U.P.B. Sci. Bull., Series D, Vol. 72, Iss. 2, 2010 ISSN 1454-2358

TRANSFER MATRIX METHOD FOR FORCED VIBRATIONS OF BARS

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Lucrarea prezintă metoda matricelor de transfer, aplicată la vibrațiile forțate ale barelor drepte. Se consideră cazul vibrațiilor axiale, ale barelor cu secțiune variabilă discontinuu, acționate de forțe perturbatoare concentrate. Sunt, de asemenea, prezentate câteva aplicații relativ simple.

The paper presents the transfer matrix method, applied to forced vibrations of straight bars. The case of axial vibrations is considered, of bars with discontinuously variable cross-section, acted upon by concentrated perturbation forces. Some relatively simple applications are also presented.

Keywords: vibration of bars, transfer matrix.

1. Introduction

Compared to the finite element method, the transfer matrix method is used more and more in the study of continuous system vibrations [1], [2], [6], [7]. The method is used for longitudinal, torsional and bending vibrations, as well as for any of their combinations [6], [7], [10], [12].

Some applications of the method have already been presented by the authors in references [2], [3]. This paper studies the forced vibrations of bars. The case of axial vibrations is considered, of bars with discontinuously variable cross-section, acted upon by concentrated perturbation forces. Some relatively simple applications are also presented.

2. Presentation of the method

The transfer matrix method is based on establishing relations between state vectors in two sections, by means of field matrixes.

Thus, for the study of free axial vibrations (fig. 1), the relation is [1], [7], [10], [11], [12]:

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$$\begin{Bmatrix} u_2 \\ N_2 \end{Bmatrix} = \begin{bmatrix} \cos\frac{\omega}{c} l_{1,2} & \frac{1}{EA} \cdot \frac{c}{\omega} \sin\frac{\omega}{c} l_{1,2} \\ -EA \cdot \frac{\omega}{c} \sin\frac{\omega}{c} l_{1,2} & \cos\frac{\omega}{c} l_{1,2} \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ N_1 \end{Bmatrix}, \tag{1}$$

where the following notations have been introduced:

– propagation speed of longitudinal waves,

$$c = \sqrt{\frac{E}{\rho}}; \tag{2}$$

– A – cross-section area; E – Young's modulus; ω – circular frequency of the vibration.

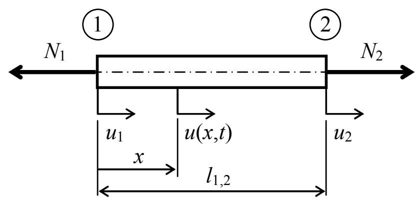

Fig. 1

By using the quantities [2], [3]

$$\frac{\omega}{c} = \alpha, \tag{3}$$

respectively,

$$q_u = u, \quad q_N = \frac{N}{EA\alpha}, \tag{4}$$

relation (1) becomes:

$$\begin{Bmatrix} q_u^{(2)} \\ q_N^{(2)} \end{Bmatrix} = \begin{bmatrix} \cos\alpha l_{1,2} & \sin\alpha l_{1,2} \\ -\sin\alpha l_{1,2} & \cos\alpha l_{1,2} \end{bmatrix} \cdot \begin{Bmatrix} q_u^{(1)} \\ q_N^{(1)} \end{Bmatrix}. \tag{5}$$

In relations (5), the status vectors

$$\{q\}^{(1)} = \begin{Bmatrix} q_u^{(1)} \\ q_N^{(1)} \end{Bmatrix} = \begin{Bmatrix} u_1 \\ \frac{N_1}{EA\alpha} \end{Bmatrix}, \quad \{q\}^{(2)} = \begin{Bmatrix} q_u^{(2)} \\ q_N^{(2)} \end{Bmatrix} = \begin{Bmatrix} u_2 \\ \frac{N_2}{EA\alpha} \end{Bmatrix} \tag{6}$$

have the dimension of a length.

Non-dimensional notations can be also used. Thus, by choosing a reference displacement, $\tilde{u}_0$, with notations

$$\tilde{u}_u = \frac{u}{u_0}, \quad \tilde{u}_N = \frac{N}{u_0 EA\alpha}, \tag{7}$$

relation (5) becomes

$$\{\tilde{q}_u\}^{(2)} = [A]^{(1,2)} \{\tilde{q}_N\}^{(1)}, \tag{8}$$

where field matrix $[A]^{(1,2)}$, also called transfer matrix from section (1) to section (2), has the same expression as in relation (5), i.e.

$$[A]^{(1,2)} = \begin{bmatrix} \cos\alpha\, l_{1,2} & \sin\alpha\, l_{1,2} \\ -\sin\alpha\, l_{1,2} & \cos\alpha\, l_{1,2} \end{bmatrix}. \tag{9}$$

By using notations (4) and (6), relation (5) can be written more concentrated:

$$\{q\}^{(2)} = [A]^{(1,2)}\{q\}^{(1)}. \tag{10}$$

In previous papers, [2], [3] the transfer matrix method has been applied for the study of free longitudinal, torsional and bending vibrations, of straight bars with constant, discontinuously variable and continuously variable cross-section.

3. Forced vibrations

For the study of forced vibrations, first the case of a bar acted upon by a concentrated perturbation force is considered.

The perturbation force is chosen in the form $F_i = F_{i,0}\cos\Omega t$ (fig. 2).

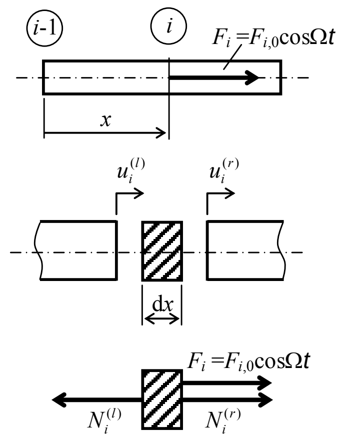

Fig. 2

In section i, an element of infinitesimal length dx is considered, with a section to the left of the application point of the force $\left(i^{(l)}\right)$ and one to the right $\left(i^{(r)}\right)$.

Relations between the status quantities are:

$$\begin{cases} u_i^{(r)} = u_i^{(l)} \\ N_i^{(r)} = N_i^{(l)} - F_0\cos\Omega t. \end{cases} \tag{11}$$

The second relation (11) can be written

$$\frac{N_i^{(r)}}{EA\alpha}=\frac{N_i^{(l)}}{EA\alpha}-\frac{F_{i,0}}{EA\alpha}\cos\Omega t \tag{12}$$

or

$$q_{N_i}^{(r)}=q_{N_i}^{(l)}-q_{F_i}\cos\Omega t\,, \tag{13}$$

where

$$q_{F_i}=\frac{F_{i,0}}{EA\alpha}\,, \tag{14}$$

$$\frac{\Omega}{c}=\alpha\,. \tag{15}$$

Relations (11) can be written, following the method presented in references [3] and [7], by means of a 3×3 jump matrix, for the axial force:

$$\begin{Bmatrix} u_i^{(r)} \\ \dfrac{N_i^{(r)}}{EA\alpha} \\ 1 \end{Bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -q_{F_i}\cos\Omega t \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{Bmatrix} u_i^{(l)} \\ \dfrac{N_i^{(l)}}{EA\alpha} \\ 1 \end{Bmatrix}. \tag{16}$$

For an element situated between sections $i-1$ and i, for which the perturbation force $F_{p,i}=F_{0,i}\cos\Omega t$ is applied in section i the following relation can be written

$$\{q^*\}^{(i,r)}=[S_F]^{(i)}[A^*]^{(i-1,i)}\{q^*\}^{(i-1,r)}, \tag{17}$$

where

$$\{q^*\}^{(i-1,r)}=\begin{Bmatrix} q_u^{(i-1)} \\ q_N^{(i-1,r)} \\ 1 \end{Bmatrix},\quad \{q^*\}^{(i,r)}=\begin{Bmatrix} q_u^{(i)} \\ q_N^{(i,r)} \\ 1 \end{Bmatrix}, \tag{18}$$

$$[A^*]^{(i-1,i)}=\begin{bmatrix} \cos\alpha l_{1-2} & \sin\alpha l_{1-2} & 0 \\ -\sin\alpha l_{1-2} & \cos\alpha l_{1-2} & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad [S_F]^{(i)}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -q_{F_i}\cos\Omega t \\ 0 & 0 & 1 \end{bmatrix}. \tag{19}$$

In the case of forced vibrations, the permanent solution (the particular solution) must be determined, which, for $F_{0,i}\cos\Omega t$, has the expression:

$$u(x,t)=X(x)\cdot[A\cos\Omega t+B\sin\Omega t]. \tag{20}$$

The constants A and B, as well as the function $X(x)$, can be determined according to the boundary conditions, by writing the relations between status matrixes at the two ends of the bar.

4. Applications

Two relatively simple applications are presented, in order to illustrate the transfer matrix method.

4.1. Bar fixed at one end, with perturbation force applied at the free end

In this case (fig. 3), since the perturbation force is applied at the end of the bar, it can be introduced by boundary conditions and the artifice presented above is not necessary. However, the method is used as it was described.

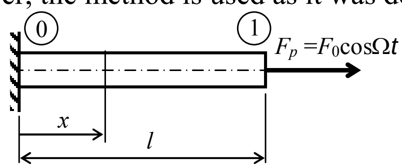

Fig. 3

With the notations in Fig. 3 and from (6) and (14), it follows:

$$\begin{Bmatrix} q_u^{(1)} \\ q_N^{(1,r)} \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -q_F \cos\Omega t \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha l & \sin\alpha l & 0 \\ -\sin\alpha l & \cos\alpha l & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} q_u^{(0)} \\ q_N^{(0)} \\ 1 \end{Bmatrix}, \tag{21}$$

which is equivalent to

$$\begin{cases} q_u^{(1)} = q_u^{(0)} \cos\alpha l + q_N^{(0)} \sin\alpha l \\ q_N^{(1,r)} = -q_u^{(0)} \sin\alpha l + q_N^{(0)} \cos\alpha l - q_F \cos\Omega t \\ 1 = 1. \end{cases} \tag{21'}$$

For the bar fixed at one end and free at the other, the boundary conditions are

$$u_0 = 0, \quad N_1 = 0 \tag{22}$$

or

$$q_u^{(0)} = 0, \quad q_N^{(1,r)} = 0 \;. \tag{22'}$$

System (21') becomes, with the initial conditions:

$$\begin{cases} u_1 = \dfrac{N_0}{EA\alpha} \sin\alpha l \\ 0 = \dfrac{N_0}{EA\alpha} \cos\alpha l - \dfrac{F_0}{EA\alpha} \cos\Omega t. \end{cases} \tag{23}$$

It follows:

$$\begin{cases} N_0 = \dfrac{F_0}{\cos \alpha l} \cos \Omega t \\ u_1 = F_0 \operatorname{tg} \alpha l \cdot \cos \Omega t . \end{cases} \tag{24}$$

Observation. In the case

$$\cos \alpha l = 0 , \tag{25}$$

respectively,

$$\cos \frac{\omega}{c} l = 0 , \tag{25'}$$

system (23) leads to $u_1 \to \infty$, $N_0 \to \infty$. This happens due to the resonance phenomenon, for which the particular solution is not in the form $A \cos \Omega t + B \sin \Omega t$, but in the form $A_1 t \cos \Omega t + B_1 t \sin \Omega t$.

Relation (25') represents the equation of the eigenfrequencies, for free undamped vibrations. This equation has the solution

$$\omega_k = (2k - 1) \frac{\pi c}{l} . \tag{26}$$

4.2. Bar fixed at both ends, with perturbation force applied at the middle

With notations in Figure 4, it results for the segment $(0 - 1)$,

$$\begin{Bmatrix} q_u^{(1)} \\ q_N^{(1,l)} \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos \alpha \frac{l}{2} & \sin \alpha \frac{l}{2} & 0 \\ -\sin \alpha \frac{l}{2} & \cos \alpha \frac{l}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} q_u^{(0)} \\ q_N^{(0)} \\ 1 \end{Bmatrix} , \tag{27}$$

or, concentrated,

$$\{q^*\}^{(1,l)} = [A^*]^{(0,1)} \{q^*\}^{(0)} . \tag{27'}$$

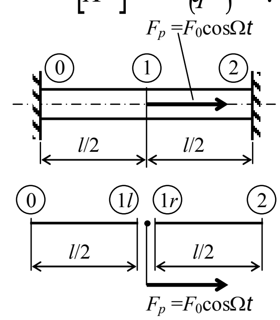

Fig. 4

In section (1) it results

$$\begin{Bmatrix} q_u^{(1)} \\ q_N^{(1,r)} \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -q_F \cos\Omega t \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} q_u^{(1)} \\ q_N^{(1,l)} \\ 1 \end{Bmatrix}, \tag{28}$$

or, concentrated,

$$\{q^*\}^{(1,r)} = [S_F]^{(1)} \{q^*\}^{(1,l)}. \tag{28'}$$

For segment $(1-2)$,

$$\begin{Bmatrix} q_u^{(2)} \\ q_N^{(2,l)} \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\alpha\frac{l}{2} & \sin\alpha\frac{l}{2} & 0 \\ -\sin\alpha\frac{l}{2} & \cos\alpha\frac{l}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} q_u^{(1)} \\ q_N^{(1,r)} \\ 1 \end{Bmatrix}, \tag{29}$$

respectively,

$$\{q^*\}^{(2)} = \left[A^*\right]^{(1,2)} \{q^*\}^{(1,r)}. \tag{29'}$$

From relations (27'), (28') and (29'), it results:

$$\{q^*\}^{(2)} = \left[A^*\right]^{(1,2)} \cdot [S_F]^{(1)} \cdot \left[A^*\right]^{(0,1)} \cdot \{q^*\}^{(0)}. \tag{30}$$

Relations (30) are accompanied by the boundary conditions:

$$q_u^{(0)} = 0, \quad q_u^{(2)} = 0 . \tag{31}$$

In relations (30) and (31), the unknowns are $q_N^{(0)}$, $q_N^{(2)}$, i.e. $N_0(t)$ and $N_2(t)$. The time function can be easily identified as: $\cos\Omega t$. Hence:

$$N_0(t) = \hat{N}_0 \cos\Omega t, \quad N_2(t) = \hat{N}_2 \cos\Omega t , \tag{32}$$

so that only the scalars $\hat{N}_0$ and $\hat{N}_2$ remain as unknowns.

The equation of the eigenfrequencies can be also established, by equating with zero the determinant of the linear system in $\hat{N}_0$ and $\hat{N}_2$, resulted from equation (30) with conditions (31).

The matrix form of the equations in $\hat{N}_0$ and $\hat{N}_2$ facilitates the use of computer codes, such as MATLAB, Mathcad or even Excel, in order to solve the problem.

Indeed, by choosing $l = 1\,\text{m}$, $A = 4\cdot 10^{-4}\,\text{m}^2$, $\Omega = 10\,\text{s}^{-1}$, $F_0 = 100\,\text{N}$,

$E = 7.86 \dfrac{\text{kg}}{\text{m}^3}$, $\rho = 2.1 \cdot 10^6 \dfrac{\text{N}}{\text{m}^2}$, the following values have been obtained: $\hat{N}_0 = 52.43\,\text{N}$ and $\hat{N}_2 = -52.43\,\text{N}$.

5. Conclusions

Transfer matrix method can be used also for solving problems of forced vibrations of bars. By using the artifice presented in relations (17), the matrix form of the equations can be easily written in order to determine the time function and the values of the unknown quantities.

The examples presented above show how the method can be applied.

6. Acknowledgement

This study was supported by research contract No. 81-031 / 2007 (SIMOCA), funded by the National Authority for Scientific Research in Romania (Ministry of Education, Research and Youth) through the National Center for Projects Management (CNMP).

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