VARIOUS SHADOWING PROPERTIES FOR DYNAMICAL SYSTEMS

Nan Wang[1], Yingcui Zhao[2], Lidong Wang[3], Shaoyun Shi[4]

We study syndetic shadowing and cofinite shadowing properties for dynamical systems and discuss the relationships among different kinds of shadowing properties. We have proven that syndetic shadowing, thick shadowing, and family shadowing all have the characteristic of being iterative invariant. Moreover, we have demonstrated that if two continuous self-mappings have the ergodic (cofinite) shadowing property, then the product of these two mappings will also have the ergodic (cofinite) property.

Keywords: Syndetic shadowing property, cofinite shadowing property, thick shadowing property, ergodic shadowing property, family shadowing property.

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1. **Introduction**

The notion of pseudo-orbit was first introduced to dynamical system in the works of Birkhoff [3]. Pseudo-orbit are approximate solutions obtained through numerical simulation methods, while shadowing refers to the ability of a system to track a given target trajectory. The famous shadowing property which is proposed by Anosov, Bowen and Ruelle [1, 6], also known as the pseudo-orbit tracing property, says that every pseudo-orbit can be approximated by an exact orbit. Therefore, in the study of dynamical systems, the concepts of pseudo-orbit and shadowing can be combined. By analyzing the shadowing property of pseudo-orbit, we can evaluate the stability and tracking property of the system, and further improve the system's control strategies and algorithms to enhance its shadowing. With the introduction of the classical

[1]School of Mathematics, Jilin University, Changchun 130012, China, e-mail: wangnanchaos@126.com

[2]School of Disciplinary Basics and Applied Statistics, Zhuhai College of Science and Technology, Zhuhai 519041, China, e-mail: zycchaos@126.com

[3]Corresponding author. School of Disciplinary Basics and Applied Statistics, Zhuhai College of Science and Technology, Zhuhai 519041, China, e-mail: wld07chaos@126.com

[4]School of Mathematics and State Key Laboratory of Automotive Simulation and Control, Jilin University, Changchun 130012, China, e-mail: shisy@jlu.edu.cn

shadowing property and based on different understandings of pseudo-orbits (δ-average-pseudo-orbit, δ-asymptotic-average-pseudo-orbit, asymptotic-average-pseudo-orbit, and δ-ergodic pseudo-orbit) and tracking methods (average shadowing, asymptotic average shadowing, weak asymptotic average shadowing, d-shadowing, ergodic shadowing, and $\mathscr{F}$-shadowing), different concepts of shadowing properties have been defined and studied by scholars. Blank introduced the average tracking property in 1988 using the average pseudo-orbit and average tracking method[4], and proved that $f|_\Lambda$ has the average shadowing property provided that Λ is a basic set of a diffeomorphism f satisfying Axiom A. For a continuous surjection f on a compact metric space with infinite elements, Yang [20] studied study the relationships between f have the pseudo-orbit-tracing property and f is chaotic in the sense of Ruelle-Takens, topological mixing and have property P. Lee and Sakai compared various shadowing properties for (positively) expansive maps in [14, 17] and proved that the continuous shadowing property, the Lipschitz shadowing property, the limit shadowing property and the strong shadowing property are all equivalent to the (usual) shadowing property for expansive homeomorphisms on compact metric spaces (see also [13]). The asymptotic average shadowing property was introduced by Gu[12], which followed the same framework as Blank in[4], but with the limit shadowing property in place of the shadowing property as the starting point for generalization. Meanwhile, he showed that every surjective dynamical system with asymptotic average shadowing is chain transitive and every $\mathcal{L}$-hyperbolic homeomorphism with asymptotic average shadowing is topologically transitive. It is interesting that partial shadowing may have strong influence on dynamics. For example Fakhari proves in[9] that the ergodic shadowing property equivalently characterizes shadowing with topological mixing in surjective dynamical systems. Carvalho[7] explore the notion of two-sided limit shadowing property, and characterize the C^1-interior of the set of diffeomorphisms with such a property on closed manifolds as the set of transitive Anosov diffeomorphisms. It proves that all codimention-one Anosov diffeomorphisms have the two-sided limit shadowing property. Wu et al. [19] presented asystematic study of shadowing properties with average error in tracing such as (asymptotic) average shadowing, $\overline{d}$-shadowing, $\underline{d}$-shadowing, and almost specification. Garg and Das [11] introduced and studied average chain transitivity, average chain mixing, total average chain transitivity, and almost average shadowing property and also discussed their interrelations. Motivated by the above works, we introduce the notions of syndetic shadowing and cofinite shadowing properties and discuss the relationships among different kinds of shadowing properties.

In section 2, we give prerequisites required for remaining sections of the paper. In section 3, we prove that the syndetic shadowing and cofinite shadowing property have the iterative invariant property. Moreover, we obtain that the cofinite shadowing property of two continuous self-mappings implies cofinite property of the product of these two mappings. In section 4, we show that the thick shadowing and family shadowing properties have the iterative invariant property and prove that the ergodic shadowing property of two continuous self-mappings implies ergodic property of the product of these two mappings.

2. **Preliminaries**

Throughout this paper, a dynamical system is a pair (X, f), where X is a compact metric space with a metric d and $f\colon X \to X$ is a continuous map. Let $\mathbf{N} = \{0, 1, 2, \cdots\}$ and $\mathbf{Z} = \{1, 2, 3, \cdots\}$. A set $A \subset \mathbf{N}$ is

(1) a thick set if it contains intervals of natural numbers of arbitrarily length.
(2) a syndetic set if $\mathbf{N} \setminus A$ is not a thick set.
(3) a cofinite set if $\mathbf{N} \setminus A$ is a finite set.

A collection $\mathscr{F}$ of subsets of $\mathbf{N}$ is called family, if it is upward hereditary, that is $A \subset B$, $A \in \mathscr{F} \Rightarrow B \in \mathscr{F}$. Put $d(A) = \lim\limits_{n\to\infty} \frac{1}{n}|A \cap \{0, 1, \cdots, n-1\}|$, for any $A \subset \mathbf{N}$, where $|A|$ is the cardinal number of A.

The shadowing property is one of the most important notions in dynamical systems (see [5]), which is related to chaos and stability of systems (see, for instance, [9, 13, 14, 12, 20, 19]). A sequence $\{x_i\}_{i\geq 0}$ in X is called an orbit of f if for every $i \in \mathbf{N}$ we have $x_{i+1} = f(x_i)$. For $\delta > 0$, $\{x_i\}_{i\geq 0}$ is a δ-pseudo-orbit of f if for every $i \in \mathbf{N}$,

$$d(f(x_i), x_{i+1}) < \delta.$$

The continuous map f has shadowing property if for any $\varepsilon > 0$ there is $\delta > 0$ such that each δ-pseudo-orbit $\{x_i\}_{i\geq 0}$ is ε-shadowed by the orbit $\{f^i(z) \mid i \in \mathbf{N}\}$, for some $z \in X$, i. e., for any $i \in \mathbf{N}$ we have

$$d(f^i(z), x_i) < \varepsilon.$$

The definition of ergodic shadowing, thick shadowing and family shadowing property are introduced in [9], [8] and [16], respectively. For $\delta > 0$, a sequence $\{x_i\}_{i\geq 0}$ in X is said to be a δ-ergodic-pseudo-orbit of f if

$$\lim_{n\to\infty} \frac{1}{n}|\{i \mid 0 \leq i < n, d(f(x_i), x_{i+1}) < \delta\}| = 1.$$

The continuous map f has ergodic shadowing property (resp. thick shadowing, $\mathscr{F}$ shadowing property) if for any $\varepsilon > 0$ there is $\delta > 0$ such that each δ-ergodic-pseudo-orbit $\{x_i\}_{i\geq 0}$ is ε-ergodic-shadowed (resp. ε-thick-shadowed, ε-$\mathscr{F}$-shadowed) by the orbit $\{f^i(z) \mid i \in \mathbf{N}\}$, for some $z \in X$, i. e., for any $i \in \mathbf{N}$ we have

$$\lim_{n\to\infty} \frac{1}{n}|\{i \mid 0 \leq i < n, d(f^i(z), x_i) < \varepsilon\}| = 1,$$

(resp. $\{i \mid d(f^i(z), x_i) < \varepsilon\}$ is a thick set, $\{i \mid d(f^i(z), x_i) < \varepsilon\} \in \mathscr{F}$). The $\mathscr{F}$ shadowing property is also called family shadowing property. For more results on ergodic shadowing, thick shadowing and $\mathscr{F}$ shadowing property, one can refer to [9, 8, 16, 18, 19]. For example in[8], the main focus of the study is δ-ergodic pseudo-orbit. If any δ-ergodic pseudo-orbit of a system is shadowed by a point along a set of positive lower density then we show that this system is chain mixing and if it is minimal then it is topologically weakly mixing.Oprocha et al.[16] provided necessary and sufficient conditions for shadowing over a set with positive density (or syndetic). In[18], It has been proven that for a dynamical system (X, f), the following starements are equivalent: f has $\overline{d}$-shadowing property(resp., $\underline{d}$-shadowing property); f^k has $\overline{d}$-shadowing property(resp., $\underline{d}$-shadowing property) for all $k \in \mathbb{N}$; f^k has $\overline{d}$-shadowing property($\underline{d}$-shadowing property) for some $k \in \mathbb{N}$.

Definition 2.1. *The continuous map f has syndetic shadowing (resp. cofinite shadowing) property, if for any $\varepsilon > 0$, there is $\delta > 0$ such that each δ-pseudo-orbit $\{x_i\}_{i\geq 0}$ is ε-syndetic-shadowed (resp. ε-cofinite-shadowed) by the orbit $\{f^i(z) \mid i \in \mathbf{N}\}$, for some $z \in X$, i. e., $\{i \mid d(f^i(z), x_i) < \varepsilon\}$ is a syndetic (resp. cofinite) set.*

In the context of dynamical systems, shadowing refers to the property where a sequence of points closely follows the trajectory of an orbit. It implies that even though the observed points may not exactly coincide with the orbit's points, they stay sufficiently close over an extended period of time.

The term "syndetic" comes from set theory and topology. A subset of integers is said to be syndetic if it has bounded gaps. In other words, the gaps between consecutive elements are bounded. In the context of dynamical systems, a syndetic set of indices indicates that the corresponding orbit points cluster together in some sense.

In set theory and number theory, a set is called "cofinite" if its complement (the elements not in the set) is finite. In other words, a cofinite set contains "almost all" the elements from a larger set.

Therefore, "syndetic shadowing" means that the observed sequence of points closely follows the trajectory of an orbit in a way that the index set where the points deviate from the orbit forms a syndetic set. This term captures the idea that the deviations from the true orbit are "bounded" or "clustered", which helps in understanding the stability and behavior of dynamical systems.

"Cofinite shadowing" means that the observed sequence of points closely follows the trajectory of an orbit in a way that the index set where deviations occur is a cofinite set. In this context, "cofinite" indicates that the set of indices where the observed sequence deviates from the orbit is "almost all" of the index set, or in simpler terms, the deviations occur only at a finite number of points.

The term "cofinite shadowing" reflects the idea that the observed sequence of points is very similar to the orbit, with deviations happening only at a finite number of instances. This concept is valuable in characterizing the stability and behavior of dynamical systems in a more relaxed manner compared to exact shadowing.

In summary, "syndetic shadowing" highlights the property that observed sequences of points in a dynamical system closely follow the behavior of an orbit, with deviations forming a bounded or clustered set of indices. while "cofinite shadowing" highlights the property that an observed sequence of points in a dynamical system closely follows the behavior of an orbit, with deviations occurring at only a finite number of instances, allowing for more flexibility in capturing the system's behavior.

We know, based on the definitions, that shadowing $\Rightarrow$ cofinite shadowing $\Rightarrow$ syndetic shadowing.

As we all know, if f has shadowing property, then for any positive integer $k \in \mathbf{Z}$, so does f^k [2]. It is shown in [15] that the average shadowing property of f implies it of f^k, $\forall k \in \mathbf{Z}$. Gu [12] studied that the asymptotic average shadowing property of f implies it of f^k, $\forall k \in \mathbf{Z}$. It is proposed in [10] that if f has almost average shadowing property, then for any $k \in \mathbf{Z}$, the same holds for f^k. In this paper, we prove if f has the cofinite shadowing, syndetic shadowing, thick shadowing and $\mathscr{F}$ shadowing property, then correspondingly so does f^k, $\forall k \in \mathbf{Z}$((respectively see Theorems 3.1, 3.2, 4.1, and 4.2).

Let (X, f), (Y, g) be two dynamical systems. Kwietniak and Oprocha [13] mentioned if f has the shadowing property, then the same holds for $f \times f$. It is proved in [15] if f and g have the average shadowing property, then so does the product $f \times g$. In the present paper, we study the cofinite shadowing, thick shadowing, ergodic shadowing property of f, g implies the corresponding property of the product $f \times g$ (respectively see Theorem 3.3, Theorem 4.3). At the end of this paper, we investigate the thick shadowing property by some examples.

3. The Syndetic shadowing and cofinite shadowing property

The following two theorems show if a dynamical system has the cofinite (resp. syndetic) shadowing property, then so does any number of iterations of the mapping.

Theorem 3.1. *If a dynamical system (X, f) has the cofinite shadowing property, then, for any $k > 1$, (X, f^k) has the cofinite shadowing property.*

Proof. Fix $k > 1$. Given any $\varepsilon > 0$, suppose that $\delta > 0$ is obtained for ε by the cofinite shadowing property of f. Let $\{y_i\}_{i \geq 0}$ be the δ-pseudo-orbit of f^k, that is,

$$d(f^k(y_i), y_{i+1}) < \delta, \text{ for all } i \in \mathbf{N}.$$

Define $x_{mk+r} = f^r(y_m),$ for all $m \geq 0, 0 \leq r < k$, i.e.,

$$\begin{aligned}\{x_i\}_{i\geq 0} =&\{y_0, f(y_0), \cdots, f^{k-1}(y_0), y_1, f(y_1), \cdots,\\ &f^{k-1}(y_1), \cdots, y_i, f(y_i)\cdots, f^{k-1}(y_i), \cdots\}.\end{aligned}$$

It is easy to know that $d(f(x_i), x_{i+1}) < \delta$, for all $i \in \mathbf{N}$, i.e., $\{x_i\}_{i\geq 0}$ is the δ-pseudo-orbit of f. By the cofinite shadowing property, there exists $z \in X$ such that $\{i \mid d(f^i(z), x_i) < \varepsilon\}$ is also a cofinite set. Therefore, there is $N \in \mathbf{Z}$ such that for any $i > N$, $d(f^i(z), x_i) < \varepsilon$. Next, we will show that $\{i \mid d(f^{ki}(z), y_i) < \varepsilon\}$ is a cofinite set. Put $M = [\frac{N}{k}] + 1$, for all $i > M$(i.e. $ik > N$), we have $d(f^{ki}(z), x_{ik}) < \varepsilon$, i.e., $d(f^{ki}(z), y_i) < \varepsilon$. Consequently, f^k has the cofinite shadowing property. □

Theorem 3.2. *If a dynamical system (X, f) has the syndetic shadowing property, then, for any $k > 1$, (X, f^k) has the syndetic shadowing property.*

Proof. Fix $k > 1$. Given any $\varepsilon > 0$, by the compactness of X and the continuity of f, there exists $0 < \eta < \varepsilon$ satisfying

$$d(u, v) < \eta \Rightarrow d(f^s(u), f^s(v)) < \varepsilon, 0 \leq s \leq k.$$

Suppose $\delta > 0$ is obtained for η by the syndetic shadowing property of f. Let $\{y_i\}_{i\geq 0}$ be the δ-pseudo-orbit of f^k. Define $x_{mk+r} = f^r(y_m),$ for all $m \geq 0, 0 \leq r < k$. Obviously, $\{x_i\}_{i\geq 0}$ is a δ-pseudo-orbit of f. By the syndetic shadowing property of f, there exists $z \in X$ such that $\{i \mid d(f^i(z), x_i) < \eta\}$ is a syndetic set. That is to say, there is $N \in \mathbf{Z}$ such that for any l, we have

$$\{l, l+1, \cdots, l+N\} \cap \{i \mid d(f^i(z), x_i) < \eta\} \neq \emptyset. \tag{1}$$

Put $M = [\frac{N}{k}] + 1$. Assume that if there exists i_0 satisfying

$$\{i_0, i_0+1, \cdots, i_0+M\} \cap \{i \mid d(f^{ik}(z), y_i) < \varepsilon\} = \emptyset.$$

That is, $d(f^{ki_0}(z), y_{i_0}) \geq \varepsilon$, $d(f^{k(i_0+1)}(z), y_{(i_0+1)}) \geq \varepsilon$, $\cdots$, $d(f^{k(i_0+M)}(z), y_{i_0+M}) \geq \varepsilon$.
Therefore,

$$\begin{gathered}d(f^{ki_0}(z), x_{ki_0}) \geq \eta, d(f^{ki_0+1}(z), x_{ki_0+1}) \geq \eta, \cdots, d(f^{k(i_0+1)}(z), x_{k(i_0+1)}) \geq \eta,\\ d(f^{k(i_0+1)+1}(z), x_{k(i_0+1)+1}) \geq \eta, \cdots, d(f^{k(i_0+2)}(z), x_{k(i_0+2)}) \geq \eta,\\ \cdots, d(f^{k(i_0+M)}(z), x_{k(i_0+M)}) \geq \eta,\end{gathered}$$

which is a contradiction with (1). And thus, $\{i \mid d(f^{ik}(z), y_i) < \varepsilon\}$ is a syndetic set. Consequently, f^k has the syndetic shadowing property. □

Let (X, d_1), (Y, d_2) be two compact metric spaces and $f : X \to X$, $g : Y \to Y$ be two continuous map. For any (x_1, y_1), $(x_2, y_2) \in X \times Y$, define the metric of them as follows:

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

Theorem 3.3. *If two dynamical systems (X, f) and (Y, g) have the cofinite shadowing property, then $(X \times Y, f \times g)$ has the cofinite shadowing property.*

Proof. Given any $\varepsilon > 0$, suppose that $\delta > 0$ is obtained for ε by the cofinite shadowing property of both f and g. Let $\{(x_i, y_i)\}_{i\geq 0}$ be the δ-pseudo-orbit of $f \times g$, i.e.

$$d((f \times g)(x_i, y_i), (x_{i+1}, y_{i+1})) < \delta, \forall i \in \mathbf{N}.$$

Furthermore, as $\{x_i\}_{i\geq 0}$ and $\{y_i\}_{i\geq 0}$ are the δ-pseudo-orbit of f and g, respectively, there exist $u \in X, v \in Y$ such that both $\{i \mid d_1(f^i(u), x_i) < \varepsilon\}$ and $\{i \mid d_2(g^i(v), y_i) < \varepsilon\}$ are the cofinite set. That is, there exist $N_1, N_2 > 0$ such that for any $i > N_1$, $d_1(f^i(u), x_i) < \varepsilon$ and for any $i > N_2$, $d_2(g^i(v), y_i) < \varepsilon$. Choose $N = \max\{N_1, N_2\}$, then for any $i > N$,

$$d((f \times g)^i(u, v), (x_i, y_i)) < \varepsilon.$$

Thus, $\{i \mid d((f \times g)^i(u, v), (x_i, y_i)) < \varepsilon\}$ is a cofinite set. Furthermore, $f \times g$ has the cofinite shadowing property. □

4. The thick shadowing, ergodic shadowing and $\mathscr{F}$-shadowing property

The following two theorems show that if a continuous self-mapping has the thick (resp. family) shadowing property, then so does any number of iterations of the mapping, extending the main result of Fakhari and Ghane [9], showing that the ergodic-shadowing property of f implies the ergodic-shadowing property of f^k, for all $k \in \mathbf{Z}$.

Theorem 4.1. *If a dynamical system (X, f) has the thick shadowing property, then, for any $k > 1$, (X, f^k) has the thick shadowing property.*

Proof. Fix $k > 1$. Given any $\varepsilon > 0$, by the compactness of X and the continuity of f, there exists $0 < \eta < \varepsilon$ satisfying

$$d(u, v) < \eta \Rightarrow d(f^s(u), f^s(v)) < \varepsilon, 0 \leq s \leq k.$$

Suppose that $\delta > 0$ is obtained for η by the thick shadowing property of f. Let $\{y_i\}_{i\geq 0}$ be the δ-ergodic-pseudo-orbit of f^k, then

$$\lim_{n\to\infty} \frac{1}{n} |\{i \mid 0 \leq i < n, d(f^k(y_i), y_{i+1}) < \delta\}| = 1.$$

Define $x_{mk+r} = f^r(y_m)$, for any $m \geq 0, 0 \leq r < k$. Obviously, $\{x_i\}_{i\geq 0}$ is a δ-ergodic-pseudo-orbit of f. By the thick shadowing property of f, there exists $z \in X$ such that $\{i \mid d(f^i(z), x_i) < \eta\}$ is a thick set. For any $l \in \{j \mid d(f^j(z), x_j) < \eta\}$, there are $m_l \geq 0, 0 \leq r_l < k$ such that $l = m_l k + r_l$. Then

$$d(f^{(m_l+1)k}(z), f^{k-r_l}(x_{m_l k+r_l})) = d(f^{(m_l+1)k}(z), y_{m_l+1}) < \varepsilon.$$

The above discussion implies that if $l \in \{j \mid d(f^j(z), x_j) < \eta\}$, there exits $m_l \in \mathbf{N}$ such that $m_l + 1 \in \{j \mid d(f^{kj}(x), y_j) < \varepsilon\}$. So, for any given positive integer M, there is $j_0 \in \mathbf{Z}$ satisfying

$$j_0, j_0 + 1, \cdots, j_0 + Mk \in \{j \mid d(f^j(z), x_j) < \eta\}.$$

Let $j_0 = m_0k + r_0, m_0 \geq 0, 0 \leq r < k$, then

$$m_0 + 1, m_0 + 2, \cdots, m_0 + M + 1 \in \{j \mid d(f^{jk}(z), y_j) < \varepsilon\}.$$

Thus, f^k has the thick shadowing property. □

Theorem 4.2. *Let $\mathscr{F}_1$ be a family. If a dynamical system (X, f) has the $\mathscr{F}_1$-shadowing property, then, for any $k > 1$, there is a family $\mathscr{F}_2$ such that (X, f^k) has the $\mathscr{F}_2$-shadowing property.*

Proof. Fix $k > 1$. Given any $\varepsilon > 0$, by the compactness of X and the continuity of f, there exists $0 < \eta < \varepsilon$ satisfying

$$d(u, v) < \eta \Rightarrow d(f^s(u), f^s(v)) < \varepsilon, 0 \leq s \leq k.$$

Suppose that $\delta > 0$ is obtained for η by the $\mathscr{F}_1$-shadowing property of f. Let $\{y_i\}_{i\geq 0}$ be the δ-pseudo-orbit of f^k. Define $x_{mk+r} = f^r(y_m)$, for any $m \geq 0, 0 \leq r < k$. Obviously, $\{x_i\}_{i\geq 0}$ is the δ-pseudo-orbit of f. By the $\mathscr{F}_1$-shadowing property of f, there exists $z \in X$ such that $\{i \mid d(f^i(z), x_i) < \eta\} \in \mathscr{F}_1$. For any $l \in \{j \mid d(f^j(z), x_j) < \eta\}$, there are $m_l \geq 0, 0 \leq r_l < k$ such that $l = m_lk + r_l$. Then $d(f^{(m_l+1)k}(z), f^{k-r_l}(x_{m_lk+r_l})) = d(f^{(m_l+1)k}(z), y_{m_l+1}) < \varepsilon$. The above discussion implies that if $l \in \{j \mid d(f^j(z), x_j) < \eta\}$, there exits $m_l \in \mathbf{N}$ such that $m_l + 1 \in \{j \mid d(f^{kj}(x), y_j) < \varepsilon\}$. Denote that $C = \{m + 1 \mid mk + r \in \{j \mid d(f^j(z), x_j) < \eta\}$ for $m \geq 0, 0 \leq r < k - 1\}$. Obviously, $C \subset \{j \mid d(f^{kj}(z), y_j) < \varepsilon\}$

Define $\mathscr{F}_2 = \{A \mid A \supset C\}$, then we have $\{j \mid d(f^{kj}(z), y_j) < \varepsilon\} \in \mathscr{F}_2$. Then we will show $\mathscr{F}_2$ is a family. Set $A \subset B, A \in \mathscr{F}_2$, then $C \subset A$. Hence $C \subset B$, i. e. $B \in \mathscr{F}_2$. Thus, $\mathscr{F}_2$ is a family.

Consequently, f^k has the $\mathscr{F}_2$-shadowing property. □

Next, we will show that ergodic shadowing property are invariable under the product. For $\mathscr{F}$-shadowing property, one can refer to [16]. Next, we first prove a lemma.

Let (X, d_1), (Y, d_2) be two compact metric spaces and $f : X \to X$, $g : Y \to Y$ be two continuous map. For any $(x_1, y_1), (x_2, y_2) \in X \times Y$, define the metric of them as follows:$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$.

Theorem 4.3. *If two dynamical systems (X, f) and (Y, g) have the ergodic shadowing property, then so does the product dynamical system $(X \times Y, f \times g)$.*

Proof. For any given $\varepsilon > 0$. Suppose that $\delta > 0$ satisfies thick shadowing property of f and g for ε. Let $\{(x_i, y_i)\}_{i\geq 0}$ be the δ-ergodic-pseudo-orbit of $f \times g$, then, $\{x_i\}_{i\geq 0}$ and $\{y_i)\}_{i\geq 0}$ are the δ-ergodic-pseudo-orbit of f and g, respectively.

By the hypothesis, there exist u and v such that

$$\lim_{n\to\infty}\frac{1}{n}\left|\{0\le i<n \mid d_1(f^i(u),x_i)<\varepsilon\}\right|=1$$

and

$$\lim_{n\to\infty}\frac{1}{n}\left|\{0\le i<n \mid d_2(g^i(v),y_i)<\varepsilon\}\right|=1.$$

Then we have

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\left|\{0\le i<n \mid d_1(f^i(u),x_i)\ge\varepsilon\}\right|\\ =&\lim_{n\to\infty}\frac{1}{n}\left|\{0\le i<n \mid d_2(g^i(v),y_i)\ge\varepsilon\}\right|\\ =&0.\end{aligned}$$

It is easy to have that

$$\begin{aligned}0\le&\lim_{n\to\infty}\frac{1}{n}\left|\{0\le i<n \mid d((f\times g)^i(u,v),(x_i,y_i))\ge\varepsilon\}\right|\\ \le&\lim_{n\to\infty}\frac{1}{n}\left|\{0\le i<n \mid d_1(f)^i(u),(x_i)\ge\varepsilon\}\right|+\lim_{n\to\infty}\frac{1}{n}\left|\{0\le i<n \mid d_2(g^i(v),y_i)\ge\varepsilon\}\right|\\ =&\,0\end{aligned} \tag{2}$$

Therefore

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\left|\{0\le i<n \mid d((f\times g)^i(u,v),(x_i,y_i))<\varepsilon\}\right|\\ =&1-\lim_{n\to\infty}\frac{1}{n}\left|\{0\le i<n \mid d((f\times g)^i(u,v),(x_i,y_i))\ge\varepsilon\}\right|\\ =&1.\end{aligned} \tag{3}$$

Consequently, $f\times g$ has the ergodic shadowing property. □

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REFERENCES

[1] *D.V. Anosov*, Geodesic flows on closed Riemann manifolds with negative curvature, American Mathematical Society, (1969).

[2] *N. Aoki and K. Hiraide*, Topological theory of dynamical systems, North-Holland Mathematical Library, **52**(1994).

[3] *G.D. Birkhoff*, An extension of Poincaré's last geometric theorem, Acta Mathematica, **47**(1926), 297-311.

[4] *M.L. Blank*, Metric properties of -trajectories of dynamical systems with stochastic behaviour, Ergodic Theory and Dynamical Systems, **3**(1988), 365-378.

[5] *M.L. Blank*, Small perturbations of chaotic dynamical systems, Russian Mathematical Surveys, **44**(1989), 1-33.

[6] *R. Bowen and D. Ruelle*, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Lecture Notes in Mathematics, **470**(2008), 1-83.

[7] *B. Carvalho*, Hyperbolicity, transitivity and the two-sided limit shadowing property, Proceedings of the American Mathematical Society, **143**(2015), 657-666.

[8] *D.A. Dastjerdi and M. Hosseini*, Sub-shadowings, Nonlinear Analysis: Theory, Methods and Applications, **72**(2010), 3759-3766.

[9] *A. Fakhari and F.H. Ghane*, On shadowing: Ordinary and ergodic, Journal of Mathematical Analysis and Applications, **364**(2010), 151-155.

[10] *M. Garg and R. Das*, Average chain transitivity and the almost average shadowing property, Communications of the Korean Mathematical Society, **32**(2017), 201-214.

[11] *M. Garg and R. Das*, Relations of the almost average shadowing property with ergodicity and proximality, Chaos solitons and Fractals, **91**(2016), 430-433.

[12] *R.B. Gu*, The asymptotic average shadowing property and transitivity, Nonlinear Analysis-theory Methods and applications, **67**(2007), 1680-1689.

[13] *D. Kwietniak and P. Oprocha*, A note on the average shadowing property for expansive maps, Topology and its Applications, **159**(2012), 19-27.

[14] *K. Lee and K. Sakai*, Various shadowing properties and their equivalence, Discrete and Continuous Dynamical Systems, **13**(2005), 533-540.

[15] *Y. Niu*, The average-shadowing property and strong ergodicity, Journal of Mathematical Analysis and Applications, **376**(2011), 528-534.

[16] *P. Oprocha, D.A. Dastjerdi, and M. Hosseini*, on partial shadowing of complete pseudo-orbits, Journal of Mathematical Analysis and Applications, **411**(2014), 454-463.

[17] *K. Sakai*, Various shadowing properties for positively expansive maps, Topology and its Applications, **131**(2003), 15-31.

[18] *X.X. Wu*, A note on the $\overline{d}$-shadowing property, Science China(in Chinese), **45**(2015), 273-286.

[19] *X.X. Wu, P. Oprocha, and G.R. Chen*, On various definitions of shadowing with average error in tracing, Nonlinearity, **29**(2016), 1942-1972.

[20] *R. Yang*, The pseudo-orbit tracing property and chaos, Acta Mathematica Sinica, **39**(1996), 382-386.