NONLINEAR VARIATIONAL INEQUALITY PROBLEMS INVOLVING UNCERTAIN VARIABLE

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Variational inequality problems have been extensively studied and applied in the area of optimization theory, but much of it has been developed in a deterministic domain. In this paper, the uncertain variational inequality is studied. Under the compact uncertain event space, we establish an uncertain nonlinear expected residual minimization (UNERM) model. Considering that uncertain variable usually has no density function, we propose integration by parts method to solve UNERM model. Under reasonable assumptions and Slater's constraint qualification, two kinds of convergence, such as convergence of global optimal solutions and convergence of stationary points, are investigated.

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1. Introduction

A nonempty closed convex set $S \in R^n$ and a mapping $f : S \to R^n$, in a deterministic environment, the classical variational inequality problem (denoted by VIP(f, S) below) is to find a vector $x^* \in S$ such that

$$(x - x^*)^T f(x^*) \geq 0, x \in S.$$

Variational inequality problem is one of the basic problems in the optimization theory. In 2010, Yao et al. [1] Studied minimum-norm solutions of variational inequalities. In 2012, Postolache et al. [2] introduced an iterative scheme to find a common element of the set of solution of a pseudomonotone, Lipschitz-continuous variational inequality problem and fixed points. In 2013, Yao et al. [3] put forward composite schemes for variational inequalities over equilibrium problems and variational inclusions. Postolache et al. [4] studied a new class of generalized extended nonlinear quasi-variational inequality problems involving set-valued relaxed monotone operators and proved it equivalent to the fixed point problem. Postolache et al. [5] proposed a variant extra gradient-type method to solve monotone variational inequalities. In 2016, Postolache et al. [6] constructed algorithms for a class of monotone variational inequalities. In 2017, Yao et al. [7] considered the split variational inequality and fixed point problem. In the same year, Yao et al. [8] considered the split variational inequality problem under a nonlinear transformation. However, there are many examples in reality that the basic problem not only involves deterministic data, but also contains some stochastic data. For example random signal process, supply chain, etc. Therefore, in order

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to reflect those stochastic phenomenons, many researchers extend to the stochastic variational inequality problem (SVIP for short). SVIP was investigated involving many aspects. In 2008, Jiang et al. [9] applied stochastic approximation approaches to the stochastic variational inequality problems. In the same year, Shapiro et al. [10] discussed the sample average approximation method applied to a class of stochastic mathematical programs with variational (equilibrium) constraints. In 2010, Xu [11] further investigated sample average approximation methods for a class of stochastic variational inequality problems. In 2011, Luo et al. [12] did the research about stochastic variational inequality problems with additional constraints. Unlucky, there are some circumstances when no historical data are available in many practical problems of the real world, many examples inevitably contain some uncertain data, such as a new stock, emergencies, devastating military experiments, etc. In computer science, uncertain data is the data that contains noise that makes it deviate from the correct, intended or original values. In uncertainty theory, uncertain data is the data that contains belief degree representing the strength with which we believe the event will happen. It is usually described as uncertain variable. Hence, in order to rationally deal with belief degree, uncertainty theory was founded by Liu [14], subsequently the uncertain variational inequality problem was proposed by Chen and Zhu [15]. In their paper,they propose a new class of variational inequality problems, which is to find $x^* \in S$

$$\mathscr{M}\{\gamma \in \Gamma | (x - x^*)^T F(x^*, \xi(\gamma)) \geq 0, \forall x \in S\} = 1, \tag{1}$$

where $\xi(\gamma)$ is the uncertain variable, Γ is a nonempty set, $F : R^n \times \Gamma \to R^n$ is a mapping. They solve expected value model based on uncertainty theory.
In a recent work [16], a new model presented by Li and Jia minimized an expected residual given by the regularized gap function based on uncertainty theory. The model is written as

$$\min\ \theta(x) := E[g(x, \xi)] = \int_T g(x, t) d\Phi(t),\ s.t.\ x \in S, \tag{2}$$

here, $\xi \in B$, E stands for the expectation with respect to the uncertain variable ξ, $\Phi(t)$ stands for the uncertain distribution function with respect to the uncertain variable ξ. T stands for domain of $\Phi(t)$. Recall that, for any $x \in R^n$ and any $\xi \in B$,

$$g(x, \xi) = (x - H(x, \xi))^T F(x, \xi) - \frac{\alpha}{2} \|x - H(x, \xi)\|_G^2, \tag{3}$$

where

$$H(x, \xi) := Proj_{S.G}(x - \alpha^{-1} G^{-1} F(x, \xi))$$

$F : R^n \times R \to R^n$ is a mapping. α is a positive parameter, G is an $n \times n$ symmetric positive-definite matrix, and $\|.\|_G$ means the G-norm defined by $\|x\|_G = \sqrt{x^T G x}$ for $x \in R^n$.

In [16], Li and Jia considered an linear uncertain variational inequality problem. The properties and convergence analysis of the ERM problem were discussed. Integration by parts method is proposed to solve (2). The purpose of this paper is to introduce UNERM model for dealing with nonlinear uncertain variational inequality problem.

The paper is organized as following. We recall some preliminary results about uncertainty theory and other preliminaries in Section 2. Then, the convergence of global optimal solutions and convergence of stationary points of UNERM model are discussed in Section 3.

2. **Preliminaries**

Definition 2.1. *(Liu* [14]*) Suppose ξ is an uncertain variable. Then the uncertainty distribution of ξ is defined by*

$$\Phi(t) = \mathscr{M}\{\xi \leq t\}$$

for any real number t.

Definition 2.2. *(Liu [14]) Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathscr{M}\{\xi \geq t\}dt - \int_{-\infty}^0 \mathscr{M}\{\xi \leq t\}dt$$

provided that at least one of the two integrals is finite.

Theorem 2.1. *(Liu [14]) Let ξ be an uncertain variable with uncertainty distribution Φ. If the expected value exists, then*

$$E[\xi] = \int_{-\infty}^{+\infty} t d\Phi(t).$$

Corollary 2.1. *(Liu [14]) Let ξ be an uncertain variable with uncertainty distribution Φ. and let f(t) be a strictly monotone function , then we have*

$$E[f(\xi)] = \int_{-\infty}^{+\infty} f(t) d\Phi(t).$$

Following from the continuity of $(F, \nabla_x F)$ and Theorem 4.2 of [16] that the function θ is continuously differentiable over S and

$$\nabla\theta(x) = E[\nabla_x g(x, \xi)], \forall x \in S. \tag{4}$$

Uncertain variable usually has no density function, so $\Phi(x)$ usually is not differentiable in uncertainty theory. But we can use the differentiable properties of $g(x, \xi)$. By the results given in [17], $g(\cdot, \xi)$ is continuously differentiable over S for any $\xi \in B$, and

$$\nabla_x g(x, \xi) = F(x, \xi) - (\nabla_x F(x, \xi) - \alpha G)(H(x, \xi) - x) \tag{5}$$

Definition 2.3. [16] *Let $\theta_k(x)$ minimum be as follows:*

$$\min\ \theta_k(x) = g(x, t)\Phi(t)|_{t\in T} - \frac{1}{N_k}\sum_{t_i\in T_k} \Phi(t_i)g'(x, t_i) \tag{6}$$

where $T_k = \{t_i | i = 1, 2, \cdots, N_k\}$ be a set of observations generated by [18] satisfying $N_k \to \infty$ as $k \to \infty$, we call T uncertain event space, theoretically T is domain of $\Phi(t)$, $g(x, t)\Phi(t)|_{t\in T}$ is a function merely related to x.

we will study the behavior of the following approximations to the ERM problem (2) as follows:

$$\theta_k(x) = g(x, t)\Phi(t)|_{t\in T} - \frac{1}{N_k}\sum_{t_i\in T_k} \Phi(t_i)g'(x, t_i),\ s.t.\ x \in S. \tag{7}$$

We consider the limiting behavior of problems (7) in latter section.

Theorem 2.2. *Through [18], we can get the following conclusion*

$$\lim_{k\to\infty} \frac{1}{N_k}\sum_{t_i\in T_k} \Phi(t_i)g'(x, t_i) = \int_T \Phi(t)g'(x, t)dt \tag{8}$$

For any $x \in R^n$, we also have

$$\sqrt{\lambda_{min}}\|x\| \leq \|x\|_G \leq \sqrt{\lambda_{max}}\|x\| \tag{9}$$

where λ_{min} and λ_{max} indicate the smallest and largest eigenvalues of the positive definite matrix G, respectively. For a given matrix A, we let $\|A\|$ denote its spectral norm and $\|A\|_{\mathscr{F}}$ denote its Frobenius matrix norm. The relationship between $\|A\|$ and $\|A\|_{\mathscr{F}}$, $\|x\|$ and $\|x\|_G$ are as follows,

$$\|A\| \leq \|A\|_{\mathscr{F}}. \tag{10}$$

$$\|A\|_{\mathscr{F}} \leq \sum_{j=1}^{n} \|A_j\|. \tag{11}$$

where A_j is the jth column vector of A.

3. Convergence Analysis

3.1. ERM Model Hypothesis

In this paper, we still consider model (2). It has been mentioned that the function F is supposed to be affine in [16]. Some assumptions such as square integrability and positive definiteness are given, see [16] for details. The difference of this article is that we suppose that F of (3) is a nonlinear function.

To prove the latter theorem, we need suppose that the uncertain distribution function $\Phi(t)$ is continuous on $t \in T$, three-order derivative of the function F exists (denoted by F'''_{xtx}). Owing to $H(x,t) := Proj_{S.G}(x - \alpha^{-1}G^{-1}F(x,t))$ and $H'(x,t) := Proj_{S.G}(x - \alpha^{-1}G^{-1}F'(x,t))$, it is easy to get H, H'_x and H'_t are continuous.

3.2. Convergence of Global Optimal Solutions

For convenience, we denote by S^* and S_k^* the sets of optimal solutions of problems (2) and (7), respectively.

Lemma 3.1. *For any fixed $x \in S$, it holds that $\theta(x) = \lim_{k\to\infty} \theta_k(x)$.*

Proof *It is known from the definition of $\theta(x)$ function of* (2) *and integration by parts method that*

$$\begin{aligned}\theta(x) = E[g(x,\xi)] &= \int_T g(x,t)d\Phi(t)\\ &= g(x,t)\Phi(t)|_{t\in T} - \int_T \Phi(t)g'(x,t)dt,\end{aligned}$$

Thus we can get the conclusion from (7) *and* (8) *easily.*

Theorem 3.1. *Assume that $x_k \in S_k^*$ for each sufficiently large k. and x^* is an accumulation point of $\{x_k\}$. Then, we have $x^* \in S^*$*

Proof *Let x^* be an accumulation point of $\{x_k\}$. Without loss of generality, we assume that $\{x_k\}$ converges to x^*. It is obvious that $x^* \in S$. We first show that*

$$\lim_{k\to\infty} (\theta_k(x_k) - \theta_k(x^*)) = 0.$$

Let $B \subset S$ be a compact convex set containing the sequence x_k. By the continuity of g''_{tx} on the compact set $B \times T$, there exists a constant $C > 0$ such that

$$\|g''_{tx}(y_{ki}, t_i)\| \leq C, \forall (x,\xi) \in B \times T. \tag{12}$$

Moreover, we have from the mean-value theorem that, for each x_k and each ξ_i, there exists $y^{ki} = \alpha_{ki}x_k + (1-\alpha_{ki})x^* \in B$ with $\alpha_{ki} \in [0,1]$ such that

$$|g'_t(x_k, t_i) - g'_t(x^*, t_i)| = |g''_{tx}(y_{ki}, t_i)^T(x_k - x^*)|.$$

So we have

$$\begin{aligned}|\theta_k(x_k) - \theta_k(x^*)| =& |g(x_k,t)\Phi(t)|_{t\in T} - \frac{1}{N_k}\sum_{t_i\in T_k} \Phi(t_i)g'_t(x_k,t_i)\\ &- g(x^*,t)\Phi(t)|_{t\in T} - \frac{1}{N_k}\sum_{t_i\in T_k} \Phi(t_i)g'_t(x^*,t_i)|\\ \leq& |g(x_k,t)\Phi(t)|_{t\in T} - g(x^*,t)\Phi(t)|_{t\in T}|\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)|(g'_t(x_k,t_i)-g'_t(x^*,t_i))|\\
=&|g(x_k,t)\Phi(t)|_{t\in T}-g(x^*,t)\Phi(t)|_{t\in T}|\\
&+\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t)|g''_{tx}(y_{ki},t_i)^T(x_k-x^*)|\\
\le&|g(x_k,t)\Phi(t)|_{t\in T}-g(x^*,t)\Phi(t)|_{t\in T}|\\
&+C\|(x_k-x^*)\|\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)
\end{aligned}
$$

Because the facts that the sequence $\{x_k\}$ converges to $\{x^*\}$, thus

$$|\theta_k(x_k)-\theta_k(x^*)|\xrightarrow{k\to\infty}0, \tag{13}$$

On the other hand, noting that

$$|\theta_k(x_k)-\theta(x^*)|\le|\theta_k(x_k)-\theta_k(x^*)|+|\theta_k(x^*)-\theta(x^*)|, \tag{14}$$

so we have from lemma 3.1 and (13) *that*

$$\lim_{k\to\infty}\theta_k(x_k)=\theta(x^*). \tag{15}$$

Since, for each sufficiently large k, $x_k\in S_k^*$, then exist $\epsilon>0$ such that

$$\theta_k(x_k)\le\theta_k(x)+\epsilon \tag{16}$$

holds for any $x\in S$. Letting $k\to+\infty$ in (16) *and taking* (15) *and lemma 3.1 into account, we get $\theta(x^*)\le\theta(x)+\epsilon,\forall x\in S$, which means $x^*\in S^*$.*

3.3. Convergence of Stationary Points

Theorem 3.2. *If $\lim_{k\to\infty}x_k=x^*$, then $\lim_{k\to\infty}\nabla\theta_k(x_k)=\nabla\theta(x^*)$.*

Proof *Let $B\subseteq S$ be a compact convex set containing the sequence x_k. By the continuity of F'_t,F''_{tx},H,H'_t, and $\nabla_x^2F_j$ on the compact set $B\times T$, there exists a constant $C\ge sup\{\|x_k\|,k=1,2,\dots\}$ such that, for any $(x,t)\in B\times T$,*

$$\|F'_t\|\le C, \tag{17}$$

$$\|F''_{tx}\|\le C, \tag{18}$$

$$\|F'''_{xtx}\|\le C, \tag{19}$$

$$\|H\|\le C,j=1,\cdots,n, \tag{20}$$

$$\|H'_t\|\le C,j=1,\cdots,n, \tag{21}$$

where F''_{tx} denotes the derivative of $F(x,t)$ with respect to x,t. We first show that

$$\lim_{k\to0}\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)\|F'_t(x_k,t_i)-F'_t(x^*,t_i)\|=0. \tag{22}$$

In fact, from (10) *and* (11), *we have*

$$\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)\|F'_t(x_k,t_i)-F'_t(x^*,t_i)\| \tag{23}$$

$$\le\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)\|F'_t(x_k,t_i)-F'_t(x^*,t_i)\|_{\mathscr{F}} \tag{24}$$

$$\le \sum_{j=1}^{n} \frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) \| F'_t(x_k, t_i)_j - F'_t(x^*, t_i)_j \|. \tag{25}$$

Moreover, for each x_k, ξ_i and any fixed j, from the mean-value theorem, there exists $y_{kij} = \alpha_{kij} x_k + (1 - \alpha_{kij})\bar{x} \in B$ with $\alpha_{kij} \in [0, 1]$ such that

$$\frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) \| F'_t(x_k, t_i)_j - F'_t(x^*, t_i)_j \| \tag{26}$$

$$\le \frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) \| F''_{tx}(y_{kij}, t_i) \| \| x_k - x^* \| \tag{27}$$

$$\le C \| x_k - x^* \| \frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) \tag{28}$$

$$\xrightarrow{k \to \infty} 0, \tag{29}$$

where the second inequality follows from (18). (22) *holds immediately from* (23)-(29). *In a similar way, it holds that*

$$\lim_{k \to 0} \frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) \| F(x_k, t_i) - F(x^*, t_i) \| = 0 \tag{30}$$

and

$$\lim_{k \to 0} \frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) \| F'_x(x_k, t_i) - F'_x(x^*, t_i) \| = 0 \tag{31}$$

and

$$\lim_{k \to 0} \frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) \| F''_{xt}(x_k, t_i) - F''_{xt}(x^*, t_i) \| = 0. \tag{32}$$

It then follows from (9), (30) *and the non-expansive property of $Proj_{S,G}$ that*

$$\begin{aligned}
&\frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) \| H(x^*, t_i) - H(x_k, t_i) \| \\
\le& \lambda_{min}^{-\frac{1}{2}} \frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) \| H(x^*, t_i) - H(x_k, t_i) \|_G \\
\le& \lambda_{min}^{-\frac{1}{2}} \frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) \| (x^* - \alpha^{-1} G^{-1} F(x^*, t_i) \\
&- (x_k - \alpha^{-1} G^{-1} F(x_k, t_i) \|_G \\
\le& \lambda_{max}^{\frac{1}{2}} \lambda_{min}^{-\frac{1}{2}} \frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) (\| x_k - x^* \| \\
&+ \alpha^{-1} \| G^{-1} \| \| F(x_k, t_i) - F(x^*, t_i) \|)
\end{aligned}$$

So

$$\frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) \| H(x^*, t_i) - H(x_k, t_i) \| \xrightarrow{k \to \infty} 0. \tag{33}$$

On the other hand, by (18), (20), (32) *and* (33), *we have*

$$\frac{1}{N_k} \sum_{t_i \in T_k} \Phi(t_i) \| F''_{xt}(x_k, t_i) H(x_k, t_i) - F''_{xt}(x^*, t_i) H(x^*, t_i) \|$$

$$
\begin{aligned}
=&\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)\|F''_{xt}(x_k,t_i)(H(x_k,t_i)-H(x^*,t_i))\\
&+(F''_{xt}(x_k,t_i)-F''_{xt}(x^*,t_i)H(x^*,t_i)\|\\
\le&\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)(\|F''_{xt}(x_k,t_i)\|(H(x_k,t_i)-H(x^*,t_i)\|\\
&+\|F''_{xt}(x_k,t_i)-F''_{xt}(x^*,t_i)\|\|H(x^*,t_i)\|)\\
\le&C\cdot\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)(\|(H(x_k,t_i)-H(x^*,t_i)\|\\
&+\|F''_{xt}(x_k,t_i)-F''_{xt}(x^*,t_i)\|)\xrightarrow{k\to\infty}0.
\end{aligned}
$$

Noting that $C\ge sup\{\|x_k\|,k=1,2,\ldots\}$, *from* (19) *and* (33), *it implies that*

$$
\begin{aligned}
&\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)\|F''_{xt}(x_k,t_i)x_k-F''_{xt}(x^*,t_i)x^*\|\\
\le&\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)(\|F''_{xt}(x_k,t_i)x_k-F''_{xt}(x^*,t_i)\|\|x_k\|)+\|F''_{xt}(x^*,\xi_i)\|\|x_k-x^*\|\\
\le&C\cdot\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)(\|F''_{xt}(x_k,t_i)-F''_{xt}(x^*,t_i)\|+\|x_k-x^*\|)\xrightarrow{k\to\infty}0.
\end{aligned}
$$

By the same way as (33), *we have*

$$
\lim_{k\to 0}\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)\|H'_t(x_k,t_i)-H'_t(x^*,t_i)\|=0. \tag{34}
$$

Thus, following from (17), (21), (31) *and* (34), *we can get*

$$
\begin{aligned}
&\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)\|F'_x(x_k,t_i)H'_t(x_k,t_i)-F'_x(x^*,t_i)H'_t(x^*,t_i)\|\\
=&\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)\|F'_x(x_k,t_i)(H'_t(x_k,t_i)-H'_t(x^*,t_i))\\
&+(F'_x(x_k,t_i)-F'_x(x^*,t_i)\|H'_t(x^*,t_i)\|\\
\le&\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)(\|F'_x(x_k,t_i)\|(H'_t(x_k,t_i)-H'_t(x^*,t_i)\|\\
&+\|F'_x(x_k,t_i)-F'_x(x^*,t_i)\|\|H'_t(x^*,t_i)\|)\\
\le&C\cdot\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)(\|(H'_t(x_k,t_i)\\
&-H'_t(x^*,t_i)\|+\|F'_x(x_k,t_i)-F'_x(x^*,t_i)\|).
\end{aligned}
$$

Hence

$$
\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)\|F'_x(x_k,t_i)H'_t(x_k,t_i)-F'_x(x^*,t_i)H'_t(x^*,t_i)\|\xrightarrow{k\to\infty}0. \tag{35}
$$

By (33)-(35), *we have following naturally*

$$
\begin{aligned}
&\|\nabla\theta_k(x_k)-\nabla\theta_k(x^*)\|\\
=&\|\nabla_x g(x_k,t)\Phi(t)|_{t\in T}-\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)g''_{xt}(x_k,t_i)
\end{aligned}
$$

$$
\begin{aligned}
&-\nabla_x g(x^*,t)\Phi(t)|_{t\in T}+\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)g''_{xt}(x^*,t_i)\|\\
\leq&|\nabla_x g(x_k,t)\Phi(t)|_{t\in T}-\nabla_x g(x^*,t)\Phi(t)|_{t\in T}|\\
&+\|\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)(g''_{xt}(x_k,t_i)-g''_{xt}(x^*,t_i))\|\\
\leq&|\nabla_x g(x_k,t)\Phi(t)|_{t\in T}-\nabla_x g(x^*,t)\Phi(t)|_{t\in T}|\\
&+\|\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)([F'_t(x_k,t_i)-F''_{xt}(x_k,t_i)(H(x_k,t_i)-x_k)\\
&-(F'_x(x_k,t_i)-\alpha G)H'_t(x_k,t_i)]-[F'_t(x^*,t_i)-F''_{xt}(x^*,t_i)(H(x^*,t_i)-x^*)\\
&-(F'_x(x^*,t_i)-\alpha G)H'_t(x^*,t_i)])\|\\
\leq&|(\nabla_x g(x_k,t)-\nabla_x g(x^*,t))\Phi(t)|_{t\in T}|+\|\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)(\|F'_t(x_k,t_i)-F'_t(x^*,t_i)\|\\
&+\|F''_{xt}(x_k,t_i)H(x_k,t_i)-F''_{xt}(x^*,t_i)H(x^*,t_i)\|\\
&+\|F''_{xt}(x_k,t_i)x_k)-F''_{xt}(x^*,t_i)x^*)\|\\
&+\|(F'_x(x_k,t_i)H'_t(x_k,t_i)-(F'_x(x^*,t_i)H'_t(x^*,t_i)\|\\
&+\alpha\|G\|\|H'_t(x_k,t_i)-H'_t(x^*,t_i)])\|\\
&\xrightarrow{k\to\infty}0.
\end{aligned}
$$

Through integration by parts, we also know that

$$
\begin{aligned}
\nabla\theta(x^*)=&E[\nabla_x g(x^*,\xi)]=\int_T\nabla_x g(x^*,t)d\Phi(t)\\
=&\nabla_x g(x^*,t)\Phi(t)|_{t\in T}-\int_T\Phi(t)g''_{xt}(x^*,t)dt.
\end{aligned}
$$

Notice that

$$
\nabla\theta_k(x^*)=\nabla_x g(x^*,t)\Phi(t)|_{t\in T}-\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)g''_{xt}(x^*,t_i).
$$

Owing to

$$
\lim_{k\to\infty}\frac{1}{N_k}\sum_{t_i\in T_k}\Phi(t_i)g''_{xt}(x^*,t_i)=\int_T\Phi(t)g''_{xt}(x^*,t)dt.
$$

So it is easy to see that $\lim_{k\to\infty}\nabla\theta_k(x^*)=\nabla\theta(x^*)$. *Thus, It is clear that*

$$
\lim_{k\to\infty}\nabla\theta_k(x_k)=\nabla\theta(x^*).
$$

Definition 3.1. *Suppose that* $S=\{x\in R^n|c(x)\leq 0\}$, *where* $c_i:R^n\to R, i=1,2,\cdots,m$, *are all continuously differentiable convex functions. A point* x_k *is said to be stationary of* (7) *if there exists a Lagrange multiplier vector* $\mu_k\in R^m$ *such that*

$$
\nabla\theta_k(x_k)+\sum_{i=1}^m(\mu_i)_k\nabla c_i(x_k)=0, \tag{36}
$$

$$
0\leq\mu_k, c(x_k)\leq 0 \text{ and } (\mu_k)^T c(x_k)=0. \tag{37}
$$

x^* *is said to be a stationary point of* (2) *if there exists a Lagrange multiplier vector* $\mu^*\in R^m$ *such that*

$$
\nabla\theta(x^*)+\sum_{i=1}^m\mu_i^*\nabla c_i(x^*)=0, \tag{38}
$$

$$0 \le \mu^*, c(x^*) \le 0 \text{ and } (\mu^*)^T c(x^*) = 0. \tag{39}$$

Definition 3.2. *The Slater's constraint qualification holds if there exists a vector* $y \in R^n$ *such that* $c_i(y) < 0$ *for each* $i = 1, 2, \cdots, m$.

Theorem 3.3. *Let* x_k *be stationary to* (7) *for each* k *and* x^* *be an accumulation point of* $\{x_k\}$. *If the Slater constraint qualification holds, then* x^* *is stationary to problem* (2).
Proof *Without loss of generality, we assume that* $\lim_{k\to\infty} x_k = x^*$. *Let* μ_k *be the corresponding multiplier vector satisfying* (36)-(37).
(i) We first show that the sequence $\{\mu_k\}$ *is bounded. To this end, we denote*

$$\upsilon_k := \sum_{i=1}^{m} (\mu_i)_k. \tag{40}$$

Let $\{\mu_k\}$ *be unbounded, which means* $\lim_{k\to\infty} \upsilon_k = +\infty$. *Taking a subsequence, we may assume that the limits* $\mu_i^* := \lim_{k\to\infty} \frac{(\mu_i)_k}{\upsilon_k}$ $(i = 1, 2, \cdots, m)$ *exist. For every* $i \notin \Upsilon(x^*) := \{i | c_i(x^*) = 0, 1 \le i \le m\}$, *it holds* $c_i(x^*) \le 0$ *by* (37), *further more* $(\mu_i^*)^T c_i(x^*) = 0$, *it holds* $\mu_i^* = 0$. *Then, from* (40),

$$\sum_{i\in\Upsilon(x^*)} \mu_i^* = \sum_{i=1}^{m} \mu_i^* = 1. \tag{41}$$

Note that ∇c_i *is continuous for each* i *and* $\{\nabla\theta_k(x_k)\}$ *is convergent by theorem 3.2. Because of* $\lim_{k\to\infty} \upsilon_k = +\infty$, $\lim_{k\to\infty} \frac{\nabla\theta_k(x_k)}{\upsilon_k} \to 0$. *Dividing* (36) *by* υ_k *and taking a limit, we obtain*

$$\sum_{i\in\Upsilon(x^*)} \mu_i^* \nabla c_i(x^*) = \sum_{i=1}^{m} \mu_i^* \nabla c_i(x^*) = 0. \tag{42}$$

Owing to the Slater's constraint qualification, there exists a vector $y \in R^n$ *such that* $c_i(y) < 0$ *for each* $i = 1, 2, \cdots, m$. *Noting that each* c_i *is convex, we have*

$$(y - x^*)^T \nabla c_i(x^*) \le c_i(y) - c_i(x^*) = c_i(y) < 0, \tag{43}$$

$$\forall i \in \Upsilon(x^*). \tag{44}$$

From (42) *and* $\mu_i^* \ge 0$ *for each* i *by* (37), *we get* $\mu_i^* \nabla c_i(x^*) = 0$. *Furthermore,* $\nabla c_i(x^*) \ne 0$ *from* (43), *it implies that* $\mu_i^* = 0$ *for each* $i \in \Upsilon(x^*)$. *This contradicts* (41). *Hence* $\{\mu_k\}$ *is bounded.*
(ii) By (i), μ_k *exists a subsequence such that* $\mu^* := \lim_{k\to\infty} \mu_k$, *we still denote it as* μ_k. *Note that both* c_i *and* ∇c_i *are continuous for each* i, *by Theorem 3.2, it holds* $\lim_{k\to\infty} \nabla\theta_k(x^*) = \nabla\theta(x^*)$. *Taking a limit in* (36) *and* (37), *we obtain* (38) *and* (39) *immediately. Therefore,* x^* *is stationary to problem* (2).

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