NEW FUZZY h-IDEALS IN HEMIRINGS

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The concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideals in hemirings are introduced. Some new characterization theorems of these kinds of fuzzy h-ideals are also given. In particular, we investigate prime and strong prime $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideals in hemirings. Finally, we show that the h-hemiregular and h-semisimple hemirings can be described by using these kinds of fuzzy h-ideals.

Keywords: (Prime) h-ideal; h-interior ideal; h-hemiregular (h-semisimple) hemiring; $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ($h$-, h-interior) ideal.

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1. Introduction

Multi-valued logic has been considered by model phenomena because both uncertainty and vagueness are involved. In applications, the non-classical logic including multi-valued logic and fuzzy logic takes the advantage of the classical logic so that they can be used to handle information with various facets of uncertainty (see [23, 24]) such as fuzziness, randomness and so on. It is noted that non-classical logic has nowadays become the formal and useful tools in computer science because they deal with fuzzy information as well as uncertain information. One of the most general class of multi-valued logic is the BL-logic which is defined as the logic of continuous t-norms. Since BL-logic can be described as a commutative lattice-ordered semiring, the Łukasiewicz logic, Gödel logic and the Product logic are therefore special BL-logic, and as a consequence, they can be regarded as some special semirings.

Semirings, regarded as a generalization of rings, have been recently found particularly useful in solving problems in different disciplines of applied mathematics and information sciences because semiring provides an algebraic framework for modeling [9]. A special semiring with a zero and endowed with the commutative addition is said to be a *hemiring* . In applications, hemirings are useful in automata and formal languages (see [12]). It is well known that the set of regular languages forms the so-called "*star*, *semirings*". According to Kleene (see [20]), the languages or the sets of words can be recognized as the finite-state automata which are precisely those that are obtained from the letters of input alphabets by applying the operations such as the "sum "(union), "the product", and the "$\star$ ", that is, the so

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called "*Kleene closure*" . If a language is represented as a formal series with coefficients in a Boolean semiring, then the ideas of Kleene can be described by the well known Schützenberger Representation theorem. Moreover, if the set of coefficients of a semiring forms a field, then its corresponding syntactic algebra of the series has a finite rank if and only if the series are rational.

We note that the ideals of semirings play a crucial role in the structure theory, however, according to Henriksen [10], ideals in semirings do not in general coincide with the ideals of a ring . For this reason, the usage of ideals in semirings is somewhat limited. The general properties of fuzzy k-ideals of semirings were first described in [14]. In 2004, Jun [15] considered the *fuzzy h-ideals* of hemirings. The *h-hemiregular hemirings* were described by Zhan et al by using the fuzzy h-ideals [25]. Furthermore, Yin et al introduced the concepts of fuzzy h-bi-ideals and fuzzy h-quasi-ideals of hemirings in [22]. By using these fuzzy ideals, a number of characterization theorems of h-hemiregular and h-intra-hemiregular hemirings were obtained. As a continuation of these investigations, Ma et al. [18] introduced the concepts of $(\in, \in\vee\mathrm{q})$-fuzzy h-bi-ideals (resp., h-quasi-ideals) of a hemiring and investigated some of their properties. In particular, they showed that the h-hemiregular hemirings and h-intra-hemiregular hemirings can be described by some of their generalized fuzzy h-ideals. Finally, the implication-based fuzzy h-bi-ideals (resp., h-quasi-ideals) of a hemiring were considered. The other important results related with fuzzy k-ideals and h-ideals of a hemiring were given in [3, 4, 5, 6, 7, 8, 13, 19, 21].

After the concept of fuzzy sets introduced by Zadeh [23], there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroup, that is, the $(\in, \in\vee\mathrm{q})$-fuzzy subgroup, was introduced by Bhakat and Das in [1] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets. In fact, the $(\in, \in\vee\mathrm{q})$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is natural to investigate some similar generalizations of the existing fuzzy subsystems by considering some other structures. With this objective in mind, Davvaz et al. in [2] obtained some results in near-rings. In particular, Ma et al. discussed the properties of generalized interval-valued fuzzy h-ideals of hemirings in [17]. On the other hand, Dudek et al. [5] introduced the concepts of fuzzy ideals of fuzzy rings and (α, β)-fuzzy h-ideals of hemirings. Fruitful results have been obtained in the literature.

This paper is organized as follows. In Section 2, we first give some basic definitions and results of hemirings. Then in Section 3, we introduce the concepts of fuzzy h-ideals and h-interior ideals in hemirings. Some new characterization theorems of fuzzy h-ideals of a hemiring are given. In Section 4, we investigate prime and strong prime $(\in_\gamma, \in_\gamma\vee\mathrm{q}_\delta)$-fuzzy h-ideals in hemirings. Finally, we show that the h-hemiregular and h-semisimple hemirings can be described by using these kinds of fuzzy h-ideals in Section 5.

2. **Preliminaries**

Recall that a semiring is an algebraic system $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws

$$a(b+c)=ab+ac \quad \text{and} \quad (a+b)c=ac+bc$$

are satisfied for all $a,b,c \in S$.

By zero of a semiring $(S,+,\cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0+x=x+0=x$ for all $x \in S$. A semiring with zero and a commutative semigroup $(S,+)$ is called a *hemiring*.

Throughout this paper, S is a hemiring. We also write $a \wedge b$ for $\min\{a,b\}$ and $a \vee b$ for $\max\{a,b\}$, where a and b are real numbers.

A left ideal of a semiring is a subset A of S closed with respect to the addition and such that $SA \subseteq A$. A subset A of a semiring S is called an *interior ideal* of S if A is closed under addition and multiplication such that $SAS \subseteq A$. An ideal P of R is called prime if $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for all ideals I and J of R.

A left ideal (right ideal, ideal and interior ideal) A of S is called a left h-ideal of S (right h-ideal, h-ideal and h-interior ideal) of S, respectively,if for any $x,z \in S$ and $a,b \in A$ from $x+a+z=b+z$ it follows $x \in A$.

The h-closure $\overline{A}$ of A in S is defined by

$$\overline{A}=\{x \in S \mid x+a_1+z=a_2+z \ \text{ for some } \ a_1,a_2 \in A, z \in S\}.$$

Clearly, if A is a left ideal of S, then $\overline{A}$ is the smallest left h-ideal of S containing A. We also have $\overline{\overline{A}}=\overline{A}$ for each $A \subseteq S$. Moreover, $A \subseteq B \subseteq S$ implies $\overline{A} \subseteq \overline{B}$.

We next state some fuzzy logic concepts. Recall that a fuzzy set is a function $\mu : S \to [0,1]$. We denote by $\mathcal{F}(S)$ the set of all fuzzy sets of S. For any $A \subseteq S$, we denote the characteristic function of A by χ_A.

For any $t \in (0,1]$, define a fuzzy set t_A of S by

$$t_A(x)=\begin{cases} t & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A, \end{cases}$$

for all $x \in S$.

A fuzzy set μ of S of the form

$$\mu(y)=\begin{cases} t(\neq 0) & \text{if} \quad y=x, \\ 0 & \text{if} \quad y \neq x, \end{cases}$$

is said to be a *fuzzy point with support x and value t* and is denoted by x_t.
A fuzzy point x_t is said to be "*belong to*" (resp.," *quasi* $-$ *coincident with*") a fuzzy set μ, written as $x_t \in \mu$ (resp., $x_t q \mu$) if $\mu(x) \geq t$ (resp., $\mu(x)+t>1$).
If $x_t \in \mu$ or $x_t \, q \, \mu$, then we write $x_t \in\vee\, \mathrm{q}\mu$. If $\mu(x)<t$ (resp., $\mu(x)+t \leq 1$), then we say that $x_t \,\overline{\in}\mu$ (resp., $x_t \ \overline{q} \ \mu$).

We note here that the symbol $\overline{\in\vee\,\mathrm{q}}$ means that $\in\vee\,\mathrm{q}$ does not hold.

Let $\gamma,\delta \in [0,1]$ be such that $\gamma<\delta$. For a fuzzy point x_r and μ of S, we say
(1) $x_r \in_\gamma \mu$ if $\mu(x) \geq r > \gamma$.
(2) $x_r q_\delta \mu$ if $\mu(x)+r>2\delta$.
(3) $x_r \in_\gamma \vee q_\delta \mu$ if $x_r \in_\gamma \mu$ or $x_r q_\delta \mu$.

For any two fuzzy sets μ,ν of S and $\gamma,\delta \in [0,1]$, $\gamma<\delta$. Define a new ordered relation "$\subseteq \vee q_{(\gamma,\delta)}$" as follows:

$x_r \in_\gamma \mu \Longrightarrow x_r \in_\gamma \vee q_\delta \nu$ for all $x \in S$.

Define a relation "$\sim$" on $\mathcal{F}(S)$ by

$\mu \sim \nu \Longleftrightarrow \mu \subseteq \vee q_{(\gamma,\delta)}\nu$ and $\nu \subseteq \vee q_{(\gamma,\delta)}\mu$.

Lemma 2.1. *Let μ and ν be any two fuzzy sets of S. Then $\mu \subseteq \vee q_{(\gamma,\delta)}\nu \iff \nu(x) \vee \gamma \geq \mu(x) \wedge \delta$ for all $x \in S$.*

Proof. Let $\mu \subseteq \vee q_{(\gamma,\delta)}\nu$. If there exists $x \in S$ such that $\nu(x) \vee \gamma < r < \mu(x) \wedge \delta$, that is, $x_r \in_\gamma \mu$, but $x_r \overline{\in_\gamma \vee q_\delta}\nu$, a contradiction.

Conversely, let $\nu(x) \vee \gamma \geq \mu(x) \wedge \delta$ for all $x \in S$. If $\mu \overline{\subseteq \vee q_{(\gamma,\delta)}}\nu$, then there exists $x \in S$ and $r > \gamma$ such that $x_r \in_\gamma \mu$, but $x_r \overline{\subseteq \vee q_{(\gamma,\delta)}}\nu$, and so $\mu(x) \geq r, \nu(x) < r$ and $\nu(x) + r < 2\delta$. Thus, $\nu(x) \vee \gamma < \mu(x) \wedge \delta$, a contradiction. □

The following is obvious.

Lemma 2.2. $\mu \subseteq \vee q_{(\gamma,\delta)}\nu \subseteq \vee q_{(\gamma,\delta)}\omega \Longrightarrow \mu \subseteq \vee q_{(\gamma,\delta)}\omega$.

Note that Lemma 2.1 gives that
$\mu \sim \nu \iff (\mu(x) \wedge \delta) \vee \gamma = (\nu(x) \wedge \delta) \vee \gamma$
for all $x \in S$ and it follow from Lemmas 2.1 and 2.2 that "$\sim$" is an equivalence.

Definition 2.1. Let μ and ν be fuzzy sets of S.

(i) The h-sum of μ and ν is

$$(\mu +_h \nu)(x) = \bigvee_{x+a_1+b_1+z=a_2+b_2+z} \mu(a_1) \wedge \mu(a_2) \wedge \nu(b_1) \wedge \nu(b_2)$$

and $(\mu +_h \nu)(x) = 0$ if x cannot be expressed as $x + a_1 + b_1 + z = a_2 + b_2 + z$.

(ii) The h-interior product of μ and ν is

$$(\mu \odot_h \nu)(x) = \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a'_j b'_j + z} \mu(a_i) \wedge \mu(a'_j) \wedge \nu(b_i) \wedge \nu(b'_j)$$

and $(\mu \odot_h \nu)(x) = 0$ if x cannot be expressed as $x + \sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a'_j b'_j + z$.

Proposition 2.1. *Let $A, B \subseteq S$. Then we have*

(1) $A \subseteq B \Leftrightarrow \chi_A \subseteq \vee\ q_{(\gamma,\delta)}\ \ \chi_B$;
(2) $\chi_A \cap \chi_B = \chi_{A \cap B}$;
(3) $\chi_A \odot_h \chi_B = \chi_{\overline{AB}}$;
(4) $\chi_A +_h \chi_B = \chi_{\overline{A+B}}$.

3. $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideals

In this Section, we introduce the concepts of fuzzy h-ideals and h-interior ideals in hemirings. Some new characterization theorems of fuzzy h-ideals of a hemiring are given.

Definition 3.1. Let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$. A fuzzy set μ of S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) h-ideal of S if for all $t, r \in (\gamma, 1]$ and $x, z \in S$,

(F1a) $\mu +_h \mu \subseteq \vee q_{(\gamma,\delta)}\mu$;
(F1b) $\chi_S \odot_h \mu \subseteq \vee q_{(\gamma,\delta)}\mu$ (resp., $\mu \odot_h \chi_S \subseteq \vee q_{(\gamma,\delta)}\mu$);
(F1c) $x + a + z = b + z, a_t, b_r \in_\gamma \mu \Longrightarrow x_{t\wedge r} \in_\gamma \vee q_\delta \mu$ for all $a, b, x, z \in S$.

A fuzzy set of S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal if it is both an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right h-ideal.

Example 3.1. Let $S = \{0, a, b\}$ be a hemiring with the Cayley table as follows:

$+$	0	a	b
0	0	a	b
a	a	0	b
b	b	b	0

.	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

Define a fuzzy set μ of S by $\mu(0) = 0.6$, $\mu(a) = 0.8$ and $\mu(b) = 0.2$. Then μ is an $(\in_{0.2}, \in_{0.2} \vee q_{0.6})$-fuzzy h-ideal of S, but it is not a fuzzy h-ideal of S.

Theorem 3.1. *A fuzzy set μ of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) h-ideal of S if and only if it satisfies:*

(F2a) $\mu(x+y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$ for all $x, y \in S$;

(F2b) $\mu(xy) \vee \gamma \geq \mu(y) \wedge \delta$ (resp., $\mu(xy) \vee \gamma \geq \mu(x) \wedge \delta$);

(F2c) $x + a + z = b + z \Longrightarrow \mu(x) \vee \gamma \geq \mu(a) \wedge \mu(b) \wedge \delta$.

Proof. We only consider the case for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideals.

Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal of S. If there exist $x, y \in S$ such that $\mu(x+y) \vee \gamma < t < \mu(x) \wedge \mu(y) \wedge \delta$, then $\mu(x) > t$, $\mu(y) > t$, $\mu(x+y) < t < \delta$, that is, $(x+y)_t \overline{\in_\gamma \vee q_\delta} \mu$. On the other hand, $\mu(0) \vee \gamma \geq \mu(x) \wedge \delta \geq t \wedge \delta = t > \gamma$, and so $\mu(0) \geq \mu(x) \wedge \delta$. Thus,

$$\begin{aligned}(\mu +_h \mu)(x+y) = & \bigvee_{x+y+a_1+b_1+z=a_2+b_2+z} \mu(a_1) \wedge \mu(a_2) \wedge \mu(b_1) \wedge \mu(b_2) \\ \geq & \mu(0) \wedge \mu(x) \wedge \mu(y) \\ \geq & \mu(x) \wedge \mu(y) \wedge \delta \\ > & t \wedge \delta = t,\end{aligned}$$

which implies, $(x+y)_t \in_\gamma \mu +_h \mu$, and hence $(x+y)_t \in_\gamma \vee q_\delta \mu$, contradiction. Thus (F2a) holds.

Similarly, we can see that (F2b) holds.

Finally, if there exist $a, b, x, z \in S$ with $x + a + z = b + z$ and $\gamma \in [0,1]$ such that $\mu(x) \vee \gamma < r < \mu(a) \wedge \mu(b) \wedge \delta$, then $\mu(a) \geq r$, $\mu(b) \geq r$ and $\mu(x) < r < \delta$, and so $a_r, b_r \in_\gamma \mu$ and $x_r \overline{\in_\gamma \vee q_\delta} \mu$, contradiction. This proves (F2c) holds.

Conversely, assume that the conditions (F2a) (F2b) and (F2c) hold. If $x_t \in_\gamma \mu +_h \mu$, but $x_t \overline{\in_\gamma \vee q_\delta} \mu$, then $\mu(x) < t$ and $\mu(x) < \delta$. For any $a_1, a_2, b_1, b_2, x, z \in S$ such that $x + a_1 + b_1 + z = a_2 + b_2 + z$, then by (F2a) and (F2c), we have

$$\begin{aligned}\delta > \mu(x) \vee \gamma \geq & \mu(a_1 + b_1) \wedge \mu(a_2 + b_2) \wedge \delta \\ \geq & (\mu(a_1 + b_1) \vee \gamma) \wedge (\mu(a_2 + b_2) \vee \gamma) \wedge \delta \\ \geq & \mu(a_1) \wedge \mu(b_1) \wedge \mu(a_2) \wedge \mu(b_2) \wedge \delta,\end{aligned}$$

which implies, $\mu(x) \geq \mu(a_1) \wedge \mu(b_1) \wedge \mu(a_2) \wedge \mu(b_2)$. Thus,

$$\begin{aligned}t \leq (\mu +_h \mu)(x) = & \bigvee_{x+a_1+b_1+z=a_2+b_2+z} \mu(a_1) \wedge \mu(a_2) \wedge \mu(b_1) \wedge \mu(b_2) \\ = & \bigvee_{x+a_1+b_1+z=a_2+b_2+z} \mu(x) \\ = & \mu(x),\end{aligned}$$

contradiction. Thus (F1a) holds.

Similarly, we can prove (F1b) holds.

Finally, if there exist $a, b, x, z \in S$ and $t, r \in [0,1]$ with $x + a + z = b + z$ and $a_t, b_r \in_\gamma \mu$ such that $x_{t\wedge r}\overline{\in_\gamma \vee q_\delta}\mu$, then $\mu(a) \geq t$, $\mu(b) \geq r$, $\mu(x) \vee \gamma < t \wedge r < \delta$, contradiction. □

Remark 3.1. *For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) h-ideal μ of S, we see that*

(i) If $\gamma = 0$ and $\delta = 1$, then μ is the fuzzy left (resp.,right) h-ideal of S (see [15]*).*

(ii) If $\gamma = 0$ and $\delta = 0.5$, then μ is the $(\in, \in \vee q)$-fuzzy left (resp.,right) h-ideal of S (see [18]*).*

Theorem 3.2. *Let μ be a fuzzy set of S. Then*

(1) μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) h-ideal of S if and only if $\mu_r^\gamma (\neq \emptyset)$ is a left (resp., right) h-ideal of S for all $r \in (\gamma, \delta]$.

(2) If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) h-ideal of S if and only if $\mu_r^\delta (\neq \emptyset)$ is a left (resp., right) h-ideal of S for all $r \in (\delta, 1]$.

(3) If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) h-ideal of S if and only if $[\mu]_r^\delta (\neq \emptyset)$ is a left (resp., right) h-ideal of S for all $r \in (\gamma, 1]$.

Proof. (1) We only consider the fuzzy left h-ideals. Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal of S.

(i) If $x, y \in \mu_r^\gamma$ for all $r \in (\gamma, \delta]$, then $\mu(x) \geq r$ and $\mu(y) \geq r$. It follows that $\mu(x + y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta \geq r \wedge \delta > r > \gamma$, and so $\mu(x + y) \geq r$, that is, $(x + y)_r \in_\gamma \mu$.

(ii) If $y \in \mu_r^\gamma$, then $\mu(y) \geq r$ and so $\mu(xy) \vee \gamma \geq \mu(y) \geq r > \gamma$, and so $\mu(xy) \geq r$, that is, $(xy)_r \in_\gamma \mu$.

(iii) Let $x, z \in S$ and $a, b \in \mu_r^\gamma$ be such that $x + a + z = b + z$. Then $\mu(x) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta \geq r \wedge \delta = r > \gamma$, and so $\mu(x) \geq r$, that is $x_r \in_\gamma \mu$. Thus, μ_r^γ is a left h-ideal of S.

Conversely, assume that μ_r^γ is a left h-ideal of S for all $r \in (\gamma, \delta]$. Let $x, y \in S$. If $\mu(x + y) \wedge \gamma < r = \mu(x) \wedge \mu(y) \wedge \delta$, then $x_r \in_\gamma \mu$,$y_r \in_\gamma \mu$, but$(x + y)_r\overline{\in_\gamma \vee q_\delta}\mu$, and so $x, y \in \mu_r^\gamma$. Since μ_r^γ is a left h-ideal of S, we have $x + y \in \mu_r^\gamma$, a contradiction. Thus (F2a) holds. Similarly, we can prove (F2b) and (F2c) hold.

(2) It is similar to (1).

(3) Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal of S and $r \in (\gamma, 1]$. Then for all $x, y \in [\mu]_r^\delta$, we have $x_r, y_r \in_\gamma \vee q_\delta \mu$, that is, $\mu(x) \geq r > \gamma$ or $\mu(x) > 2\delta - r > 2\delta - 1 = \gamma$, and $\mu(y) \geq r > \gamma$ or $\mu(y) > 2\delta - r > 2\delta - 1 = \gamma$. Since μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal of S, then $\mu(x + y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta > \gamma \wedge \delta = \gamma$. and so $\mu(x + y) \geq \mu(x) \wedge \mu(y) \wedge \delta$.

Case 1: $r \in (\gamma, \delta]$. Then $2\delta - r \geq \delta \geq r$, and so
$\mu(x + y) \geq r \wedge r \wedge \delta = r$
or $\mu(x + y) \geq r \wedge (2\delta - r) \wedge \delta = r$
or $\mu(x + y) \geq (2\delta - r) \wedge (2\delta - r) \wedge \delta = \delta > r$. Hence, $(x + y)_r \in_\gamma \mu$.

Case 2: $r \in (\delta, 1]$. Then $2\delta - r < \delta < r$, and so
$\mu(x + y) \geq r \wedge r \wedge \delta = \delta > 2\delta - r$
or $\mu(x + y) \geq r \wedge (2\delta - r) \wedge \delta = 2\delta - r$
or $\mu(x + y) \geq (2\delta - r) \wedge (2\delta - r) \wedge \delta = 2\delta - r$. Hence, $(x + y)_r q_\delta \mu$.

Thus in any case, $(x+y)_r \in_\gamma \vee q_\delta \mu$, that is, $(x+y)_r \in [\mu]_r^\delta$. Similarly, we can prove the others of left h-ideals. Hence, $[\mu]_r^\delta$ is a left h-ideal of S.

Conversely, assume that $[\mu]_r^\delta$ is a left h-ideal of S for all $r \in (\gamma, \delta]$. If $x, y \in S$ such that $\mu(x+y) \vee \gamma < r = \mu(x) \wedge \mu(y) \wedge \delta$, then $x_r \in_\gamma \mu$, $y_r \in_\gamma \mu$, but $(x+y)_r \overline{\in_\gamma \vee q_\delta} \mu$, and so $x, y \in [\mu]_r^\delta$. Since $[\mu]_r^\delta$ is a left h-ideal of S, we have $x+y \in [\mu]_r^\delta$, a contradiction. Thus (F2a) holds.

Similarly, we can prove (F2b) and (F2c) hold. Hence μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal of S. □

If we take $\gamma = 0$ and $\delta = 0.5$ in Theorem 3.2, we can conclude the following results:

Corollary 3.1. *Let μ be a fuzzy set of S. Then*

(1) μ is an $(\in, \in \vee q)$-fuzzy left (resp., right) h-ideal of S if and only if $\mu_r (\neq \emptyset)$ is a left (resp., right) h-ideal of S for all $r \in (0, 0.5]$ (see [18]*).*

(2) μ is an $(\in, \in \vee q)$-fuzzy left (resp., right) h-ideal of S if and only if $Q(\mu; r)(\neq \emptyset)$ is a left (resp., right) h-ideal of S for all $r \in (0.5, 1]$, where $Q(\mu, r) = \{x \in S \mid x_r q \mu\}$.

(3) μ is an $(\in, \in \vee q)$-fuzzy left (resp., right) h-ideal of S if and only if $[\mu]_r (\neq \emptyset)$ is a left (resp., right) h-ideal of S for all $r \in (0, 1]$ (see [18]*).*

Definition 3.2. Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. A fuzzy set μ of S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideal of S if it satisfies (F1a), (F1c) and

(F3a) $\mu \odot_h \mu \subseteq \vee q_{(\gamma,\delta)} \mu$;

(F3b) $\chi_S \odot_h \mu \odot \chi_S \subseteq \vee q_{(\gamma,\delta)} \mu$.

It is easy to get the following result:

Theorem 3.3. *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideal.*

Remark 3.2. *The converse of Theorem 3.3 is not true.*

Example 3.2. Let $S = \{0, 1, 2, 3\}$ be a hemiring with Cayley tables as follows:

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

.	0	1	2	3
0	0	0	0	0
1	0	2	0	2
2	0	0	0	0
3	0	2	0	2

Define a fuzzy set μ of S by $\mu(0) = \mu(1) = 0.6$ and $\mu(2) = \mu(3) = 0.2$. Thus, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideal of S, but it is not an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal of S.

Theorem 3.4. *A fuzzy set μ of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideal of S if and only if it satisfies (F2a), (F2c) and*

(F4a) $\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$;

(F4b) $\mu(xyz) \vee \gamma \geq \mu(y) \wedge \delta$.

Proof. It is similarly to that of Theorem 3.1. □

Remark 3.3. *For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideal μ of S, we have*

(i) If $\gamma = 0$ and $\delta = 1$, then μ is the fuzzy h-interior ideal of S (see [21]*).*

(ii) If $\gamma = 0$ and $\delta = 0.5$, then μ is the $(\in, \in \vee q)$ fuzzy h-interior ideal of S (see [21]).

Theorem 3.5. *Let μ be a fuzzy set of S. Then*

(1) μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideal of S if and only if $\mu_r^\gamma (\neq \emptyset)$ is an h-interior ideal of S for all $r \in (\gamma, \delta]$.

(2) If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideal of S if and only if $\mu_r^\delta (\neq \emptyset)$ is an h-interior ideal of S for all $r \in (\delta, 1]$.

(3) If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideal of S if and only if $[\mu]_r^\delta (\neq \emptyset)$ is an h-interior ideal of S for all $r \in (\gamma, 1]$.

Proof. It is similarly to that of Theorem 3.2. □

If we take $\gamma = 0$ and $\delta = 0.5$ in Theorem 3.5, we can conclude the following results.

Corollary 3.2. *Let μ be a fuzzy set of S. Then*

(1) μ is an $(\in, \in \vee q)$-fuzzy h-interior ideal of S if and only if $\mu_r (\neq \emptyset)$ is an h-interior ideal of S for all $r \in (0, 0.5]$ (see [21]).

(2) μ is an $(\in, \in \vee q)$-fuzzy h-interior ideal of S if and only if $Q(\mu; r)(\neq \emptyset)$ is an h-interior ideal of S for all $r \in (0.5, 1]$.

(3) μ is an $(\in, \in \vee q)$-fuzzy h-interior ideal of S if and only if $[\mu]_r (\neq \emptyset)$ is an h-interior ideal of S for all $r \in (0, 1]$.

4. Prime $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideals

In this Section, we investigate prime and strongly prime $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy h-ideals in hemirings.

Definition 4.1. An $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal of S is called prime if for all $x, y \in S, t \in (\gamma, 1]$, we have

(P) $(xy)_t \in_\gamma \mu \Rightarrow x_t \in_\gamma \vee q_\delta \mu$ or $y_t \in_\gamma \vee q_\delta \mu$.

Example 4.1. Consider the hemiring $(N_0, +, \cdot)$, where N_0 is the set of all non-negative integers. Define a fuzzy set μ of N_0 by

$$\mu(x) = \begin{cases} 0.7 & \text{if } x \in \langle 2 \rangle, \\ 0.2 & \text{otherwise.} \end{cases}$$

Then, one easily checks that μ is a prime $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal of N_0.

Theorem 4.1. *An $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal of S is prime if for all $x, y \in S$, it holds,*

(P') $\mu(x) \vee \mu(y) \vee \gamma \geq \mu(xy) \wedge \delta$.

Proof. Let μ be a prime $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy h-ideal of S. If there exist $x, y \in S$ such that $\mu(x) \vee \mu(y) \vee \gamma < t = \mu(xy) \wedge \delta$, then $\gamma < t \leq \delta, (xy)_t \in_\gamma \mu$, but $x_t \overline{\in_\gamma} \mu$ and $y_t \overline{\in_\gamma} \mu$. Since $\mu(x) + t < 2t \leq 2\delta$, and $\mu(y) + t < 2t \leq 2\delta$, then $x_t \overline{q_\delta} \mu$ and $y_t \overline{q_\delta} \mu$, and hence, we have $x_t \overline{\in_\gamma \vee \mathrm{q}_\delta} \mu$ and $y_t \overline{\in_\gamma \vee \mathrm{q}_\delta} \mu$, which is a contradiction. Thus, (P') holds.

Conversely, suppose that the condition (P') holds. Let $(xy)_t \in_\gamma \mu$. Then $\mu(xy) \geq t$ and so $\mu(x) \vee \mu(y) \geq \mu(xy) \wedge \delta \geq t \wedge \delta$. We consider the following two cases:

(i) If $t \in (\gamma, \delta]$, then $\mu(x) \geq t$ or $\mu(y) \geq t$, that is, $x_t \in_\gamma \mu$ or $y_t \in_\gamma \mu$. Thus, $x_t \in_\gamma \vee \mathrm{q}_\delta \mu$ or $y_t \in_\gamma \vee \mathrm{q}_\delta \mu$.

(ii) If $t \in (\delta, 1]$, then $\mu(x) \vee \mu(y) \geq \delta$, and so, $\mu(x) \geq \delta$ or $\mu(y) \geq \delta$. Hence, $\mu(x) + t > 2\delta$ or $\mu(y) + t > 2\delta$, that is, $x_t q_\delta \mu$ or $y_t q_\delta \mu$. Thus, $x_t \in_\gamma \vee \mathrm{q}_\delta \mu$ or $y_t \in_\gamma \vee \mathrm{q}_\delta \mu$.

This proves that μ is prime. □

Theorem 4.2. *An $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal μ of S is prime if and only if $\mu_t^\gamma (\neq \emptyset)$ is a prime h-ideal of S for all $t \in (\gamma, \delta]$.*

Proof. Let μ be a prime $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy h-ideal of S and $t \in (\gamma, \delta]$. Then, by Theorem 3.2(1), μ_t^γ is an h-ideal of S for all $\gamma < t \leq \delta$. Let $xy \in \mu_t^\gamma$. By Theorem 4.1, we have $\mu(x) \vee \mu(y) \geq \mu(xy) \wedge \delta \geq t \wedge \delta = t$, and so $\mu(x) \geq t$ or $\mu(y) \geq t$. Thus, $x \in \mu_t^\gamma$ or $y \in \mu_t^\gamma$. This shows that μ_t^γ is a prime h-ideal of S for all $t \in (\gamma, \delta]$.

Conversely, assume that $\mu_t^\gamma (\neq \emptyset)$ is a prime h-ideal of S for all $t \in (\gamma, \delta]$. Then by Theorem 3.2(1), μ is an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy h-ideal of S. Let $(xy)_t \in_\gamma \mu$. Then $xy \in \mu_t^\gamma$. Since μ_t^γ is prime, $x \in \mu_t^\gamma$ or $y \in \mu_t^\gamma$, that is, $x_t \in_\gamma \mu$ or $y_t \in_\gamma \mu$. Thus, $x_t \in_\gamma \vee \mathrm{q}_\delta \mu$ or $y_t \in_\gamma \vee \mathrm{q}_\delta \mu$. Therefore, μ must be a prime $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy h-ideal of S. □

If $\gamma = 0$ and $\delta = 0.5$ in Theorem 4.2, we can conclude the following result.

Corollary 4.1. *μ is a prime $(\in, \in \vee q)$-fuzzy h-ideal of S if and only if $\mu_t (\neq \emptyset)$ is a prime h-ideal of S for all $t \in (0, 0.5]$ (see [19]).*

Theorem 4.3. *If $2\delta = 1 + \gamma$, then a fuzzy set μ of S is a prime $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal of S if and only if $\mu_t^\delta (\neq \emptyset)$ is a prime h-ideal of S for all $t \in (\delta, 1]$.*

Proof. Let μ be a prime $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy h-ideal of S and $t \in (\delta, 1]$. Then by Theorem 3.2(2), μ_t^δ is an h-ideal of S and $t > \delta > 2\delta - t$. To prove μ_t^δ is prime, let $xy \in \mu_t^\delta$. Hence $\mu(xy) > 2\delta - t$. Since μ is a prime $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy h-ideal of S, we have $\mu(x) \vee \mu(y) \vee \gamma \geq \mu(xy) \wedge \delta > 2\delta - t \geq 2\delta - 1 = \gamma$. It follows that $\mu(x) \vee \mu(y) \geq 2\delta - t$, and so $x \in \mu_t^\delta$ or $y \in \mu_t^\delta$.

Therefore, μ_t^δ is a prime h-ideal of S.

Conversely, let $\mu_t^\delta (\neq \emptyset)$ be a prime h-ideal of S for all $t \in (\delta, 1]$, then by Theorem 3.2(2), we know μ is an $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy h-ideal of S. Now, if there exist $x, y \in S$ such that $\mu(x) \vee \mu(y) \vee \gamma < t = \mu(xy) \wedge \delta$. Then $t \leq \delta, \mu(xy) \geq t, \mu(x) < t, \mu(x) + t < 2t \leq 2\delta, \mu(y) < t$ and $\mu(y) + t < 2t \leq 2\delta$. It follows that $(xy)_t \in_\gamma \mu$, but $x_t \overline{\in_\gamma \vee \mathrm{q}_\delta} \mu$ and $y_t \overline{\in_\gamma \vee \mathrm{q}_\delta} \mu$, a contradiction. Therefore, $\mu(x) \vee \mu(y) \vee \gamma \geq \mu(xy) \wedge \delta$ for all $x, y \in S$ and so μ is a prime $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy h-ideal of S. □

Theorem 4.4. *If $2\delta = 1 + \gamma$, then a fuzzy set μ of S is a prime $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal if and only if $[\mu]_t^\delta (\neq \emptyset)$ is a prime h-ideal of S for all $t \in (\gamma, 1]$.*

Proof. Let μ be a prime $(\in_\gamma, \in_\gamma \vee \mathrm{q}_\delta)$-fuzzy h-ideal of S. Then by Theorem 3.2(3), $[\mu]_t^\delta$ is an h-ideal of S for all $t \in (\gamma, 1]$.

To prove $[\mu]_t^\delta$ is prime, let $xy \in [\mu]_t^\delta$. Since $[\mu]_t^\delta = \mu_t^\delta \cup \mu_t^\gamma$, we have $xy \in \mu_t^\delta$ or $xy \in \mu_t^\gamma$.

Case 1: $xy \in \mu_t^\delta - \mu_t^\gamma$. Then $\mu(xy) + t > 2\delta$ and $\mu(xy) < t$.

(1) If $\mu(xy) \leq \delta$, then $\mu(x) \vee \mu(y) + t \geq \mu(xy) \wedge \delta + t = \mu(xy) + t > 2\delta$, which implies, $\mu(x) + t > 2\delta$ or $\mu(y) + t > 2\delta$, that is, $x \in \mu_t^\delta \subseteq [\mu]_t^\delta$ or $y \in \mu_t^\delta \subseteq [\mu]_t^\delta$.

(2) If $\mu(xy) > \delta$, then $\delta < \mu(xy) < t$. Thus, $\mu(x) \vee \mu(y) + t \geq \mu(xy) \wedge \delta + t = \delta + t > 2\delta$. Hence $x \in \mu_t^{\delta} \subseteq [\mu]_t^{\delta}$ or $y \in \mu_t^{\delta} \subseteq [\mu]_t^{\delta}$.

Case 2: $xy \in \mu_t^{\gamma}$. Then $\mu(xy) \geq t$.

(1) If $t \leq \delta$, then $\mu(x) \vee \mu(y) \geq \mu(xy) \wedge \delta \geq t$, which implies, $x \in \mu_t^{\gamma} \subseteq [\mu]_t^{\delta}$ or $y \in \mu_t^{\gamma} \subseteq [\mu]_t^{\delta}$.

(2) If $t > \delta$, then $\mu(x) \vee \mu(y) \geq t \wedge \delta = \delta$, which implies, $\mu(x) \vee \mu(y) + t > 2\delta$. Hence, $x \in \mu_t^{\delta} \subseteq [\mu]_t^{\delta}$ or $y \in \mu_t^{\delta} \subseteq [\mu]_t^{\delta}$.

Therefore, $[\mu]_t^{\delta}$ is a prime h-ideal of S.

Conversely, let $[\mu]_t^{\delta}(\neq \emptyset)$ be a prime h-ideal of S for all $t \in (\gamma, 1]$, then by Theorem 3.2(3), we know μ is an $(\in_{\gamma}, \in_{\gamma} \vee \mathrm{q}_{\delta})$-fuzzy h-ideal of S. Let $(xy)_t \in_{\gamma} \mu$, then $xy \in \mu_t^{\gamma} \subseteq [\mu]_t^{\delta}$. Since $[\mu]_t^{\delta}$ is prime, we have $x \in [\mu]_t^{\delta}$ or $y \in [\mu]_t^{\delta}$. This implies, $x_t \in_{\gamma} \vee \mathrm{q}_{\delta}\mu$ or $y_t \in_{\gamma} \vee \mathrm{q}_{\delta}\mu$. Therefore, μ is a prime $(\in_{\gamma}, \in_{\gamma} \vee \mathrm{q}_{\delta})$-fuzzy h-ideal of S. □

If $\gamma = 0$ and $\delta = 0.5$ in Theorem 4.4, we can conclude the following result.

Corollary 4.2. *μ is a prime $(\in, \in \vee q)$-fuzzy h-ideal of S if and only if $[\mu]_t(\neq \emptyset)$ is a prime h-ideal of S for all $t \in (0, 1]$ (see* [18]*).*

Definition 4.2. An $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy h-ideal ρ of S is called strongly prime if for every $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy h-ideals μ, ν of S, we have

(P") $\mu \odot_h \nu \subseteq \rho \Rightarrow \mu \subseteq \rho$ or $\nu \subseteq \rho$.

Theorem 4.5. *Let μ be a strongly prime $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy h-ideal of S. Then $\mu_t^{\gamma}(\neq \emptyset)$ is a prime h-ideal of S for all $t \in (\gamma, \delta]$.*

Proof. Let $t \in (\gamma, \delta]$ be such that μ_t^{γ} is nonempty. Then μ_t^{γ} is an h-ideal of S by Theorem 3.1(1). Now, we show that μ_t^{γ} is prime.

Let I and J be two h-ideals of S such $IJ \subseteq \mu_t^{\gamma}$, then it is easy to see that t_I and t_J are two $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy h-ideals of S and $t_I \odot_h t_J \subseteq \mu$. In fact, let $x \in S$.

If $(t_I \odot_h t_J)(x) = 0$, then $(t_I \odot_h t_J)(x) = 0 \leq \mu(x)$. Otherwise, there exist $a_i, b_i, a'_j, b'_j (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n), z \in S$ such that

$$x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z.$$

and $t_I(a_i) \wedge t_I(a'_j) \wedge t_J(b_i) \wedge t_J(b'_j) \neq 0$.

This implies $a_i, a'_j \in I$ and $b_i, b'_j \in J$, and so $\sum_{i=1}^{m} a_i b_i, \sum_{j=1}^{n} a'_j b'_j \in IJ$. Hence $x \in IJ \subseteq \mu_t^{\gamma}$, that is, $\mu(x) \geq t$. Thus

$$\begin{aligned}(t_I \odot_h t_J)(x) = & \bigvee_{x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} t_I(a_i) \wedge t_I(a'_j) \wedge t_J(b_i) \wedge t_J(b'_j) \\ \leq & t \leq \mu(x).\end{aligned}$$

Therefore $t_I \odot_h t_J \subseteq \mu$. Since μ is a strongly prime $(\in_{\gamma}, \in_{\gamma} \vee \mathrm{q}_{\delta})$-fuzzy h-ideal of S, we have $t_I \subseteq \mu$ or $t_J \subseteq \mu$, this implies, $I \subseteq \mu_t^{\gamma}$ or $J \subseteq \mu_t^{\gamma}$. This completes the proof. □

Theorem 4.6. *Every strongly prime* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy* h*-ideal is prime* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy* h*-ideal.*

Proof. It follows from Theorems 4.5 and 4.2. □

Remark 4.1. *The converse of Theorem 4.6 is not true.*

Example 4.2. Consider the hemiring $(N_0, +, \cdot)$, where N_0 is the set of all non-negative integers. Define a fuzzy set μ of N_0 by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.2 & \text{if } x = 2k \neq 0, \\ 0.4 & \text{if } x = 2k+1. \end{cases}$$

Thus μ is a prime $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal of N_0, but it is not strong prime.

5. Characterization of h-hemiregular and h-hemisimple hemirings

In this Section, we show that the h-hemiregular and h-semisimple hemirings can be described by using these kinds of fuzzy h-ideals.

Definition 5.1. [25]A hemiring S is said to be h-hemiregular if for each $a \in S$, there exist $x_1, x_2, z \in S$ such that $a + ax_1a + z = ax_2a + z$.

Lemma 5.1. [25] *Let* μ *be a fuzzy set of* L*. Then an* $\in$*-soft set* (F, A) *over* L *with* $A = (\alpha, \beta] \subset (0, 1]$ *is a filteristic soft MTL-algebra over* L *if and only if* μ *is a fuzzy filter with thresholds* $(\alpha, \beta]$*.*

Theorem 5.1. *A hemiring* S *is* h*-hemiregular if and only if for any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy right* h*-ideal* μ *and* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left* h*-ideal* ν*, we have*

$$\mu \cap \nu \sim \mu \odot_h \nu.$$

Proof. Let S be a h-hemiregular hemiring, μ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right h-ideal and ν an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal of S.

Then

$$\mu \odot_h \nu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \chi_S \subseteq \vee q_{(\gamma,\delta)} \mu$$

and

$$\mu \odot_h \nu \subseteq \vee q_{(\gamma,\delta)} \chi_S \odot \nu \subseteq \vee q_{(\gamma,\delta)} \nu.$$

Thus, $\mu \odot_h \nu \subseteq \vee q_{(\gamma,\delta)} \mu \cap \nu$.

For any $x \in S$, there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$ since S is h-hemiregular.

Thus, we have

$$\begin{aligned}(\mu \odot_h \nu) \vee \gamma = & \bigvee_{x + \sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} \mu(a_i) \wedge \mu(a_{j'}) \wedge \nu(b_i) \wedge \nu(b_{j'}) \vee \gamma \\ \geq & (\mu(xa) \wedge \mu(xa') \wedge \nu(x)) \vee \gamma \\ = & (\mu(xa) \vee \gamma) \wedge (\mu(xa') \vee \gamma) \vee (\nu(x) \vee \gamma) \\ \geq & \mu(x) \wedge \nu(x) \wedge \delta \\ = & (\mu \cap \nu)(x) \wedge \delta.\end{aligned}$$

This implies, $\mu\cap\nu\subseteq\vee q_{(\gamma,\delta)}\mu\odot_h\nu$. Whence $\mu\cap\nu\sim\mu\odot_h\nu$.

Conversely, let A and B, resp., a right h-ideal and a left h-ideal of S. Then χ_A and χ_B are $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right h-ideal and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left h-ideal of S, respectively.

Thus,

$$\chi_{\overline{AB}}=\chi_A\odot_h\chi_B\sim\chi_{A\cap B}=\chi_A\cap\chi_B,$$

which implies, $\overline{AB}=A\cap B$. It follows from Lemma 5.1 that S is h-hemiregular. □

Definition 5.2. [21]A subset A of S is called idempotent if $A=\overline{A}_2$.

Definition 5.3. [21]A hemiring S is called h-semisimple if every h-ideal is idempotent.

Lemma 5.2. [21] *A hemiring S is h-semisimple is if and only if one of the following holds:*

(1) There exist $c_i,d_i,e_i,f_i,c'_j,d'_j,e'_j,f'_j,z\in S$ such that

$$x+\sum_{i=1}^{m}c_ixd_ie_ixf_i+z=\sum_{j=1}^{n}c'_jxd'_je'_jxf'_j+z$$

for all $x\in S$;

(2) $x\in\overline{SxSxS}$ for all $x\in S$;

(3) $A\subseteq\overline{SASAS}$ for all $A\in S$.

Theorem 5.2. *Let S be an h-semisimple hemiring, μ a fuzzy set of S. Then μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy h-ideal if and only if it is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy h-interior ideal.*

Proof. Let μ be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy h-ideal of S, it follows from Theorem 3.3 that it is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy h-interior ideal.

Conversely, Let μ be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy h-interior ideal of S. For any $x,y\in S$, since S is h-semisimple, by Lemma 5.2, there exist $c_i,d_i,e_i,f_i,c'_j,d'_j,e'_j,f'_j,z\in S$ such that $x+\sum\limits_{i=1}^{m}c_ixd_ie_if_i+z=\sum\limits_{j=1}^{n}c'_jxd'_je'_jxf'_j+z$ and so $xy+\sum\limits_{i=1}^{m}c_ixd_ie_if_iy+zy=\sum\limits_{j=1}^{n}c'_jxd'_je'_jxf'_jy+zy$. Thus,

$$\begin{aligned}\mu(xy)\vee\gamma\geq&(\mu(\sum_{i=1}^{m}c_ixd_ie_ixf_iy)\wedge\mu(\sum_{j=1}^{n}c'_jxd'_je'_jxf'_jy)\wedge\delta)\vee\gamma\\ \geq&(\mu(\sum_{i=1}^{m}c_ixd_ie_ixf_iy)\vee\gamma)\wedge(\mu(\sum_{j=1}^{n}c'_jxd'_je'_jxf'_jy)\vee\gamma)\wedge(\delta\vee\gamma)\\ \geq&\mu(x)\wedge\delta.\end{aligned}$$

Thus, μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right h-ideal of S. Similarly, μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left h-ideal of S. Hence μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy h-ideal of S. □

Corollary 5.1. [21] *Let S be an h-semisimple hemiring, μ a fuzzy set of S. Then μ is an $(\in,\in\vee q)$-fuzzy h-ideal if and only if it is an $(\in,\in\vee q)$-fuzzy h-interior ideal.*

Theorem 5.3. *A hemiring S is h-semisimple if and only if for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideals μ and ν, we have*

$$\mu \cap \nu \sim \mu \odot_h \nu.$$

Proof. Let S be an h-semisimple hemiring, μ and ν $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideals of S. Then, by Theorem 5.2, μ and ν are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideals of S. Thus,

$\mu \odot_h \nu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \chi_S \subseteq \vee q_{(\gamma,\delta)} \mu$
and $\mu \odot_h \nu \subseteq \vee q_{(\gamma,\delta)} \chi_S \odot_h \nu \subseteq \vee q_{(\gamma,\delta)} \nu$.

This proves, $\mu \odot_h \nu \subseteq \vee q_{(\gamma,\delta)} \mu \cap \nu$. For any $x, y \in S$, since S is h-semisimple, there exist $c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j, z \in S$ such that $x + \sum_{i=1}^{m} c_i x d_i e_i f_i + z = \sum_{j=1}^{n} c'_j x d'_j e'_j x f'_j + z$ for all $x \in S$. Thus

$$\begin{aligned}
(\mu \odot_h \nu)(x) \vee \gamma =& \Big(\bigvee_{x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \mu(a_i) \wedge \mu(a_j) \wedge \nu(b_i) \wedge \nu(b'_j) \Big) \vee \gamma \\
\geq& \mu(c_i x d_i) \wedge \mu(c'_j x d'_j) \wedge \nu(e_i x f_i) \wedge \nu(e'_j x f'_j) \vee \gamma \\
=& (\mu(c_i x d_i) \vee \gamma) \wedge (\mu(c'_j x d'_j) \vee \gamma) \wedge (\nu(e_i x f_i) \vee \gamma) \wedge (\nu(e'_j x f'_j) \vee \gamma) \\
\geq& \mu(x) \wedge \nu(x) \wedge \delta \\
=& (\mu \cap \nu)(x) \wedge \delta.
\end{aligned}$$

Thus, $\mu \cap \nu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \nu$, and so, $\mu \cap \nu \sim \mu \odot_h \nu$.

Conversely, let A be any h-ideal of S, then it is an h-interior ideal. Thus, its characteristic functions χ_A is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideal. Thus, we have $\chi_A = \chi_A \cap \chi_A \sim \chi_A \odot_h \chi_A = \chi_{\overline{A^2}}$, and so, $A = \overline{A^2}$. Thus, S is h-semisimple. □

Corollary 5.2. [21] *A hemiring S is h-semisimple if and only if for any $(\in, \in \vee q)$-fuzzy h-interior ideals μ and ν, $\mu \cap \nu \sim \mu \odot_h \nu$.*

6. Conclusions

In the study of a fuzzy algebraic system, we notice that the (fuzzy) ideals with special properties play an important role. By using these kinds of fuzzy h-ideals in hemirings, we are able to characterize the hemirings, in particular, we are able to show that the h-hemiregular hemirings can be characterized by their fuzzy h-ideals.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of hemirings. In the future study of fuzzy hemirings, perhaps the following topics are worth to be considered:

(1) To describe the n-ary-hemirings;
(2) To establish an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy spectrum of n-ary-hemirings;
(3) To discuss soft n-ary-hemirings and some of its application in computer science.

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