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COMPARATIVE ANALYSIS OF ORTHOTROPIC PLATES

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In this article are proposed some new analytical solutions for solving plate displacements problem and corresponding modal analysis for thin orthotropic rectangular plates, having clamped edges. The considered reasons for the solutions were to exactly satisfy the boundary conditions and are compared to those found in the literature. Also, a complex comparative study for analytical, numerical and experimental results is performed, aiming to validate the proposed analytical solutions.

Key words: thin plates, orthotropic, analytical model, clamped edges

1. Introduction

Thin-walled structures are frequently found in engineering practice due to technical and economic advantages that they have. In these structures are found flat thin plates or with very small curvatures. Such plates are subjected to mechanical stress differently in different directions, which explains the importance of the study of composite materials. In the particular case, orthotropic plates are created with different stiffness in two directions, their use improved ratio between stiffness and weight.

Given the complexity of design of fiber-reinforced materials and heterogeneous character, modeling of their mechanical response under different external stress is particularly difficult to deal in the absence of simplifying assumptions. In the literature there are many papers that propose different models with varying degrees of approximation [1], [2], [3].

There are also numerical approaches of applications involving composites, especially regarding impact problems by SPH [5].

However, analytical models and solutions remain a prospective tool, even they are obtained generally for quite simple or particular cases of geometries and/or loadings. They are still very useful to validate a numerical solution, they are less expensive in time and calculation volume than the numerical ones, they can emphasize how the solution depends on data, etc.

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Thus, this article proposed new analytical solutions that consider both solving the problem in displacements and modal analysis of orthotropic thin rectangular plates with clamped edges.

The considered reasons for the solutions were to exactly satisfy the boundary conditions and to verify as close as possible the differential equation of the plate. The weighted residue method was considered to optimise the chosen analytical solutions. Interesting evaluations were performed for different types of functions, especially with respect to the orthotropic answer of the plate. The purposed solutions were compared with those obtained by Reddy in [2]. Finally, the solutions were critically analysed considering a FEM solution and experimental data.

Thorough comparison between analytical solutions, numerical results and experimental data reveals a good agreement of the results.

2. Assumptions

The following assumptions are used in the analysis of thin plate model:

i) The constitutive law is orthotropic elasticity.

ii) Strain-displacement relation is linear, i.e. geometrical linearity.

The classical laminated plate theory is applied considering Kirchhoff-Love hyphotheses:

iii) The inextensibility of normal is imposed, implying that during deformation the normal to the median of the plate remain straight, i.e.the transverse displacement is independent of the transverse (thickness) coordinate.

iv) During deformation the plate thickness remains constant, equivalent with $\varepsilon_z = 0$.

v) During deformation the transverse normals remain perpendicular to the midsurface, i.e. $\varepsilon_{xz} = 0, \varepsilon_{yz} = 0$.

3. Orthotropic plate equations

For an orthotropic elastic thin rectangular plate subjected to an uniform distributed pressure *p* on the bottom face, plate with thickness *h*, lenghth *a* and width *b*, the orthotropic elastic constitutive equations is:

$$[\sigma] = \bar{Q} \cdot [\varepsilon],$$

where $\bar{Q} = R(\theta) \cdot Q$, $R(\theta) \in M_{6\times4}(\mathbb{R})$, is the matrix that defines the rotation angle fibers θ to the axis Ox, $Q = (Q_{11}, Q_{22}, Q_{12}, Q_{66})^T$ and

$$Q_{11} = \frac{E_1}{1 - \nu_{12} \cdot \nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12} \cdot \nu_{21}}, \; Q_{12} = \frac{\nu_{21} \cdot E_1}{1 - \nu_{12} \cdot \nu_{21}}, Q_{66} = G_{12},$$

where E_1, E_2 are elasticity moduli in the longitudinal and transversal directions, respectively, G_{12} is the shear modulus in the plane of the ply, and ν_{12} is the Poisson coefficient.

Replacing constitutive equation in the equilibrium equation and according to the hypothesis (i)-(v) it is obtained the partial differential equation for *w* (the displacement on *Oz* direction):

$$\overline{Q}_{11}\cdot\frac{\partial^4 w}{\partial x^4}+\overline{Q}_{22}\cdot\frac{\partial^4 w}{\partial y^4}+2\cdot\left(\overline{Q}_{12}+2\cdot\overline{Q}_{66}\right)\cdot\frac{\partial^4 w}{\partial x^2\partial y^2}+$$
$$+4\cdot\overline{Q}_{26}\cdot\frac{\partial^4 w}{\partial x\partial y^3}+4\cdot\overline{Q}_{16}\cdot\frac{\partial^4 w}{\partial x^3\partial y}=\frac{12\cdot p}{h^3}. \tag{1}$$

In particular case, when *Ox*-axis is oriented along the fiber direction, $\theta=0^\circ$, equation (1) becomes:

$$Q_{11}\cdot\frac{\partial^4 w}{\partial x^4}+Q_{22}\cdot\frac{\partial^4 w}{\partial y^4}+2\cdot\left(Q_{12}+2\cdot G_{12}\right)\cdot\frac{\partial^4 w}{\partial x^2\partial y^2}=\frac{12\cdot p}{h^3}. \tag{2}$$

For modal analysis, in orthotropic plate equations as a uniformly distributed load p is considered inertial force, resulting equation corresponding free vibrations:

$$\frac{h^3}{12}\cdot A\tilde{w}+\rho\cdot h\cdot\frac{\partial^2\tilde{w}}{\partial t^2}=0, \tag{3}$$

where $A=Q_{11}\frac{\partial^4}{\partial x^4}+Q_{22}\frac{\partial^4}{\partial y^4}+2\left(Q_{12}+2\cdot Q_{66}\right)\frac{\partial^4}{\partial x^2\partial y^2}$ for $\theta=0^\circ$ and

$$A=\overline{Q}_{11}\frac{\partial^4}{\partial x^4}+\overline{Q}_{22}\frac{\partial^4}{\partial y^4}+2\left(\overline{Q}_{12}+2\overline{Q}_{66}\right)\frac{\partial^4}{\partial x^2\partial y^2}+4\overline{Q}_{16}\frac{\partial^4}{\partial x^3\partial y}+4\overline{Q}_{26}\frac{\partial^4}{\partial x\partial y^3}$$ for

$\theta\neq 0^\circ$.

4. Analytical solutions

For a rectangular orthotropic plate with clamped edges, must be solved equations (2) for $\theta=0^\circ$, respectively (1), for $\theta\neq 0^\circ$, with boundary conditions:

$$\begin{cases} w(0,y)=w(a,y)=0, & \dfrac{\partial w}{\partial x}(0,y)=\dfrac{\partial w}{\partial x}(a,y) \\ w(x,0)=w(x,b)=0, & \dfrac{\partial w}{\partial y}(x,0)=\dfrac{\partial w}{\partial y}(x,b)=0 \end{cases}. \tag{4}$$

Let us mention that Reddy proposes solution for equation (2) with boundary conditions (4) as [2], [4], [7]:

$$w(x,y)=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}\cdot\left(\frac{x}{a}\right)^{i+1}\cdot\left(\frac{y}{b}\right)^{j+1}\cdot\left(1-\frac{x}{a}\right)^{2}\cdot\left(1-\frac{y}{b}\right)^{2}. \qquad (5)$$

For this solution is observed that the terms corresponding to the plate edges $x=a$ and $y=b$ are not included in the sum.

In this article, using Ritz method and compared to Reddy's solution, it is proposed solution [6], [7]:

$$w(x,y)=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}\cdot\left(\frac{x}{a}\right)^{i+1}\cdot\left(\frac{y}{b}\right)^{j+1}\cdot\left(1-\frac{x}{a}\right)^{i+1}\cdot\left(1-\frac{y}{b}\right)^{j+1}. \qquad (6)$$

The coefficients $\left(c_{ij}\right)_{i,j}$ will be determined using the weighted residue method.

According to this method, the following linear system is solved:

$$\iint_D\left[A(w(x,y))-\frac{12p}{h^3}\right]\cdot\Phi_{ij}(x,y)\,\mathrm{d}x\,\mathrm{d}y=0,\quad i=\overline{1,n},\quad j=\overline{1,m}$$

where $\Phi_{ij}=\left(\frac{x}{a}\right)^{i+1}\cdot\left(\frac{y}{b}\right)^{j+1}\cdot\left(1-\frac{x}{a}\right)^{i+1}\cdot\left(1-\frac{y}{b}\right)^{j+1}, i=\overline{1,n}, j=\overline{1,m}$,

are weighting functions (Galerkin method) and D is the domain of the composite plate.

For a comparative analysis between solutions (5) and (6), it is considered a thin rectangular plate with $a=300\,\text{mm}$, $b=200\,\text{mm}$, $h=1.45\,\text{mm}$, $E_1=22051\,\text{MPa}$, $E_2=18512\,\text{MPa}$, $G_{12}=8642\,\text{MPa}$, $\nu_{12}=0.071$, subjected to a uniform pressure $p=0.00419\,\text{MPa}$, using Maple we get the maximum value of the deflection $w\left(\frac{a}{2},\frac{b}{2}\right)\simeq 3.053\,\text{mm}$, using both solutions, (5) and (6).

Note that in particular, for an isotropic thin plate, of $300\times300\times1.45$ with $E=22051\,\text{MPa}$, $\nu=0.071$, subjected to a uniform pressure $p=0.00419$ MPa, the maximum value of the deflection using the new solution (6), is

$$w\left(\frac{a}{2},\frac{b}{2}\right)=0.0012653\cdot\frac{p\cdot a^4}{D},\ \text{where}\ D=\frac{E\cdot h^3}{12\cdot\left(1-\nu^2\right)}.$$

This result is high close of the exact solution obtained by Timoshenko and Woinowsky, $w\left(\frac{a}{2},\frac{b}{2}\right)=0.00126\cdot\frac{p\cdot a^4}{D}$.

In the same time, using Reddy's solution, it result

$$w\left(\frac{a}{2},\frac{b}{2}\right)=0.00133\cdot\frac{p\cdot a^4}{D}.$$

Note that a good agreement between the two solutions (5) and (6), but a faster convergence is obtained using solution (6).

In case for angle $\left(\theta\neq 0^\circ\right)$ must be solved equation (1) with boundary conditions (4).

In this article it is proposed a solution of form:

$$w(x,y)=w_0(x,y)+\sum_{i=1}^{r}\sum_{j=1}^{s}k_{ij}\cdot f_i(x,\theta)\cdot g_j(y,\theta) \tag{7}$$

where w_0 is the proposed solution (6), and the last terms have been added to provide asymmetric solution and to verify boundary conditions.

For $r=s=1$,

$$f_1(x,\theta)=\operatorname{tg}\theta\cdot\left(\frac{x}{a}\right)^2\cdot\left(1-\frac{x}{a}\right)^3,\quad g_1(y,\theta)=\left(\frac{y}{b}\right)^2\cdot\left(1-\frac{y}{b}\right)^3$$

$$f_2(x,\theta)=\operatorname{tg}\theta\cdot\left(\frac{x}{a}\right)^2\cdot\left(1-\frac{x}{a}\right)^3,\quad g_2(y,\theta)=\left(\frac{y}{b}\right)^3\cdot\left(1-\frac{y}{b}\right)^2.$$

For the same orthotropic plate, with $\theta=20^\circ$, subjected to a uniform pressure $p=0.00419\,\text{MPa}$, using Maple we get $w\left(\frac{a}{2},\frac{b}{2}\right)=3.0138\text{ mm}$.

The proposed analytical solution (7), for orthotropic rectangular plate with clamped edges, when $\left(\theta\neq 0^\circ\right)$, has the same theory as presented in [7], except that it starts from the solution (6), not from (5) proposed by Reddy in [2].

Note that both new solutions, (6) and (7), were critically analysed considering a FEM solution (SHELL63) and experimental data, using an experimental device.

Taking into account the necessity of several sets of measurements of orthotropic plate deflection, under the imposed boundary conditions as mentioned previously, it is chosen to apply the uniform distributed pressure perpendicular on the bottom face of the plate, a pressurized air chamber. This technical solution allows repeated measurements and application of pressure change in a convenient way. To measure the deformed plate deflection, an inductive displacement transducer is used.

In fig. 1 it is presented the comparative results between the experimental measurements, numerical and analytical results, for the orthotropic plate of mentioned dimension, having clamped edges, for $\theta = 0^\circ$, subjected to the uniform pressure $p = 0.00419$ MPa.

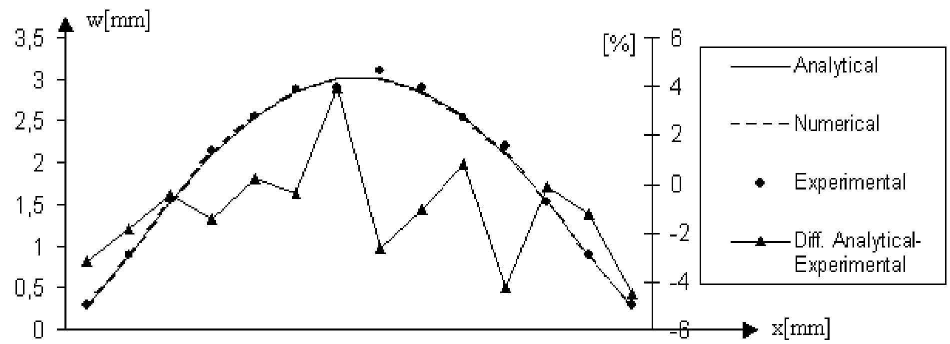

Fig. 1 Comparative results for orthotropic plate, when $\theta = 0^\circ$, for $y = 95$ mm

In fig. 2 shows the comparative results for the orthotropic plate, having clamped edges, for $\theta = 20^\circ$, subjected to the uniform pressure $p = 0.00419$ MPa.

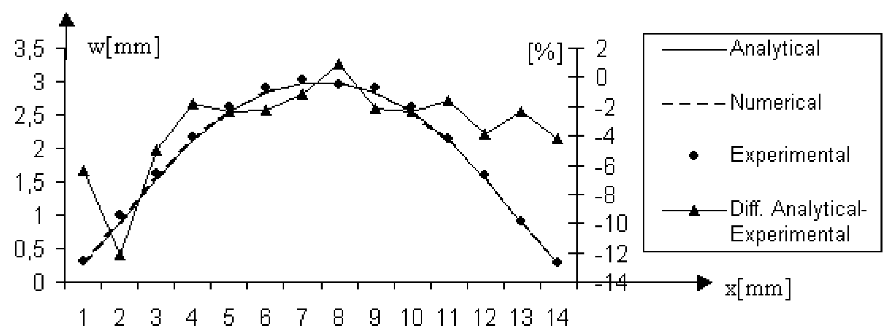

Fig. 2 Comparative results for orthotropic plate, when $\theta = 20^\circ$, for $y = 95$ mm

Following the analysis of the results in fig. 1 and 2 is observed small differences $(< 3\%)$ between theoretical and experimental values near the center of the plate, where deformations are maximum. Also, analyzing the results presented in Fig. 1 and 2, we can see the influence of the angle θ on the results of plate deformation. These differences are very small if the board is required pressures low enough so as to be verified assumptions mentioned above.

To perform modal analysis, to solve equation (3) where differential operator is $A = Q_{11}\frac{\partial^4}{\partial x^4} + Q_{22}\frac{\partial^4}{\partial y^4} + 2\left(Q_{12} + 2\cdot Q_{66}\right)\frac{\partial^4}{\partial x^2 \partial y^2}$, with boundary conditions (4), in this article it is considered a solution of form

$$w(x,y) = \sum_{n\geq 1}\sum_{m\geq 1} c_{nm}\cdot w_0\left(x,y\right)\cdot \sin\left(\frac{n\pi x}{a}\right)\cdot \sin\left(\frac{m\pi y}{b}\right),\text{ where}$$

$$w_0\left(x,y\right) = \left(\frac{x}{a}\right)^2\cdot\left(1-\frac{x}{a}\right)^2\cdot\left(\frac{y}{b}\right)^2\cdot\left(1-\frac{y}{b}\right)^2.$$

For $m = n = 1$ it is result frequency 1, $\nu_1 = 149.99$ Hz.

In the same time, for $n = 2,\ m = 1$, results $\nu_2 = 246.465$ Hz, for $n = 1,\ m = 2$, $\nu_3 = 367.395$ Hz, for $n = 3,\ m = 1$, $\nu_4 = 382.841$ Hz and for $n = 2,\ m = 2$, results $\nu_5 = 453.596$ Hz.

These results were critically analysed considering FEM solution (SHELL63) and experimental data. To perform experimental modal analysis we used an electromagnetic excitator.

In table 1 are presented the comparative results on modal analysis, between the experimental measurements, numerical and analytical results, for the orthotropic plate of dimension mentioned, having clamped edges, with $\theta = 0^\circ$.

Table 1

Frequence	ν_1	ν_2	ν_3	ν_4	ν_5
Analytic (a)	149.99Hz	246.465Hz	367.395Hz	382.841Hz	453.596Hz
FEM (b)	153.59Hz	243.21Hz	369.50Hz	396.07Hz	447.71Hz
Experim. (c)	149Hz	226Hz	334Hz	379Hz	430Hz
Diff. (%) a)-b)	-2.4 %	1.32 %	-0.572 %	-3.455 %	1.297 %
Diff. (%) a)-c)	0.66 %	8.3 %	9.08 %	1.003 %	5.201 %

To perform experimental measurements was used the device in [7],[9]. In this article, a comprehensive analysis on the influence of reinforcement fibers, (θ), only one big orthotropic plate was manufactured at STRAERO SA, having a thickness of 1.45 mm, which was cut in two rectangular plates of size (200/300), one with an angle $\left(\theta = 0^\circ\right)$, another with $\left(\theta = 20^\circ\right)$.

5. Conclusions

The analytical solutions approximates in a proper way the deformation of a composite plate having clamped edges. Small differences in results validate the proposed analytical solutions in this article.

Is important to study the rectangular plates, their corresponding theory can be extended by a conform transformation to any surface.

R E F F E R E N C E S

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