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FRACTAL VECTOR MEASURES

Ion CHIȚESCU[1], Lucian NIȚĂ[2]

In this paper we extend the concept of the fractal measure (the Hutchinson measure, which is the unique fixed point of a contraction on the set of normalized Borel measures on a compact metric space) associated to an iterated function system. An important property of this measure is that its support is the atractor of the iterated function system. Here, an extension of this result is given for the case of vector measures taking their values in a finite dimensional space or in an arbitrary Banach space. .

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1. Introduction

The concept of *fractal set* is, nowadays, extremely important not only in mathematics, but also to help the understanding of a lot of phenomena from physics, biology, economy etc.

Many fractals appear as fixed points of some contractions on the complete metric space $(\mathbf{K}(T), \delta)$, where $\mathbf{K}(T)$ is the set of the compact and not empty subsets of a complete metric space (T, d) and δ is the Hausdorff-Pompeiu distance. We can associate to such a "fixed point" set (called *atractor*) a measure, called *the Hutchinson measure*, that is also a fixed point of an operator (called *the Markov operator*) which is a contraction on the complete metric space of the *Borel normalized* measures on T, with respect to a certain distance (see the section **2.1**). The important property of this measure is that its support is just the atractor (the fractal). For more details and proofs of the results from section **2.1**, see, for example, [1].

In this paper, we give two extensions of the Hutchinson measure : one for the case of vector measures with values in a finite dimensional vector space, and the other one for the case when the vector measures take values in an arbitrary Banach space. One of the interesting results is that, even in the case of vector

[1] Faculty of Mathematics and Computer Science, University of Bucharest, Academiei Str. 14, 010014, Bucharest, Romania
[2] Technical University of Civil Engineering, Lacul Tei Blvd., 122-124, 020396, Bucharest, Romania

measures, we can obtain the "fixed point" measure (the analogue of the Hutchinson measure) having as support the fractal. The general framework is given in section **2.2**. For more details and proofs of the results from the section **2.2**, one can consult [2], [3], [4]. For additional facts, see [5], [6], [7].

2. Preliminary facts

2.1. Hutchinson metric and measure

Let (T, d) be a compact metric space.

Definition 2.1.1. A positive measure μ defined on the Borel subsets of T is called *normalized measure* if $\mu(T) = 1$.

We will denote by **B** the set of all the normalized measures. We will also denote $Lip_1(T) = \{f : T \to \mathbf{R} : |f(x) - f(y)| \le d(x, y), \forall x, y \in T\}$.

Let $d_H : \mathbf{B} \text{x} \mathbf{B} \to \mathbf{R}$, $d_H(\mu, \nu) = \sup\{\int f d\mu - \int f d\nu : f \in Lip_1(T)\}$.

Proposition 2.1.2. d_H is a metric on **B**, called the *Hutchinson metric* (or *the Kantorovich-Rubinshtein metric*). Besides, $(\mathbf{B}, d_H)$ is a compact metric space.

Definition 2.1.3. Let N be a natural number. A set of functions $(\omega_i)_{i=1}^N$ is called *iterated function system* if, for any $i \in \{1,2,...,N\}$, the function $\omega_i : T \to T$ is a contraction of ratio $r_i < 1$.

Let us consider the numbers $p_1, p_2, ..., p_N, p_i > 0, \forall i \in \{1,...,N\}$, such that $\sum_{i=1}^{N} p_i = 1$.

Definition 2.1.4. The pair $((\omega_i)_{i=1}^N, (p_i)_{i=1}^N)$ is called *iterated function system with probabilities*.

Definition 2.1.5. The function m: $\mathbf{B} \to \mathbf{B}$, $m(\nu) = \sum_{i=1}^{N} p_i \omega_i(\nu)$, generated via $m(\nu)(A) = \sum_{i=1}^{N} p_i \nu(\omega_i^{-1})(A)$ for any Borel set $A \subset T$, is called *Markov operator.*

Theorem 2.1.6. The Markov operator is a contraction on the metric space $(\mathbf{B}, d_H)$. Consequently, there is an unique measure $\mu \in \mathbf{B}$, such that $m(\mu) = \mu$.

Definition 2.1.7. The measure μ from theorem **2.1.6** is called the *Hutchinson measure.*

Theorem 2.1.8. The support of the Hutchinson measure is the atractor of the iterated function system $(\omega_i)_{i=1}^{N}$, that is, the unique set $F \subset T$ such that $F = \bigcup_{i=1}^{N} \omega_i(F)$.

2.2 Vector measures associated with an iterated funtion system

We will use the following notations :
(T, d) is a compact metric space, as before; X is a Banach space;
$B(X)$ is the set of liniar and continuous operators on X;
If $H \in B(X)$, we will denote by $\|H\|_0$ the operatorial norm;

Definition 2.2.1. Let μ be a vector measure, defined on the Borel subsets of T, taking values in X. For any Borel set $A \subset T$ we define *the variation of A* (denoted by $|\mu|(A)$) via :

$|\mu|(A) = \sup\{\sum_{j=1}^{n} \|\mu(A_j)\|\}$, the supremmum being computed with respect to all the partitions of A built with finite families of Borel sets. If $|\mu|(T) < \infty$, we say that μ has *bounded variation.*

We denote by $cabv(X)$ the set of vector measures with bounded variation.

Definition 2.2.2. Let $\mu \in cabv(X)$ and $|\mu|$ the variation of μ .The support of $|\mu|$ is called *the support of the vector measure* μ and is denoted by supp(μ).

Proposition 2.2.3. The application $\|\ \| : cabv(X) \to [0,\infty), \|\mu\| = |\mu|(T)$ is a norm on $cabv(X)$, called *the variational norm.* The space $(cabv(X), \|\ \|)$ is a Banach space.

Let now M be a natural number, $(\omega_i)_{i=1}^{M}$ be an iterated function system and $(R_i)_{i=1}^{M} \subset B(X)$. For any $\mu \in cabv(X)$, we denote : $H(\mu) = \sum_{i=1}^{M} R_i o \omega_i(\mu)$, which means that, for any Borel subset $A \subset T$, $H(\mu)(A) = \sum_{i=1}^{M} R_i(\mu(\omega_i^{-1}(A)))$.

In the sequel, we shall briefly present the integral introduced and studied in ***Sesquilinear Uniform Vector Integral*** by Ion Chițescu, Radu Miculescu, Lucian Niță and Loredana Ioana, to appear in Proc. Indian Acad. Sci. (Math. Sci.).

We shall work with an arbitrary Hilbert space X over the scalar field K (real or complex). In particular, $X = K^n$. We will use the following function spaces :

$S(X) = \{f : T \to X, f$ simple function$\}$;

$TM(X) = \{f : T \to X, f$ totally measurable i.e. uniform limit of simple functions$\}$; $C(X) = \{f : T \to X, f$ continuous function$\}$.

For any $A \subset T$, we denote by φ_A the characteristic function of A.

Definition 2.2.4. Let $f \in S(K^n), f = \sum_{i=1}^{m} \varphi_{A_i} x_i$, with $A_i \subset T$ being Borel sets. Also, let $\mu \in cabv(K^n)$. The number $\sum_{i=1}^{m} \langle x_i, \mu(A_i) \rangle$ is called *the integral of f with respect to* μ and is denoted by $\int f d\mu$ (it is easy to see that the integral doesn't depend on the representation of f).

If $f \in TM(X)$, let $(f_p)_p \subset S(X)$ be a sequence that converges uniformly to f. We define $\int f d\mu = \lim_{p \to \infty} \int f_p d\mu$.

Now, we shall briefly present the Monge-Kantorovich norm which has been introduced by Ion Chițescu, Radu Miculescu, Lucian Niță and Loredana Ioana in a forthcoming paper. Let $a > 0$ and $B_a(X) = \{\mu \in cabv(X), \|\mu\| \le a\}$;

$BL(X) = \{f : T \to X, f$ Lipschitz function$\}$; on $BL(X)$ we have the norm : $\|f\|_{BL} = \|f\|_\infty + \|f\|_L$, $\|f\|_L$ being the Lipschitz constant of f.

$BL_1(K^n) = \{f \in BL(K^n), \|f\|_{BL} \le 1\}$; for any $\mu \in cabv(X)$, we denote : $\|\mu\|_{MK} = \sup\{\left|\int f d\mu\right|, f \in BL_1(K^n)\}$.

Lemma 2.2.5. i)The application $\| \ \|_{MK}$ is a norm on $cabv(X)$, called *the Monge-Kantorovich norm* and $\|\mu\|_{MK} \le \|\mu\|, \forall \mu \in cabv(K^n)$; ii) For any $a > 0$, the topology generated by $\| \ \|_{MK}$ on $B_a(K^n)$ is the same as the weak-* topology on $B_a(K^n)$. iii)The space $B_a(K^n)$ with the metric generated by $\| \ \|_{MK}$ is a compact (hence complete) metric space.

Lemma 2.2.6. (change of variable formula) For any $f \in C(K^n)$, we have :

$\int f dH(\mu) = \int g d\mu$, where $g = \sum_{i=1}^{M} R_i^* o f o \omega_i$.

3. Results

We shall use two general schemes.

Scheme 3.1. Let $(Y,\| \ \|)$ be a normed space (not necessarily Banach), $H \in B(Y)$ with $\|H\|_0 < 1, y^0 \in Y$ .Define $P: Y \to Y, P(y) = H(y) + y^0$. We have :

$\|P(y) - P(z)\| \le \|H\|_0 \|y - z\|$, hence P is a contraction. Let $\phi \ne A \subset Y$ such that : i) $P(A) \subset A$; ii) A is a complete metric space for the metric d induced on A by $\| \ \|$, i.e. $d(x,y) = \|x - y\|$. We obtain the contraction $\pi : A \to A$ given via $\pi(y) = P(y)$. Using the contraction principle we deduce that there exists an unique fixed point $y^* \in A$ for π :

$$\pi(y^*) = H(y^*) + y^0 = y^* \Leftrightarrow (I - H)(y^*) = y^0 \qquad (1)$$

We will use this scheme as follows.

We take $Y = cabv(K^n, \| \ \|_{MK}), H : Y \to Y$, $H(\nu) = \sum_{i=1}^{M} R_i \omega_i(\nu)$ and $A = B_a(K^n)$ for some $a > 0$.

Lemma 3.2. Let $f \in L_1(X)$. We define, as before, $g = \sum_{i=1}^{M} R_i^* o f o \omega_i$. Then, g is a Lipschitz function and $\|g\|_L \le \|R_i\|_0 r_i$. (2)

Lemma 3.3. For any $n \in \mathrm{N}, n \ge 2$, let us consider the space $(cabv(K^n), \| \ \|_{MK})$. Then $H \in B(cabv(K^n))$ and $\|H\|_0 \le \sum_{i=1}^{M} \|R_i\|_0 (1 + r_i)$.

Proof. Let $f \in BL_1(K^n)$ arbitrarily and $g = \sum_{i=1}^{M} R_i^* o f o \omega_i$. For any $t \in T$, we have : $\|g(t)\| \le \sum_{i=1}^{M} \|R_i^*\|_0 \|f(\omega_i(t))\| \le \sum_{i=1}^{M} \|R_i\|_0$, hence $\|g\|_\infty \le \sum_{i=1}^{M} \|R_i\|_0$. We now use lemma **3.2.** and get : $\|g\|_{BL} = \|g\|_\infty + \|g\|_L \le \sum_{i=1}^{M} \|R_i\|_0 (1 + r_i)$ (3)

Using the change of variable formula, we can write :

$\left|\int f dH(\mu)\right| = \left|\int g d\mu\right| \le \|g\|_{BL} \|\mu\|_{MK} \le (\sum_{i=1}^{M} \|R_i\|_0 (1 + r_i)) \|\mu\|_{MK}$ and, from this, we deduce : $\|H(\mu)\|_{MK} \le (\sum_{i=1}^{M} \|R_i\|_0 (1 + r_i) \|\mu\|_{MK}$. Hence, $\|H\|_0 \le \sum_{i=1}^{M} \|R_i\|_0 (1 + r_i)$ QED.

Let $p_1, p_2, ... p_M \in (0,1)$ such $\sum_{i=1}^{M} p_i = 1$. For the iterated function system with probabilities $((\omega_i)_{i=1}^{M}, (p_i)_{i=1}^{M})$ we have the Markov operator

$m(\nu)=\sum_{i=1}^{M} p_i\omega_i(\nu)$. We denote by μ the Hutchinson measure associated with m. We have $m(\mu)=\mu$. Besides, it is known (see [1]) that supp$(\mu)=F$ (the atractor of the iterated function system). Let now $x\in X, x\neq 0$ and $\overline{\mu}=\mu x$.

i) $\overline{\mu}\in cabv(K^n)$. Indeed, if $(A_j)_{1\le j\le n}$ is a partition of T with Borel sets, then : $\sum_{j=1}^{n}\left\|\overline{\mu}(A_j)\right\|=\|x\|\sum_{j=1}^{n}\mu(A_j)=\|x\|\mu(T)=\|x\|$, because μ is normalized. From this , we deduce that $|\overline{\mu}|(T)=\|x\|$, hence $\overline{\mu}\in cabv(X)$ and $\|\overline{\mu}\|=\|x\|$.

ii) Similar to i), we get that for any Borel set B, $|\overline{\mu}|(B)=\|x\|\mu(B)$. So, $|\overline{\mu}|(B)=0\Leftrightarrow \mu(B)=0$. We conclude that supp$(\overline{\mu})=$ supp$(\mu)=F$.

Theorem 3.4. Let us consider the space $(cabv(K^n),\|\ \|_{MK})$. We denote by $\mu^0=(I-H)(\overline{\mu})$ and, for any $\nu\in cabv(K^n)$, $P(\nu)=H(\nu)+\mu^0$. Suppose that $\sum_{i=1}^{M}\|R_i\|_0(1+r_i)<1$. We choose a real number a such that $a\ge\frac{1+\|H\|_0}{1-\|H\|_0}\|\overline{\mu}\|$. Then : 1) $\overline{\mu}\in B_a(K^n)$; 2) $P(B_a(K^n))\subset B_a(K^n)$; 3) There is an unique measure $\mu^*\in B_a(K^n)$ such that $P(\mu^*)=\mu^*$; 4) supp$(\mu^*)=F$.

Proof.1) $\|H\|_0\le\sum_{i=1}^{M}\|R_i\|(1+r_i)$ (see Lemma **3.2**), so $\frac{1+\|H\|_0}{1+\|H\|_0}>1$.

Consequently, $a\ge\frac{1+\|H\|_0}{1-\|H\|_0}\|\overline{\mu}\|>\|\overline{\mu}\|$, hence $\overline{\mu}\in B_a(K^n)$.

2) For any $\nu\in B_a(K^n))$, we have :

$$\|P(\nu)\|\le\|H(\nu)\|+\|\mu^0\|=\|H(\nu)\|+\|(I-H)(\overline{\mu})\|\le\|H\|_0\|\nu\|+(\|I\|_0+\|H\|_0)\|\overline{\mu}\|=$$

$$=\|\overline{\mu}\|+\|H\|_0(\|\nu\|+\|\overline{\mu}\|)\le\|\overline{\mu}\|+\|H\|_0(a+\|\overline{\mu}\|)=\|\overline{\mu}\|(1+\|H\|_0)+a\|H\|_0\le$$

$$\le\|\overline{\mu}\|\frac{a(1-\|H\|_0}{\|\overline{\mu}\|}+a\|H\|_0=a.$$

Hence, $P(\nu)\in B_a(K^n)$. Denote $\pi:B_a(K^n)\to B_a(K^n),\pi(\nu)=P(\nu)$

3) For any $\gamma_1,\gamma_2\in B_a(K^n)$, we have :

$\|\pi(\gamma_1)-\pi(\gamma_2)\| = \|H(\gamma_1-\gamma_2)\| \le \sum_{i=1}^{M}\|R_i\|_0(1+r_i)\|\gamma_1-\gamma_2\|$, so π is a contraction. The set $B_a(K^n)$ is weak-* compact. But, the weak-* topology coincides with the topology generated by the Monge-Kantorovich norm on $B_a(K^n)$. We deduce that $B_a(K^n)$ is compact in the topology generated by $\| \ \|_{MK}$. Hence, $B_a(K^n)$ is complete in this topology. We conclude that there is an unique measure μ^* such that $P(\mu^*)=\mu^*$.

4) We have : $(I-H)(\mu^*)=(I-H)(\overline{\mu})$. But, considering the space $(cabv(K^n),\| \ \|)$ and $H \in B(cabv(K^n))$, we have $\|H\|_0 < 1$, hence $I-H$ is invertible. We obtain that $\mu^* = \overline{\mu}$ and supp$(\mu^*)=$ supp$(\overline{\mu})=F$ QED.

Let us pass to the second scheme.

Scheme 3.5. Let $(Y,\| \ \|)$ a Banach space, $H \in B(Y), \|H\|_0 < 1$.Let, also $y^0 \in Y$ and $P: Y \to Y$, $P(y)=H(y)+y^0$; we deduce that P is a contraction. Let $\phi \ne A \subset Y$ such that : i) $P(A) \subset A$; ii) A is a closed (and, consequently, complete) set for the metric $d(x,y)=\|x-y\|$. We obtain $\pi: A \to A$ given via $\pi(y)=P(y)$ and π is, also, a contraction. Using the contraction principle, we get $z^* \in A$, the unique fixed point of π. We deduce that $H(z^*)+y^0 = z^* \Leftrightarrow (I-H)(z^*)=y^0$. But $\|H\|_0 < 1$, hence $I-H$ is invertible. We conclude that $z^* = (I-H)^{-1}(y^0)$.

We will use this scheme taking $Y=(cabv(X),\| \ \|)$ (X being a Banach space) and $A=Y$.

Lemma 3.6. Let $(cabv(X),\| \ \|)$. Then $\|H\|_0 \le \sum_{i=1}^{M}\|R_i\|_0$. (3)

Theorem 3.7. Let us consider the Banach space $(cabv(X),\| \ \|)$. Let $\mu^0 \in cabv(X)$ and $P: cabv(X) \to cabv(X)$, $P(\mu)=H(\mu)+\mu^0$. Suppose that $\sum_{i=1}^{M}\|R_i\|_0 < 1$. Then :

a) there is an unique measure $\mu^* \in cabv(X)$, such that $P(\mu^*)=\mu^*$.

b) the measure μ^0 can be chosen such that supp$(\mu^*)=F$, where F is the atractor of the iterated function system $(\omega_i)_{i=1}^{M}$.

Proof. a) For any μ, $\nu \in cabv(X)$, we have : $\|P(\mu)-P(\nu)\| = \|H(\mu)-H(\nu)\| = \|H(\mu-\nu)\| \le \|H\|_0 \|\mu-\nu\| \le (\sum_{i=1}^{M}\|R_i\|_0)\|\mu-\nu\| < \|\mu-\nu\|$

Hence, P is a contraction on the Banach space $cabv(X)$. Taking $\pi = P$ in scheme **3.5.** , we deduce that P has only one fixed point : there is an unique measure $\mu^* \in cabv(X)$ such that $P(\mu^*) = \mu^*$.

b) Let $p_1, p_2, ... p_M \in (0,1)$ such $\sum_{i=1}^{M} p_i = 1$. For the iterated function system with probabilities $((\omega_i)_{i=1}^M, (p_i)_{i=1}^M)$ we have the Markov operator $m(\nu) = \sum_{i=1}^{M} p_i \omega_i(\nu)$. We denote, as before, by μ the Hutchinson measure associated with m. Hence, $m(\mu) = \mu$. Let now $x \in X, x \ne 0$ and $\overline{\mu} = \mu x$. As in the proof of theorem **3.4.** we get that $\operatorname{supp}(\overline{\mu}) = \operatorname{supp}(\mu) = F.$

Let us denote by I the identity operator on $cabv(X)$. We have : $P(\mu^*) = \mu^* \Leftrightarrow H(\mu^*) + \mu^0 = \mu^* \Leftrightarrow (I-H)(\mu^*) = \mu^0$. Using the inequalities: $\|H\|_0 \le \sum_{i=1}^{M} \|R_i\|_0 < 1$, we deduce that the operator $I - H$ is invertible in $cabv(X)$. Therefore, $\mu^* = (I-H)^{-1}(\mu^0)$. Choosing $\mu^0 = (I-H)(\overline{\mu})$, we get $\mu^* = \overline{\mu}$, hence $\operatorname{supp}(\mu^*) = \operatorname{supp}(\overline{\mu}) = F.$ QED.

Example 3.8. i) We denote by λ the Lebesgue measure on [0,1] and let $G : [0,1]\mathrm{x}[0,1] \to K$ be a continuous function. Let M= $\sup\{|G(x,y)|, x, y \in [0,1]\}$. For any $\overline{f} \in L^2(\lambda)$ and $x \in [0,1]$, we define $g(x) = \int_0^1 G(x,y) f(y) d\lambda(y)$, for any $f \in \overline{f}$. It is easy to prove that:

a) g is a continuous function on $[0,1]$; b) $\|g\|_2 \le \|f\|_2$ M.

ii) Let us consider now $V : L^2(\lambda) \to L^2(\lambda), V(\overline{f}) = \overline{g}$, where $\overline{f}, \overline{g}$ are the classes of f and g. From (3) we get :

$\|V(\overline{f})\|_2 = \|\overline{g}\|_2 \le \mathrm{M}\|\overline{f}\|_2$, hence V is continuous and $\|V\|_0 \le$ M.

ii)We take $\mu^0 \in cabv(L^2(\lambda))$ and $G_1, G_2;[0,1]\text{x}[0,1] \to K$, continuous functions, such that $\text{M}_i = \sup\{G_i(x,y), x,y \in [0,1]\} \le \frac{1}{3}$. Let us denote $g_i(x) = \int_0^1 F_i(x,y) f(y) d\lambda(y), \forall x \in [0,1], \forall f \in \overline{f}, \forall \overline{f} \in L^2(\lambda)$. We get the operators $R_i \in B(L^2(\lambda)), R_i(\overline{f}) = g_i, f \in \overline{f}, g_i \in \overline{g_i}$.

For $i \in \{1,2\}$, we have : $\|R_i\| \le \text{M}_i \le \frac{1}{3}$, hence $\|R_1\| + \|R_2\| \le \frac{2}{3} < 1$.

Let $T = [0,1], M = 2, \omega_i : T \to T, i \in \{1,2\}, \omega_1 = \frac{t}{3}, \omega_2 = \frac{t+2}{3}$ (Cantor contractions). For any Borel subset A of T, we have :

$$\omega_1^{-1}(A) = 3A \cap [0,1], \omega_2^{-1}(A) = (3A-2) \cap [0,1].$$

Finally, let $H(\mu) = \sum_{i=1}^{2} R_i o \mu o \omega_i^{-1}$, $P(\mu) = H(\mu) + \mu^0, \forall \mu \in cabv(L^2(\lambda)$.

According to Theorem **3.7.**, there is an unique measure $\mu^* \in cabv(L^2(\lambda)$ such that $P(\mu^*) = \mu^*$, that is :

$$R_1(\mu^*(3A \cap [0,1])) + R_2(\mu^*(3A-2) \cap [0,1]) + \mu^0(A) = \mu^*(A). \quad (4)$$

For example, we can take $F_1(x,y) = \frac{x^2 y^2}{3}, F_2(x,y) = \frac{xy}{3}$, hence $\text{M}_1 = \text{M}_2 = \frac{1}{3}$. Let

$$B \subset T, R_1(\mu^*(B))(x) = \frac{1}{3}\int_0^1 x^2 y^2 f_B(y) d\lambda(y), R_2(\mu^*(B))(x) = \frac{1}{3}\int_0^1 xy f_B(y) d\lambda(y),$$

$\forall x \in [0,1], f_B \in \overline{f_B}$

(we denoted by $\overline{f_B} = \mu^*(B)$). So, the relation (4) becomes :

$$\frac{1}{3}\left(\int_0^1 x^2 y^2 f_{3A\cap T}(y) d\lambda(y) + \int_0^1 xy f_{(3A-2)\cap T}(y) d\lambda(y)\right) + \mu^0(A)(x) = \mu^*(A)(x).$$

iii) Let now $p_1 = p_2 = \frac{1}{2}$ and the Markov operator: $m(\nu) = p_1\omega_1(\nu) + p_2\omega_2(\nu)$. If we denote by μ the Hutchinson measure, $m(\mu) = \mu \Leftrightarrow \mu = p_1\omega_1(\mu) + p_2\omega_2(\mu)$, that is, for any Borel set $A \subset T$, we have :

$$\mu(3A \cap T) + \mu((3A-2) \cap T) = 2\mu(A) \quad (5)$$

Let us take $\overline{f} \in L^2(\lambda) \setminus \{0\}$ and define $\overline{\mu} = \mu\overline{f}$. Let $\mu^0 = (I - H)(\overline{\mu})$. According to Theorem **3.7.**, we get $\mu^* = \overline{\mu} = \mu\overline{f}$ and $\operatorname{supp}(\mu^*) = F$, the atractor of the iterated function system $(\omega_i)_{i=1}^2$, that is the Cantor set.

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