ON INJECTIVITY OF ACTS

Gh. Moghaddasi[1], M. Haddadi[2], S. Delavari[3]

In this paper we investigate the actions of a monoid of the form $S = G\dot{\cup}I$, where G is a group and I is an ideal of S, on sets. So, naturally, every S-act can be considered as an I^1-act. The central question here is that what is the relation between injective and weakly injective I^1-acts and injective and weakly injective S-acts?

We are going to respond this question and show that weakly (principally or finitely generated) injectivivity of an S-act A is extendable from I^1-acts to S-acts. But for injectivity we need some more hypothesis.

Keywords: S-act, injectivity, weakly (principally, finitely generated) injectivity

2000 Mathematics Subject Clasification:08B30, 08C05, 20M30

1. **Introduction**

One of the very useful notions in many branches of mathematics, as well as in computer science, is the notion of actions of a semigroup or a monoid on a set. The notion of injectivity is one of the important concepts in every category, specially in the category of acts. Injective and weakly injective acts were first studied by Bertheaume in [1], and later studied by many authors, see [7, 8]. In [4], V. Gould introduced an infinite sequence of different injectivities between principally weakly injectivity and weakly injectivity. Principally weakly injective acts were first considered by J. Luedeman, F. McMorris and S.K.Sim [6].

A. Golchin and J. Renshaw in [2, 3] have studied actions of a monoid of the form $S = G\dot{\cup}I$, in which G is a group and I is an ideal of S. They show that, for these kind of actions, flatness is extendable from I^1-acts to S-acts. That is, an S-act A is flat if it is flat as an I^1-act. Thus, it is a natural question to ask that: what is the relation between (weakly) injective I^1-acts and (weakly) injective S-acts?

[1]Department of Mathematics, Hakim Sabzevary University, Sabzevar, Iran, e-mail: r.moghadasi@hsu.ac.ir

[2]Department of Mathematics, Statistics and Computer Science, Semnan University, Semnan, Iran

[3]Department of Mathematics, Hakim Sabzevary University, Sabzevar, Iran

Here we answer this question and we show that weakly injective property is extendable from I^1-acts to S-acts in general while injectivity needs some more hypothesis.

First, we briefly recall some notions about S-acts. Given a monoid S, a (right) S-act is a set A together with a function $A\times S \to A$, mapping each (a,s) to as, such that $(i)\,(as)t = a(st)$ and $(ii)\,a1 = a$, for every $a \in A$, $s,t \in S$. A subset B of an S-act A is called an *S-subact* of A, denoted by $B \leq A$, whenever $bs \in B$, for every $b \in B$ and $s \in S$. Specially, considering, naturally, S as an S-act, the S-subacts of S are exactly the right ideals of S. A map $f : A \to B$ between two S-acts A and B is called an *S-map* or an *S-homomorphism* if, for each $a \in A$, $s \in S, f(as) = f(a)s$. The usual definitions for *monomorphisms*, *epimorphisms* and *ismorphisms* hold. We denote the *category* of all S-acts and S-homomorphisms between them by **Act-S**.

An S-act A is said to be *finitely generated* if $A = \bigcup_{i=1}^{n} a_iS$, for some $n \in \mathbb{N}$ and $a_i \in A$. So a right ideal I of a monoid S is called a *finitely generated ideal* if it is finitely generated as an S-subact of S. Also A is called a *cyclic* S-act if $A = aS$, for some $a \in A$. A right ideal I of S is said to be *principal* if it is a cyclic S-subact of S.

An element θ in an S-act A with $\theta s = \theta$, for all $s \in S$, is called a *zero* or a *fixed element* of A.

An element $s \in S$ is called a *regular element* if $sxs = s$, for some $x \in S$. One calls S a *regular monoid* if all its elements are regular.

An element $e \in S$ is called *idempotent* if $e^2 = e$. The set of all idempotent elements of S is denoted by $E(S)$. An element $s \in S$ is called *left cancellable* if $sr = st$, for $r, s \in S$, implies $r = t$. An element $a \in A$ is called *divisible by* $s \in S$ if there exists $b \in A$ such that $bs = a$.

An S-act A is called *injective* if for every S-monomorphism $i : B \rightarrowtail C$ and every S-homomorphism $f : B \to A$, there exists an S-homomorphism $\bar{f} : C \to A$ with $\bar{f}i = f$. A monoid S is called *self-injective* if it is injective as an S-act. Also an S-act A is called (*principally, finitely generated) weakly injective* if for every (principal, finitely generated) ideal K of S and any S-homomorphism $f : K \to A$, there exists an S-homomorphism $\bar{f}{:}S \to A$ which extends f, that is, $\bar{f}|_K = f$.

An S-subact A of an S-act B is called *large* in B if any S-homomorphism $f : B \to C$ whose restriction $f|_A$ to A is a monomorphism, is itself a monomorphism. An extension B of A with the embedding $f : A \rightarrowtail B$ is said to be an *essential extension* if Imf is large in B.

If the monoid S has a zero element 0, then each S-act has a zero element, too. From now on, for a monoid S with a zero element 0, we consider S-acts with unique zero θ, that is $A0 = \{\theta\}$, together with zero preserving S-homomorphisms between them. The category so obtained is denoted by $\mathbf{Act_0}$-**S**.

Throughout this paper, we take S to be a monoid of the form $S = G\dot{\cup}I$, where

G is a group and I is an ideal of S and $I^1 = I\dot{\cup}\{1\}$. It is worth noting that, since I is a subsemigroup of S, every S-act can be considered as an I^1-act .

Now, we mention the following theorems (Theorems III.3.2 and III.4.2 from [5]) used repeatedly through out the paper. But, first see the following definition:

Definition 1.1. [5] *Let A be an S-act and $a \in A$. Then, by λ_a we denote the S-homomorphism from S into A defined by $\lambda_a(s) = as$, for every $s \in S$, and by λ_s the S-homomorphism from S into S with $\lambda_s(t) = st$ for every $t \in S$. The kernel of λ_a is called the kernel equivalence (that is $s(ker\lambda_a)s'$ if and only if $\lambda_a(s) = \lambda_a(s')$ for $s, s' \in S$).*

Theorem 1.1. [5] *The following statements are equivalent for any S-act A over a monoid S:*

(1) *The S-act A is principally weakly injective;*
(2) *For every $s \in S$ and every S-homomorphism $f : sS \to A$, there exists $z \in A$ such that $f(x) = zx$ for every $x \in sS$;*
(3) *For every $s \in S$, $a \in A$ with $ker\lambda_s \leq ker\lambda_a$, one has that a is divisible by s in A, that is, $a = zs$ for some $z \in A$.*

Theorem 1.2. [5] *An S-act A is (finitely generated) weakly injective if and only if for every S-homomorphism f: $K \to A$, where $K \subseteq S$ is a (finitely generated) right ideal, there exists an element $z \in A$ such that $f(k) = zk$ for each $k \in K$.*

2. weakly injective S-acts

In this section, we show that if $S = G\dot{\cup}I$ is a monoid, then an S-act A is (principally, finitely generated) weakly injectivity if it is (principally, finitely generated) weakly injectivity as an I^1-act. We then give a criterion to recognize (principally, finitely generated) weakly injective S-acts.

Lemma 2.1. *Let S be a group. Then every S-act is principally weakly injective.*

Proof. Let A be an S-act and $a \in A$. Then $(t, t') \in ker\lambda_a$ if $(t, t') \in ker\lambda_s$, for every $s \in S$. Indeed, if $(t, t') \in ker\lambda_s$, then $\lambda_s(t) = \lambda_s(t')$ $(st = st')$, and so $s^{-1}st = s^{-1}st'$. Therefore, $at = at'$, meaning that $(t, t') \in ker\lambda_a$. Hence $ker\lambda_s \leq ker\lambda_a$, for every $s \in S, a \in A$. Thus, for every $s \in S, a \in A$, we have $a = a(s^{-1}s) = (as^{-1})s$, meaning that every element of A is divisible by every element of S. Consequently, by Theorem 1.1, A is principally weakly injective. □

Theorem 2.1. *Let $S = G\dot{\cup}I$ be a monoid and A be an S-act. Then A is principally weakly injective as an S-act whenever it is principally weakly injective as an I^1-act.*

Proof. Since A is principally weakly injective as an I^1-act, by Theorem 1.1, there exist $b \in A$ such that $a = bi$, for some $i \in I^1$ and $a \in A$ with $ker\lambda_i \leq ker\lambda_a$. Also, Lemma 2.1 ensures that A is principally weakly injective as a G-act. So, for each $g \in G$ and $a \in A$ with $ker\lambda_s \leq ker\lambda_a$, there exists $c \in A$ such that $a = cg$. Hence, for $s \in S$ and $a \in A$ with $ker\lambda_s \leq ker\lambda_a$, there exists $b \in B$ such that $a = bs$. That is, A is principally weakly injective. $\square$

Corollary 2.1. (1) *Let $S = G\dot{\cup}I$ be a monoid. Then, principally weakly injectivity of all I^1-acts implies that all S-acts are principally weakly injective.*

(2) *For all monoids of the form $S = G\dot{\cup}\{0\}$, if an S-act A is principally weakly injective as a $\{0, 1\}$-act then it is principally weakly injective as an S-act.*

Theorem 2.2. *Let $S = G\dot{\cup}I$ be a monoid and A be an S-act. Then, A is (finitely generated) weakly injective as an S-act whenever it is (finitely generated) weakly injective as an I^1-act.*

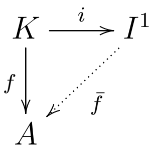
Proof. Let J be a (finitely generated) right ideal of S and $f : J \to A$ be an S-homomorphism. We consider f as an I^1-homomorphism. So, A being (finitely generated) weakly injective as an I^1-act, by Theorem 1.2, implies that there exists $a \in A$ such that $f(j) = aj$ for every $j \in J$. Also, for an S-homomorphism $f : S \to A$ we have $f(s) = f(1.s) = f(1)s$. Hence, for every S-homomorphism $f : K \to A$, where $K \subseteq S$ is a (finitely generated) right ideal, there exists an element $z \in A$ such that $f(k) = zk$, for every $k \in K$. Therefore, the result follows from Theorem 1.2. $\square$

By the above theorem, we get a useful criterion to check (finitely generated) weakly injectivity of S-acts. See the following examples.

Example 2.1. *(1) Let $S = G\dot{\cup}I$ be a monoid. Then, (finitely generated) weakly injectivity of all I^1-acts implies that of all S-acts.*

(2)Let $S = G\dot{\cup}\{0\}$. Then, an S-act is (finitely generated) weakly injective, if it is (finitely generated) weakly injective as an $\{0, 1\}$-act

(3) Let $S = (\mathbb{Q}, .)$ be the monoid of all rational numbers with the usual multiplication. Consider $S = (\mathbb{Q} - \{0\})\dot{\cup}\{0\}$, where $G = \mathbb{Q} - \{0\}$ is a group and $I = \{0\}$ is an ideal of S. The monoid $I^1 = \{0, 1\}$ has only one proper ideal, that is $K = \{0\}$. For an I^1-act A with a unique fixed element θ, there exists only one I^1-homomorphism from K into A, $f : K \to A$, with $f(0) = \theta$. Now we define I^1-homomorphism $\bar{f} : I^1 \to A$ to be $\bar{f}(0) = \bar{f}(1) = \theta$. Then we have the following commutative diagram.



That is A is (principally, finitely generated) weakly injective. Hence all I^1-acts with a unique fixed element θ are (principally, finitely generated) weakly *injective. So, by Theorems 2.1 and 2.2, all $\mathbb{Q}$-acts with a unique fixed element θ are (principally, finitely generated) weakly injective.*

(4) Analogously, one can see that all $\mathbb{R}$-acts with a unique fixed element θ are (principally, finitely generated) weakly injective, in which $\mathbb{R}$ is the monoid of all real numbers with usual multiplication.

We know that, every group is a regular monoid. Now, if I^1 is a regular monoid then $S = G\dot{\cup}I$ is a regular monoid, too. Therefore, by Theorem 4.1.6 of [5], we have the following corollary.

Corollary 2.2. *Let $S = G\dot{\cup}I$ be a monoid. Then, all S-acts are principally weakly injective whenever I^1 is a regular monoid.*

Definition 2.1. *A monoid S is called weakly left zero if for every $s \in S$ there exist $t \in S$ such that $st = s$.*

Definition 2.2. *A monoid S is called a kernel monoid if it is weakly left zero and for every $s \in S$ there exists $t \in S$ such that $ker\lambda_s \leq ker\lambda_t$.*

Theorem 2.3. *Let $S = G\dot{\cup}I$ be a monoid. Then every S-act is principally weakly injective whenever I^1 is a kernel monoid and principally weakly self injective.*

Proof. Since I^1 is a kernel monoid, for every $i \in I^1$, there exists $j \in I^1$ such that $ker\lambda_i \leq ker\lambda_j$ and $ij = i$. Also, I^1 is principally weakly self injective, so, by Theorem 1.2, j is divisible by i. That is, there exists $x \in I^1$ such that $j = xi$. Now, we have $i = ixi$, meaning that i is a regular element. Therefore, I^1 is a regular monoid. Consequently, S is regular and, by Theorem 4.1.6 of [5], every S-act is principally weakly injective. □

Corollary 2.3. *Let $S = G\dot{\cup}I$ be a monoid and I^1 be a principally weakly self injective monoid. If I^1 is a left cancelable and weakly left zero monoid then every S-acts is principally weakly injective.*

Proof. By Theorem 2.3, it is enough to show that $ker\lambda_i \leq ker\lambda_j$ for every $i, j \in I^1$. But it easily follows from the left cancelablity of I^1. □

3. **injective S-acts**

In this section, we first show that injectivity is extendable from I^1-acts to S-acts in the category of S-acts with a unique zero and zero preserving S-homomorphisms between them, denoted by $\mathbf{Act_0\text{-}S}$, and we give some examples. We then provide some properties under which injectivity can be extended from I^1-acts to S-acts in general. We also show that if there exists a nontrivial semigroup homomorphism $h : S \to I^1$ with $h(1) = 1$, then the injective S-acts are precisely the injective I^1-acts. First we mention the following theorems (Theorems 3.10 and III.1.20 and 4.4 from [4, 5, 10]).

Theorem 3.1. [4] *The following conditions are equivalent for an S-act A:*

(i) A is injective,

(ii) any consistent system of equations with constants from A has a solution in A.

Theorem 3.2. [5] *An act is injective if and only if it has no proper essential extension.*

Theorem 3.3. [10] *For a semigroup S, each S-act A is weakly injective if and only if every right ideal I of S has an idernpotent generator.*

Remark 3.1. *Given a cyclic S-act xS in which $S = G\dot{\cup}I$, one can easily check that xI is an S-subact of xS while xG is not. Suppose that $m \in xI \cap xG$. Then $m = xg, g \in G$ and $m \in xI$. Since xI is subact of xS, $(xg)g^{-1} = x(gg^{-1}) = x \in xI$. Thus, $xS \subseteq xI$. Obviously, $xI \subseteq xS$, and hence $xI = xS$. So if $xI \cap xG \neq \varnothing$ then $xI = xS$.*

Theorem 3.4. *Let $S = G\dot{\cup}I$ be a monoid with zero element 0, and $GI = \{0\}$. Then, each S-act $A \in$ $\mathbf{Act_0}$-$\mathbf{S}$ is an injective S-act whenever it is injective as an I^1-act.*

Proof. Suppose A is an injective I^1-act. Since A contains a zero θ, by Theorem 1 of [9], it is enough to show that A is injective with respect to the inclusions into cyclic right acts. So, we prove that every S-homomorphism $f : B \to A$, in the following diagram, is extended to $\bar{f}$.

$$\begin{array}{ccc} B_S & \xrightarrow{\;i\;} & (xS)_S \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \bar{f}} & \\ A_S & & \end{array} \tag{1}$$

But, the following possible cases can occur.

case1. If $xI \cap xG \neq \varnothing$ then, by the above remark, we have $xI = xS$. Now, considering xS, A and B as I^1-acts and i, f as I^1-homomorphisms, we get the following commutative diagram which is completed by an I^1-homomorphism $\bar{f} : xS \to A$, since A is an injective I^1-act.

$$\begin{array}{ccc} B_{I^1} & \xrightarrow{\;i\;} & (xS)_{I^1} \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \bar{f}} & \\ A_{I^1} & & \end{array}$$

Now, we show that $\bar{f} : xS \to A$ is in fact an S-homomorphism and commutes Diagram 1 of S-acts. Because, for every $s \in S$ and $xt \in xS = xI$, we have:

(1) if $s \in I$ then $\bar{f}((xt)s) = \bar{f}(xt)s$

(2) if $s \in G$ then $\bar{f}((xt)s) = \bar{f}(x(ts)) = \bar{f}(x)ts = \bar{f}(xt)s$.

case2. If $xI \cap xG = \varnothing$ then injectivity of A as an I^1-act implies the following commutative diagram of I^1-acts and I^1-homomorphisms.

$$\begin{array}{ccc} B_{I^1} & \xrightarrow{\ i\ } & (xS)_{I^1} \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ A_{I^1} & & \end{array}$$

Now, we define the map $\bar{f} : xS \to A$ to be:

$$\bar{f}(xs) = \begin{cases} g(xs) & , xs \in xI \\ \theta & , xs \in xG. \end{cases}$$

Clearly, $\bar{f}$ is well-defined. Also for every $xs \in xS$ and $t \in S$ we have:
(1) If $t \in I, xs \in xI$ then $\bar{f}((xs)t) = \bar{f}(x(st)) = g(x(st)) = g((xs)t) = g(xs)t = \bar{f}(xs)t$.
(2) If $t \in I, xs \in xG$ then $\bar{f}((xs)t) = \bar{f}(x(st)) = \bar{f}(x.0) = g(x.0) = g(\theta_{xS}) = \theta_{A_S} = \theta_{A_S}t = \bar{f}(xs)t$.
(3) If $t \in G, xs \in xG$ then $\bar{f}((xs)t) = \bar{f}(x(st)) = \theta_{A_S} = \theta_{A_S}t = \bar{f}(xs)t$.
(4) If $t \in G, xs \in xI$ then $\bar{f}((xs)t) = \bar{f}(x(st)) = g(x(st)) = g(x)(st) = (g(x)s)t = g(xs)t = \bar{f}(xs)t$.
These mean that $\bar{f}$ is an S-homomorphism.

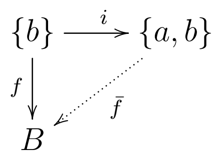
□

The above theorem gives a useful criterion to find the injective S-acts, where $S = G\dot{\cup}I$. Specially, if $S = G\dot{\cup}\{0\}$, then clearly $G\{0\} = \{0\}$. Now, the above theorem ensures that an S-act is injective if it is injective as $\{0,1\}$-act. See the following corollary.

Corollary 3.1. (1) *Let $S = G\dot{\cup}\{0\}$. Then, injectivity of each $A \in$ **Act₀-S** as a $\{0,1\}$-act implies the injectivity of it as an S-act.*

(2)*If all I^1-acts in the category **Act₀-S** are injective then all S-acts are injective.*

Example 3.1. *(1) All the $\mathbb{Q}$-acts are injective, in which $\mathbb{Q}$ is the rational numbers with the usual multiplication. Indeed, one can consider $\mathbb{Q} = (\mathbb{Q} - \{0\})\dot{\cup}\{0\}$, where $G = \mathbb{Q} - \{0\}$ is a group, $I = \{0\}$ is an ideal of S, and $I^1 = \{0,1\}$. Now, by Corollary 3.1, it is enough to show that every $\mathbb{Q}$-act B is an injective I^1-act. But, since B contains the zero element θ, by Theorem 1 of* [9]*, it is enough to show that B is injective with respect to the inclusions into cyclic acts. It worths noting that there are only two non-isomorphic cyclic I^1-acts. One is the trivial I^1-act $\Theta = \{\theta\}$ and the other is $A = \{a, b \mid a0 = b0 = b\}$. So, there exists only one proper inclusion map $\{b\} \hookrightarrow \{a,b\}$. Now, for every I^1-homomorphism $f : \{b\} \to B, (f(b) = \theta_B)$, there exists an I^1-homomorphism $\bar{f} : \{a,b\} \to B$ defined by $\bar{f}(a) = f(b) = \theta_B$, which commutes the following* *diagram.*

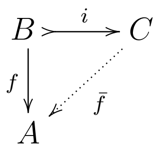


Clearly, B is injective relative to $i_1 : \{\theta\} \to \{\theta\}$ and $i_2 : \{a, b\} \to \{a, b\}$. That is, B is injective, and hence all I^1-acts are injective.

(2) Analogously, one can see that all the $\mathbb{R}$-acts are injective, in which $\mathbb{R}$ is the monoid of the real numbers with usual multiplication.

Theorem 3.5. *Let $S = G\dot{\cup}I$ be a monoid whose idempotents are central. Then, every S-act with a unique zero is injective whenever every I^1-act is so.*

Proof. Let every I^1-act be injective. Then, Proposition 4.4 of [10], ensures that every ideal of I^1 is generated by an idempotent. Hence, every ideal of S is generated by an idempotent. So, weakly injectivity of every S-acts follows from Proposition 4.4 of [10]. Now, we show that every diagram



of S-acts is completed by $\bar{f}$. To do so, consider

$$\rho = \{(X_S, h) | B_S \subseteq X_S \subseteq C_S, h : X_S \to A_S, h|_{B_S} = f\}$$

and define the relation $\leq$ on ρ to be:

$$(X_1, h_1) \leq (X_2, h_2) \Leftrightarrow X_1 \subseteq X_2, h_2|_{X_1} = h_1.$$

It is easy to check that $\leq$ is a partial order on ρ and every chain such as $(X_\alpha, h_\alpha)_{\alpha \in I}$ has the upper bound $(\cup X_\alpha, \bar{h})$, where $\bar{h}(x_\alpha) = h_\alpha(x_\alpha)$ for $x_\alpha \in X_\alpha$. Suppose (X'_S, h') is the maximal element of ρ ensured by Zorn's Lemma. We shall show that $X'_S = C_S$. Assume $X'_S \neq C_S$. So, there exists $x \in C_S \setminus X'_S$. Define J to be $\{s \in S \in X'_S\}$. Obviously, two possible cases can occur: J is an ideal of S or it is empty.

case1. If J is an ideal then there exists an idempotent $e \neq 1$ of S such that $J = eS$. Since A is weakly injective, there exists an S-homomorphism $\bar{k} : S \to A$ such that $\bar{k}|_J = k$, for every S-homomorphism $k : J \to A$ with $k(j) = h'(xj)$, for every $j \in J$. So we have:

$$k(j) = \bar{k}(j) = \bar{k}(1)j = h'(xj). \tag{2}$$

Now we define the map $l : X' \cup xS \to A$ to be:

$$l(y) = \begin{cases} h'(y) & , y \in X' \\ \bar{k}(es) & , y \in xS. \end{cases}$$

The defined l is well-defined. Indeed, if $y_1 = y_2 \in X' \cup xS$, then we have:

(1) if $y_1 = y_2 \in X'$, then $h'(y_1) = h'(y_2)$. And so, $l(y_1) = l(y_2)$.

(2) if $y_1 = y_2 \in xS$, then $y_1 = xs_1, y_2 = xs_2$, for some $s_1, s_2 \in S$. Also since e is a central idempotent, $xes_1 = xes_2$. That is, $l(y_1) = l(xs_1) = \bar{k}(1)es_1 = h'(xes_1) = h'(xes_2) = \bar{k}(1)es_2 = l(xs_2) = l(y_2)$.

(3) if $y_1 = y_2 = y \in X' \cap xS$, then $y \in xS$ implies that $y = xt$ for some $t \in S$. So, $y = xt \in X'$, and hence $t \in J$ follows from the definition of J. Now, (2) implies that $l(y) = l(xt) = h'(xt) = \bar{k}(1)t$, and if $y \in xS$ we have $l(y) = l(xt) = \bar{k}(1)et = \bar{k}(1)t$. Also, for every $y \in X' \cup xS$ and $s \in S$ we have:

(1) if $y \in X'$ then $l(ys) = h'(ys) = h'(y)s = l(y)s$.

(2) if $y \in xS$ then $l(ys) = l((xt)s) = l(x(ts)) = \bar{k}(1)ets = (\bar{k}(1)et)s = l(xt)s = l(y)s$.

Hence l is an S-homomorphism.

case2. If J is empty, then we define $l : X' \cup xS \to A$ as follows:

$$l(y) = \begin{cases} h'(y) & , y \in X' \\ \theta & , y \in xS. \end{cases}$$

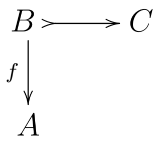
Clearly l is a well-defined and an S-homomorphism. In both cases, l has been extended to h', which is a contradiction. So $X'_S = C_S$ and A is injective. $\square$

Suppose that $S = G\dot{\cup}I$, and $h : S \to I^1$ is a nontrivial semigroup homomorphism with $h(1) = 1$. It is easy to check that every I^1-act A turns to an S-act by the following action:

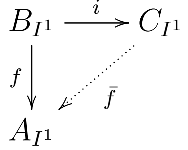
$$a.s = ah(s), a \in A, s \in S.$$

Theorem 3.6. *Let $S = G\dot{\cup}I$, and $h : S \to I^1$ be a nontrivial semigroup homomorphism with $h(1) = 1$. Then, A is an injective I^1-act if and only if it is an injective S-act.*

Proof. Necessity. Consider the diagram



of S-acts. To complete the diagram, we first consider A, B and C as the I^1-acts and f as an I^1-homomorphism. The existence of an I^1-homomorphism $\bar{f} : C \to A$ which completes the diagram follows from the hypothesis.



Now, for every $c \in C$ and $s \in S$ we have:

$$\bar{f}(c.s) = \bar{f}(c.h(s)) = \bar{f}(c)h(s) = \bar{f}(c)s.$$

This means that $\bar{f}$ is an S-homomorphism and so A is injective as an S-act.

sufficiency. Suppose A is not injective as an I^1-act. Then, by Proposition 3.1.20 of [5], A has a proper essential extension such as the I^1-act B, meaning that A is a large I^1-subact of B. Now, since every S-homomorphism $f : B \to C$ whose restriction $f|_A$ to A is a monomorphism can be considered as an I^1-homomorphism and A is a large I^1-subact of B, f is a monomorphism. Namely, A is a large S-subact of B or equivalently B is a proper essential extension of A. This contradicts with the injectivity of A as an S-act. □

REFERENCES

[1] Berthiaume, P.: *The injective envelope of S-sets.* Canad. Math. Bull., 10:261-273, (1967).

[2] Golchin, A., Renshaw, J.: *Periodic monoids over which all flat cyclic right acts satisfy condition (P).* Semigroup Forum 54(2): 261 263, (1997).

[3] Golchin, A.: *On flatness of acts.* Semigroup Forum 67:262 270, (2003).

[4] Gould, V.A.R.: *The characterization of monoids by properties of their S-systems.* Semigroup Forum 32:251-265,(1985).

[5] Kilp, M., Knauer, U., Mikhalev, A.: *Monoids, Acts and Categories.* Walter de Gruyter, Berlin, New York,(2000).

[6] Luedeman, J., McMorris, F., Sim, S.K.: *Semigroups for which every totally irreducible S-system is injective.* Comment. Math. Univ. Carolinae, 19:27-35,(1978).

[7] Shoji, K.: *Injective hulls of certain right reductive semigroups as right S-systems.* Mem. Fac. Sei., Shimane Univ., 14:25-34, (1980).

[8] Shoji, K.: *Completions and injective hulls of E-reflexive inverse semigroups.* Semigroup Forum, 36:55-68, (1987).

[9] Skornjakov, L.A.: *On homological classification of monoids.* Sib. Math. J., 10:1139-1143, 1969 (in Russian), correction ibid. 12(1971) 689.

[10] Weinert, H.J.: *S-sets and semigroups of quotients* Semigroup Forum, 19, 1-78,(1980).