Lemma 2.2. (*Lemma 2.1,* [3]) *An operator* U *is averaged if and only if its complement* $V = I - U$ *is* ν*-ism with* $\nu > \frac{1}{2}$.

Lemma 2.3. (*Lemma 1,* [12]) *Let* $\{x_n\}$ *be a sequence of Hilbert space* H*. If* $\{x_n\}$ *converges weakly to* x*, then for any* $y \in H$ *and* $y \neq x$*, we have* $\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|$.

Lemma 2.4. (*Demiclosed principle*)(*Lemma 2,* [12]) *Let* C *be a closed, convex, and nonempty subset of real Hilbert space* H*, and* $T : C \to C$ *be a nonexpansive mapping. Then* $I - T$ *is demiclosed at zero, i.e., if* $x_k \rightharpoonup x \in C$ *and* $x_k - Tx_k \to 0$*, then* $x = Tx$.

Lemma 2.5. (*Lemma 1.3,* [14]) *Let* X *be a uniformly convex Banach space and* $0 < p \leq t_n \leq q < 1$ *for all* $n \in N$*. Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of* X *such that* $\lim_{n\to\infty} \sup \|x_n\| \leq r$, $\lim_{n\to\infty} \sup \|y_n\| \leq r$ *and* $\lim_{n\to\infty} \sup \|t_n x_n + (1 - t_n)y_n\| = r$ *hold for some* $r \geq 0$*. Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3. The three-step iterative algorithm and its convergence analysis

Now, we propose our three-step iterative algorithm.

Algorithm 3.1. *For an arbitrarily initial point* $x_0 \in H_1$*, the sequence* $\{x_n\}$ *is generated by*

$$\begin{cases} u_n = (1 - \alpha_n)x_n + \alpha_n Tx_n, \\ v_n = (1 - \beta_n)u_n + \beta_n Tu_n, \\ x_{n+1} = (1 - \gamma_n)Tu_n + \gamma_n Tv_n, \end{cases} \tag{3.1}$$

where $T = P_C[I - \gamma A^*(I - P_Q)A]$*, and* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$ *are three real sequences in* $(0, 1)$.

Remark 3.1. *Since the solution set of the SFP* (1.1) *is nonempty, it is not hard to find that* $x^* \in C$ *solves* (1.1) *if and only if it solves the fixed point equation:*

$$P_C[I - \gamma A^*(I - P_Q)A]x = x, \ x \in C.$$

Then, the solution set of the SFP (1.1) *is equal to fixed points of* T*, i.e.,* $F(T) = \Omega = C \cap A^{-1}Q \neq \emptyset$*. Concrete detail can be found in* [16, 17].

Lemma 3.1. *Let operator* $T = P_C[I - \gamma A^*(I - P_Q)A]$*, where* $0 < \gamma < \frac{2}{\|A\|^2}$*. Then* T *is nonexpansive.*

Proof. Let $U = A^*(I - P_Q)A$. Firstly, we prove that U is L-Lipschitz continuous with $L = \|A\|^2$. In fact, for $\forall\ x, y \in C$, we have

$$\begin{aligned}\|Ux - Uy\|^2 =& \|A^*(I - P_Q)Ax - A^*(I - P_Q)Ay\|^2 \\ \leq& L\|(I - P_Q)Ax - (I - P_Q)Ay\|^2 \\ =& L\|Ax - Ay - (P_QAx - P_QAy)\|^2 \\ =& L(\|Ax - Ay\|^2 + \|P_QAx - P_QAy\|^2 \\ &- 2\langle Ax - Ay, P_QAx - P_QAy\rangle).\end{aligned}$$

By Lemma 2.1(ii), we obtain

$$\langle Ax - Ay, P_QAx - P_QAy\rangle \geq \|P_QAx - P_QAy\|^2.$$

Therefore,

$$\begin{aligned}\|Ux - Uy\|^2 &\leq L(\|Ax - Ay\|^2 - \|P_QAx - P_QAy\|^2) \\ &\leq L\|Ax - Ay\|^2 \\ &\leq L^2\|x - y\|^2.\end{aligned}$$

Then, U is L-Lipschitz continuous, which means that U is $\frac{1}{L}$-ism. Hence, γU is $\frac{1}{L}$-ism.

Next, we show that T is nonexpensive. By Lemma 2.2, $V = I - \gamma U$ is averaged mapping. Then, $V = (1 - t)I + tS$, where $t \in (0, 1)$, $S : C \to C$ is nonexpansive. Taking $x, y \in C$, we have

$$\begin{aligned}\|Vx - Vy\| &= \|(1 - t)x + tSx - (1 - t)y - tSy\| \\ &\leq (1 - t)\|x - y\| + t\|Sx - Sy\| \\ &\leq (1 - t)\|x - y\| + t\|x - y\| \\ &= \|x - y\|.\end{aligned}$$

Thus, V is nonexpensive mapping. Note that $T = P_CV$, P_C and V are both nonexpensive. Consequently, T is nonexpansive mapping. The proof is completed.

Lemma 3.2. *Let* $\{x_n\}$ *be the sequence generated by Algorithm 3.1. Then,* $\lim_{n\to\infty} \|x_n - x^*\|$ *exists for any* $x^* \in F(T)$*.*

Proof. Taking a point $x^* \in F(T)$. Since T is nonexpensive, by (3.1), for all $n \in N$, we have

$$\begin{aligned}\|u_n - x^*\| &= \|(1 - \alpha_n)x_n + \alpha_nTx_n - x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|Tx_n - x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^*\| \\ &= \|x_n - x^*\|,\end{aligned}$$

i.e.,

$$\|u_n - x^*\| \leq \|x_n - x^*\|. \tag{3.2}$$

Similarly, we obtain

$$\|v_n - x^*\| \leq \|x_n - x^*\|. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1-\gamma_n)Tu_n + \gamma_n Tv_n - x^*\| \\ &\leq (1-\gamma_n)\|Tu_n - x^*\| + \gamma_n\|Tv_n - x^*\| \\ &\leq (1-\gamma_n)\|u_n - x^*\| + \gamma_n\|v_n - x^*\| \\ &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\ &= \|x_n - x^*\|.\end{aligned}$$

Since x^* is chosen arbitrarily in $F(T)$, one deduces that $\{\|x_n - x^*\|\}_n$ is decreasing, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for any $x^* \in F(T)$. The proof is completed.

Lemma 3.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. By Lemma 3.2, $\lim_{n\to\infty} \|x_n - x^*\|$ exists for any $x^* \in F(T)$. Suppose that

$$\lim_{n\to\infty} \|x_n - x^*\| = a(a \geq 0). \tag{3.4}$$

By (3.2) and (3.3), we have

$$\lim_{n\to\infty} \sup \|u_n - x^*\| \leq a, \tag{3.5}$$

and

$$\lim_{n\to\infty} \sup \|v_n - x^*\| \leq a. \tag{3.6}$$

Since T is nonexpensive mapping, we obtain

$$\|Tx_n - x^*\| \leq \|x_n - x^*\|,\ \|Tu_n - x^*\| \leq \|u_n - x^*\|,\ \|Tv_n - x^*\| \leq \|v_n - x^*\|.$$

Taking the superior limit on both sides, we get

$$\lim_{n\to\infty} \sup \|Tx_n - x^*\| \leq a, \tag{3.7}$$

$$\lim_{n\to\infty} \sup \|Tu_n - x^*\| \leq a, \tag{3.8}$$

and

$$\lim_{n\to\infty} \sup \|Tv_n - x^*\| \leq a. \tag{3.9}$$

Since

$$a = \lim_{n\to\infty} \|x_{n+1} - x^*\| = \lim_{n\to\infty} \|(1-\gamma_n)(Tu_n - x^*) + \gamma_n(Tv_n - x^*)\|, \tag{3.10}$$

combining (3.8), (3.9) and (3.10), from Lemma 2.5, we infer that

$$\lim_{n\to\infty} \|Tu_n - Tv_n\| = 0.$$

Now

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1-\gamma_n)(Tu_n - x^*) + \gamma_n(Tv_n - x^*)\| \\ &\leq \|Tu_n - x^*\| + \gamma_n\|Tu_n - Tv_n\|,\end{aligned}$$

which implies that

$$a \leq \lim_{n\to\infty} \inf \|Tu_n - x^*\|. \tag{3.11}$$

From (3.8) and (3.11), we obtain

$$\lim_{n\to\infty} \|Tu_n - x^*\| = a.$$

Moreover,

$$\begin{aligned}\|Tu_n - x^*\| &\le \|Tu_n - Tv_n\| + \|Tv_n - x^*\| \\ &\le \|Tu_n - Tv_n\| + \|v_n - x^*\|,\end{aligned}$$

which implies that

$$a \le \lim_{n\to\infty} \inf \|v_n - x^*\|. \tag{3.12}$$

Combining (3.6) and (3.12), we obtain

$$\lim_{n\to\infty} \|v_n - x^*\| = a.$$

Since T is nonexpensive, by Lemma 2.4, we get

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0.$$

Due to

$$\begin{aligned}\|v_n - x^*\| &= \|(1-\beta_n)u_n + \beta_n Tu_n - x^*\| \\ &= \|(u_n - x^*) + \beta_n(Tu_n - u_n)\| \\ &\le \|u_n - x^*\| + \beta_n\|Tu_n - u_n\|,\end{aligned}$$

we have

$$a \le \lim_{n\to\infty} \inf \|u_n - x^*\|. \tag{3.13}$$

According to (3.5) and (3.13), we obtain

$$\lim_{n\to\infty} \|u_n - x^*\| = a,$$

hence,

$$\begin{aligned}a &= \lim_{n\to\infty} \|u_n - x^*\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)x_n + \alpha_n Tx_n - x^*\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Tx_n - x^*)\|,\end{aligned}$$

that is,

$$\lim_{n\to\infty} \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Tx_n - x^*)\| = a. \tag{3.14}$$

Combining (3.4), (3.7) and (3.14), from Lemma 2.5, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

The proof is completed.

Theorem 3.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then $\{x_n\}$ converges weakly to a point in Ω.*

Proof. By Remark 3.1, $\Omega = F(T) \neq \emptyset$. Hence, we only need to show that the sequence $\{x_n\}$ converges weakly to a point in $F(T)$.

Taking $x^* \in F(T)$, by Lemma 3.2, $\lim_{n\to\infty} \|x_n - x^*\|$ exists.

First, we show that the subsequences of $\{x_n\}$ only have a weak limit in $F(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$, the weak limits of $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are denoted by u and v, respectively. By Lemma 3.3, we have $\lim_{n\to\infty} \|x_{n_i} - Tx_{n_i}\| = 0$. By Lemma 2.4, $I - T$ is demiclosed at zero. Hence, we gain $Tu = u$, i.e., $u \in F(T)$. Similarly, we can prove that $v \in F(T)$.

Next, we show the uniqueness of weak limit. Since $T = P_C[I - \gamma A^*(I - P_Q)A]$ is nonexpansive mapping, by Lemma 3.2, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Suppose that $u \neq v$, according to Lemma 2.3, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{n_i\to\infty} \|x_{n_i} - u\| < \lim_{n_i\to\infty} \|x_{n_i} - v\| \\ &= \lim_{n\to\infty} \|x_n - v\| = \lim_{n_j\to\infty} \|x_{n_j} - v\| \\ &< \lim_{n_j\to\infty} \|x_{n_j} - u\| = \lim_{n\to\infty} \|x_n - u\|.\end{aligned}$$

This is clearly contradictory, hence, $u = v$. Therefore, $\{x_n\}$ converges weakly to a point in $F(T)$, that is, the sequence $\{x_n\}$ converges weakly to a point in Ω. The proof is completed.

Theorem 3.2. *Let $\{x_n\}$ be the sequence defined by Algorithm 3.1. Then $\{x_n\}$ converges to a point in Ω if and only if $\lim_{n\to\infty} \inf d(x_n, \Omega) = 0$.*

Proof. Obviously, necessity is true. We only need to prove sufficiency.

Since $\lim_{n\to\infty} \inf d(x_n, \Omega) = 0$. From Remark 3.1, we have $F(T) = \Omega \neq \emptyset$. Hence $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$. For any $x^* \in F(T)$, $\lim_{n\to\infty} \|x_n - x^*\|$ exists by Lemma 3.2, thus, $\lim_{n\to\infty} d(x_n, F(T))$ exists and $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next, we prove that $\{x_n\}$ is a Cauchy sequence in C. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for any $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$, $d(x_n, F(T)) < \frac{\varepsilon}{2}$. Meanwhile,

$$\inf\{\|x_{n_0} - x^*\| : x^* \in F(T)\} < \frac{\varepsilon}{2},$$

therefore, there exists $\overline{x} \in F(T)$ such that $\|x_{n_0} - \overline{x}\| < \frac{\varepsilon}{2}$. For $m, n \geq n_0$, we have

$$\|x_n - x_m\| \leq \|x_n - \overline{x}\| + \|x_m - \overline{x}\|.$$

In addition, from the proof of Lemma 3.2, we know that $\|x_n - x^*\|$ is decreasing for n, then

$$\|x_n - x_m\| \leq 2\|x_{n_0} - \overline{x}\| < \varepsilon,$$

which yields that $\{x_n\}$ is a Cauchy sequence in C.

Note that C is a closed subset in H_1, hence, there exists $\hat{x} \in C$ such that $x_n \to \hat{x}$. From $\lim_{n\to\infty} d(x_n, F(T)) = 0$, one deduces that $d(\hat{x}, F(T)) = 0$. Since $F(T)$ is closed set, we have $\hat{x} \in F(T)$. Again, using $F(T) = \Omega$, we obtain $\hat{x} \in \Omega$. Hence, $\{x_n\}$ converges to a point in Ω. The proof is completed.

Theorem 3.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. If there exists a nondecreasing function $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$, $f(r) > 0$, for any $r \in (0, +\infty)$,* *such that* $\|x - Tx\| \geq f(d(x, F(T))$*, for all* $x \in C$*, then* $\{x_n\}$ *converges strongly to a point in* Ω*.*

Proof. From Lemma 3.3, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

According to the assumption of T, we obtain

$$\lim_{n\to\infty} f(d(x_n, F(T)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Since $f : [0, +\infty) \to [0, +\infty)$ satisfies $f(0) = 0$, $f(r) > 0$, for any $r \in (0, +\infty)$, we can deduce that

$$\lim_{n\to\infty} d(x_n, F(T)) = 0,$$

by Remark 3.1, we have $F(T) = \Omega$, which implies that

$$\lim_{n\to\infty} d(x_n, \Omega) = 0.$$

It follows from Theorem 3.2 that $\{x_n\}$ converges strongly to a point in Ω. The proof is completed.

4. **Numerical experiment**

In this section, we provide a concrete example including numerical results and compare Algorithm 3.1 with Dang's [8] algorithm (i.e., (1.3)) to declare that our algorithm is more effective. All codes were written in Matlab 2012b.

Example 4.1. *Let* $H_1 = H_2 = R^3$, $C = \{x \in R^3 : \|x\| \leq 1\}$, $Q = \{x \in R^3 : \|x\| \leq 2\}$ *and take*

$$A = \begin{pmatrix} -3 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix}.$$

Then the projections P_C *and* P_Q *of* x *onto sets* C *and* Q *are as follows:*

$$P_C(x) = \begin{cases} x, \ \|x\| \leq 1 \\ \frac{x}{\|x\|}, \ \|x\| \geq 1 \end{cases}$$

and

$$P_Q(x) = \begin{cases} x, \ \|x\| \leq 2 \\ \frac{2}{\|x\|}x, \ \|x\| \geq 2. \end{cases}$$

Meanwhile, choose $\alpha_n = \frac{1}{3}$, $\beta_n = \frac{1}{3}$, $\gamma_n = \frac{1}{3}$, *and* $\gamma = 0.01$ *in (1.3) and (3.1). And* $\lambda_n = 0.03$ *in Dang's (1.3). Take an initial point* $x_0 = \{2, 1, 0\}$*. We take* $\|x_{n+1} - x_n\| < 10^{-6}$ *as the standard of stopping in the process of calculation.*

In the following table, n, t and $a = \|x_{n+1} - x_n\|$ denote iterative steps, CPU time and error, respectively. After the calculation, we can compare our results with Dang's as follows:

From the above table, we can find that, under the same conditions, the results of our algorithm are superior to Dang's. In short, the results of numerical experiment show that our algorithm is more efficient than Dang's.

	n	t	a
sequence (3.1)	238	0.031250	0.0000099
Dang's (1.3)	514	0.093750	0.0000100

5. **Conclusions**

We propose a new three-step iterative algorithm to solve the split feasibility problem. Under proper assumptions, our algorithm can converge strongly to a solution of the split feasibility problem (1.1). Numerical results show the effectiveness of our algorithm.

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