NEW FACTOR PAIRS FOR FACTORIZATIONS OF LAMBERT SERIES GENERATING FUNCTIONS

Mircea Merca[1], Maxie D. Schmidt[2]

We prove several new variants of the Lambert series factorization theorem established in the first article "Generating special arithmetic functions by Lambert series factorizations" by Merca and Schmidt (2017). Several characteristic examples of our new results are presented in the article to motivate the formulations of the generalized factorization theorems. Applications of these new factorization results include new identities involving the Euler partition function and the generalized sum-of-divisors functions, the Möbius function, Euler's totient function, the Liouville lambda function, von Mangoldt's lambda function, and the Jordan totient function.

Keywords: Lambert series, factorization theorem, matrix factorization, partition function, multiplicative function

MSC2020: 11A25 11P81 05A17 05A19.

1. Introduction.

We consider recurrence relations and matrix equations related to *Lambert series* expansions of the form [5, §27.7] [1, §17.10]

$$\sum_{n\geq 1} \frac{a_n\, q^n}{1-q^n} = \sum_{m\geq 1} b_m\, q^m,\ |q|<1, \tag{1}$$

for prescribed arithmetic functions $a : \mathbb{Z}^+ \to \mathbb{C}$ and $b : \mathbb{Z}^+ \to \mathbb{C}$ where $b_m = \sum_{d|m} a_d$. As in [3], we are interested in so-termed Lambert series factorizations of the form

$$\sum_{n\geq 1} \frac{a_n\, q^n}{1-q^n} = \frac{1}{C(q)} \sum_{n\geq 1} \left(\sum_{k=1}^{n} s_{n,k}\, a_k \right) q^n, \tag{2}$$

for arbitrary $\{a_n\}_{n\geq 1}$ and where specifying one of the sequences, $c_n := [q^n] 1/C(q)$ or $s_{n,k}$ with $C(0) := 1$, uniquely determines the form of the other. In effect, we have "*factorization pairs*" in the expansions of (2). The special case of

$$\big(C(q), s_{n,k}\big) \equiv \big((q;q)_\infty, s_o(n,k) - s_e(n,k)\big),$$

where $s_o(n,k)$ and $s_e(n,k)$ are respectively the number of k's in all partitions of n into an odd (even) number of distinct parts is considered in the references [3, 4, 7]. We generalize this result in two key new ways in the next sections.

Central to the definition of our factorization pairs in (2) is the next matrix identity providing a factorized representation of special arithmetic functions generated by Lambert

[1]Professor, National University of Science and Technology Politehnica Bucharest, Department of Mathematical Methods and Models, Fundamental Sciences Applied in Engineering Research Center, RO-060042 Bucharest, Romania; Academy of Romanian Scientists, RO-050044 Bucharest, Romania, e-mail: mircea.merca@upb.ro (corresponding author)

[2]Georgia Institute of Technology, School of Mathematics, Atlanta, GA 30332, USA, e-mail: maxieds@gmail.com

series expansions where

$$A_n := (s_{i,j})_{1\leq i,j\leq n} \quad \text{and} \quad A_n^{-1} := \left(s_{i,j}^{(-1)}\right)_{1\leq i,j\leq n},$$

and the one-dimensional sequence of $\{B_m\}_{m\geq 0}$ depends on the arithmetic function, b_n, implicit to the expansion of (1) and the factorization pair, $(C(q), s_{n,k})$. Thus in order to construct a valid factorization pair we require that both the fundamental factorization result in (2) hold, and that the corresponding construction provides an identity of the form

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = A_n^{-1} \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{n-1} \end{bmatrix} \tag{3}$$

for an application-dependent, suitable choice of the sequence, B_m (see below).

Significance of our new results. In the article we prove several variants and properties of the Lambert series factorization theorem defined by (2). Namely, in Section 2 and Section 3 we prove Theorem 2.1, Theorem 2.2, and then Theorem 3.1 and Theorem 3.2 which provide interesting generalized variations of the first two factorization theorem results. Each of these factorization theorems suggest new relations between sums of an arbitrary sequence $\{a_n\}_{n\geq 1}$ over the divisors of an integer n as in (1) and more additive identities involving the same sequence. Our results proved in the article relate the two branches of additive and multiplicative number theory in many interesting new ways. Moreover, our new theorems connect several famous special multiplicative functions with divisor sums over partitions which are additive in nature.

Even though there are a number of important results connection the theory of divisors with that of partitions and special classical partition functions, these results are more or less scattered in their approach. We propose to continue the study of the relationships between divisors and partitions with the goal of identifying common threads between these connections by the means of our unified factorization theorems of Lambert series generating functions. On the multiplicative number theory side, we connect the Euler partition function $p(n)$ with other important number theoretic functions including Euler's totient function, the Möbius function, Liouville's lambda function, von Mangoldt's lambda function, the Jordan totient functions, and the generalized sum-of-divisors functions by extending the results first proved in [3, 4, 7].

2. Natural generalizations of the factor pairs

Theorem 2.1. *Suppose that $C(q)$ in (2) is fixed. Then for all integers $n, k \geq 1$, we have that*

$$s_{n,k} = \sum_{i=1}^{\lfloor n/k \rfloor} [q^{n-i\cdot k}]C(q), \tag{i}$$

i.e., so that we have a generating function for the general case of $s_{n,k}$ in the form of

$$s_{n,k} = [q^n]\frac{q^k}{1-q^k}C(q). \tag{ii}$$

Proof. We rewrite (2) as

$$C(q)\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}a_k = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} s_{n,k}\,q^n\right)a_k.$$

Equating the coefficient of a_k in this identity, gives

$$\sum_{n=1}^{\infty} s_{n,k}\, q^n = \frac{q^k}{1-q^k} C(q).$$

Rewriting this relation

$$\sum_{n=1}^{\infty} s_{n,k}\, q^n = C(q) \sum_{n=1}^{\infty} q^{k \cdot n},$$

we derived the first claimed relation, which we note easily implies the second, where we have invoked the Cauchy multiplication of two power series. □

Remark 2.1. *We remark that the general factorization in* (2) *can be easily derived considering the following identity*

$$\sum_{k=1}^{n} \left(\sum_{d|k} a_d \right) [q^{n-k}] C(q) = \sum_{k=1}^{n} \left(\sum_{i \geq 1} [q^{n-i \cdot k}] C(q) \right) a_k.$$

The case of $C(q) \equiv (q;q)_\infty$ in Theorem 2.1 can be rewritten considering Euler's pentagonal number theorem, i.e.,

$$(q;q)_\infty = \sum_{j=0}^{\infty} (-1)^{\lceil j/2 \rceil} q^{G_j},$$

where the exponent

$$G_j = \frac{1}{2} \left\lceil \frac{j}{2} \right\rceil \left\lceil \frac{3j+1}{2} \right\rceil, \qquad j \geq 0,$$

is the j^{th} generalized pentagonal number. In particular, for $n, k > 0$ we have that

$$s_o(n,k) - s_e(n,k) = \sum_{k | n - G_j} (-1)^{\lceil j/2 \rceil},$$

where the sum runs over all positive multiple of k of the form $n - G_j$.

Theorem 2.1 allows us to give another very interesting special case of (2) considered in [3, 4].

Corollary 2.1. *For arbitrary $\{a_n\}_{n \geq 1}$,*

$$\sum_{n \geq 1} \frac{a_n\, q^n}{1-q^n} = (q;q)_\infty \sum_{n \geq 1} \sum_{k=1}^{n} s_{n,k}\, a_k\, q^n,$$

where $s_{n,k}$ is the number of k's in all unrestricted partitions of n.

Proof. We take into account the fact that

$$\frac{q^k}{1-q^k} \cdot \frac{1}{(q;q)_\infty}$$

is the generating function for the number of k's in all unrestricted partitions of n. This generating function implies our result. □

Example 2.1 (Applications of the Corollary)**.** *The result in Corollary 2.1 allows us to derive many special case identities involving Euler's partition function and various arithmetic* *functions. More precisely, by the well-known famous special cases Lambert series identities expanded in the introduction to* [3], *for $n \geq 1$ we have that*

$$\begin{aligned}
&\sum_{k=1}^{n} \sigma_x(k)\, p(n-k) = \sum_{k=1}^{n} k^x s_{n,k}, \\
&p(n-1) = \sum_{k=1}^{n} \mu(k)\, s_{n,k}, \\
&\sum_{k=1}^{n} k\, p(n-k) = \sum_{k=1}^{n} \phi(k)\, s_{n,k}, \\
&\sum_{k\geq 1} p(n-k^2) = \sum_{k=1}^{n} \lambda(k)\, s_{n,k}, \\
&\sum_{k=1}^{n} \Lambda(k)\, p(n-k) = \sum_{k=1}^{n} \log(k)\, s_{n,k}, \\
&\sum_{k=1}^{n} 2^{\omega(k)}\, p(n-k) = \sum_{k=1}^{n} |\mu(k)|\, s_{n,k}, \\
&\sum_{k=1}^{n} k^t\, p(n-k) = \sum_{k=1}^{n} J_t(k)\, s_{n,k},
\end{aligned}$$

where $s_{n,k}$ is the number of k's in all unrestricted partitions of n. Moreover, in the case where $a_n \equiv 1$ in the corollary, for $n > 0$ we have that

$$\sum_{k=-\infty}^{\infty} (-1)^k\, S\big(n - k(3k+1)/2\big) = \sigma_0(n),$$

and that

$$\sum_{k=1}^{n} \sigma_0(k)\, p(n-k) = S(n),$$

where $S(n)$ is number of parts in all partitions of n (also, sum of largest parts of all partitions of n). Similarly, in the special case where $a_n := n$, for $n \geq 1$ we have that

$$\sum_{k=-\infty}^{\infty} (-1)^k \big(n - k(3k+1)/2\big)\, p\big(n - k(3k+1)/2\big) = \sigma_1(n),$$

where

$$\sum_{k=1}^{n} \sigma_1(k)\, p(n-k) = n\, p(n).$$

Corollary 2.2 (A Known Factorization)**.** *For arbitrary $\{a_n\}_{n\geq 1}$, we have that*

$$\sum_{n\geq 1} \frac{a_n\, q^n}{1-q^n} = (q^2;q)_\infty \sum_{n\geq 1} \left(p(n-1)\, a_1 + \sum_{k=2}^{n} s'_{n,k}\, a_k \right) q^n,$$

where $s'_{n,k}$ is the number of k's in all unrestricted partitions of n that do not contain 1 as a part.

Proof. We consider (2) with $C(q) = (q^2;q)_\infty^{-1}$. According to Theorem 2.1, the generating function of $s'_{n,1}$ is given by

$$\frac{q}{1-q} \cdot \frac{1}{(q^2;q)_\infty} = \frac{q}{(q;q)_\infty} = \sum_{n=1}^{\infty} p(n-1)\, q^n.$$

For $k > 1$, we see that the generating function of $s'_{n,k}$ is given by

$$\frac{q^k}{1-q^k} \cdot \frac{1}{(q^2;q)_\infty} = \frac{q^k}{1-q^k} \cdot \frac{1-q}{(q;q)_\infty},$$

which is the generating function for the number of k's in all partitions of n that do not contain 1 as a part. □

Example 2.2 (More Applications of the Corollary). *We denote by $p_1(n)$ the number of partition of n that do not contain 1 as a part. For $n \geq 1$ and fixed $x \in \mathbb{C}$, we have that*

$$\sum_{k=1}^{n} \sigma_x(k)\, p_1(n-k) = p(n-1) + \sum_{k=2}^{n} k^x\, s'_{n,k},$$

$$-\,p(n-2) = \sum_{k=2}^{n} \mu(k)\, s'_{n,k},$$

$$\sum_{k=1}^{n} k\, p_1(n-k) = p(n-1) + \sum_{k=2}^{n} \phi(k)\, s'_{n,k},$$

$$\sum_{k\geq 1} p_1(n-k^2) = p(n-1) + \sum_{k=2}^{n} \lambda(k)\, s'_{n,k},$$

$$\sum_{k=1}^{n} \Lambda(k)\, p_1(n-k) = \sum_{k=2}^{n} \log(k)\, s'_{n,k},$$

$$\sum_{k=1}^{n} 2^{\omega(k)}\, p_1(n-k) = p(n-1) + \sum_{k=2}^{n} |\mu(k)|\, s'_{n,k},$$

$$\sum_{k=1}^{n} k^t\, p_1(n-k) = p(n-1) + \sum_{k=2}^{n} J_t(k)\, s'_{n,k},$$

where $s'_{n,k}$ is the number of k's in all partitions of n that do not contain 1 as a part.

Corollary 2.3 (Another Known Factorization). *For arbitrary $\{a_n\}_{n\geq 1}$, we have that*

$$\sum_{n\geq 1} \frac{a_n\, q^n}{1-q^n} = (q^3;q)_\infty \sum_{n\geq 1} \left(p_2(n-1)\, a_1 + p_1(n-2)\, a_2 + \sum_{k=3}^{n} s''_{n,k}\, a_k \right) q^n,$$

where $p_k(n)$ is the number of partition of n that do not contain k as a part and $s''_{n,k}$ is the number of k's in all unrestricted partitions of n that do not contain 1 or 2 as a part.

Proof. We consider (2) with $C(q) = (q^3;q)_\infty^{-1}$. According to Theorem 2.1, the generating function of $s''_{n,1}$ is given by

$$\frac{q}{1-q} \cdot \frac{1}{(q^3;q)_\infty} = \frac{q(1-q^2)}{(q;q)_\infty} = \sum_{n=1}^{\infty} p_2(n-1)\, q^n.$$

The generating function for $s''_{n,2}$ is

$$\frac{q^2}{1-q^2} \cdot \frac{1}{(q^3;q)_\infty} = \frac{q^2}{(q^2;q)_\infty} = \sum_{n=1}^{\infty} p_1(n-2)\, q^n.$$

For $k > 2$, we see that the generating function of $s''_{n,k}$ is given by

$$\frac{q^k}{1-q^k} \cdot \frac{1}{(q^3;q)_\infty} = \frac{q^k}{1-q^k} \cdot \frac{(1-q)(1-q^2)}{(q;q)_\infty},$$

which is the generating function for the number of k's in all partitions of n that do not contain 1 or 2 as a part. □

Corollary 2.4 (A Generalization of the Known Factorizations). *For integers $m \geq 1$ and arbitrary $\{a_n\}_{n\geq 1}$, we have a Lambert series factorization given by*

$$\sum_{n\geq 1} \frac{a_n\, q^n}{1-q^n} = \sum_{n\geq 1} \left(\sum_{i=1}^{m-1} \sum_{j=1}^{\lfloor n/i \rfloor} p_{m-1}(n - i \cdot j)\, a_i + \sum_{k=m}^{n} s_{n,k}^{(m-1)}\, a_k \right) q^n,$$

where $p_m(n)$ denotes the number of partitions of n that do not contain $1, 2, \ldots, m$ as a part and where $s_{n,k}^{(m)}$ denotes the number of k's in all unrestricted partitions that do not contain $1, 2, \ldots, m$ as a part.

Proof. The proof of Corollary 2.3 is the starting point for proving this generalized result. In particular, for the factorization pair determined by $C(q) := (q^m; q)_\infty^{-1}$ in (2), we have that for $1 \leq i < m$ the coefficient on the right-hand-side of the factorization is given by

$$\frac{q^i}{1-q^i} \frac{(1-q)(1-q^2)\cdots(1-q^{m-1})}{(q;q)_\infty} = \sum_{n\geq 1} \sum_{j=1}^{\lfloor n/i \rfloor} p_{m-1}(n - ij)\, q^n.$$

Similarly, by Theorem 2.1 for $k \geq m$ we see that the right-hand-side coefficient of a_k satisfies the following generating function over n:

$$\frac{q^k}{1-q^k} \cdot \frac{1}{(q^m;q)_\infty} = \frac{q^k}{1-q^k} \cdot \frac{(1-q)(1-q^2)\cdots(1-q^{m-1})}{(q;q)_\infty}.$$ □

Theorem 2.2 (Generalized Factorization Theorem Identities). *Suppose that the factorization pair $(c_n, s_{n,k})$ in (2) is fixed where $c_n := [q^n]1/C(q)$. Then for all integers $n, k \geq 1$ and $m \geq 0$ with $1 \leq k \leq n$, we have that*

$$s_{n,k}^{(-1)} = \sum_{d|n} c_{d-k}\, \mu(n/d) \tag{i}$$

$$c_{n-k} = \sum_{d|n} s_{n,k}^{(-1)} \tag{ii}$$

$$B_m = b_{m+1} + \sum_{k=1}^{m} [q^k] C(q)\, b_{m+1-k}. \tag{iii}$$

Proof of (i) and (ii). This result is equivalent to showing that

$$c_{n-k} = \sum_{d|n} s_{d,k}^{(-1)},$$

which we do below by mimicking the proof from the reference [3, §3]. In particular, we consider the Lambert series over the sequence of $s_{n,k}^{(-1)}$ for a fixed integer $k \geq 1$ and note its factorization from (2) in the form of

$$\sum_{d|n} s_{d,k}^{(-1)} = \sum_{m=0}^{n} \delta_{n-k,m}\, c_m = c_{n-k}.$$ □

Proof of (iii). By the matrix representation of our factorization theorem given in (3), we see by a generating function argument starting from (2) that

$$B_{n-1} = \sum_{k=1}^{n} s_{n,k}\, a_k = [q^n]C(q) \sum_{m \geq 1} b_m\, q^m = b_n + \sum_{k=1}^{n} [q^k]C(q)\, b_{n-k},$$

when $C(0) \equiv 1$ as in the factorization theorem stated in the introduction. □

Note that (i) in the proposition implies the following closed-form generating function for the Lambert series over the inverse matrix sequences by Möbius inversion:

$$\sum_{n \geq 1} \frac{s_{n,k}^{(-1)}\, q^n}{1-q^n} = \frac{q^k}{C(q)}.$$

We have additional formulas that relate the sequences implicit to the choice of a fixed factorization pair in the form of (2). Namely, we see that for $m \geq 1$

$$b_m = \sum_{d|m} a_d = \sum_{j=0}^{m} \sum_{k=1}^{j} s_{j,k}\, a_k\, c_{m-j}.$$

We also have the following determinant-based recurrence relations proved as in the reference [3, §2] between the sequences, $s_{n,k}$ and $s_{n,k}^{(-1)}$, which are symmetric in that these identities still hold if one sequence is interchanged with the other:

$$\begin{aligned} s_{n,j}^{(-1)} &= -\sum_{k=1}^{n-j} s_{n,n+1-k}^{(-1)}\, s_{n+1-k,j} + \delta_{n,j} \qquad (4) \\ &= -\sum_{k=1}^{n-j} s_{n,n-k}\, s_{n-k,j}^{(-1)} + \delta_{n,j} = -\sum_{k=1}^{n} s_{n,k-1}\, s_{k-1,j}^{(-1)} + \delta_{n,j}. \end{aligned}$$

3. **Variations of the factorization theorems**

One topic suggested by the first author as we considered generalizations of the factorization theorems both in this article and in our first article [3] is to consider what happens in the form of Theorem 2.2 part (i) when the Möbius function is replaced by any other special multiplicative function, $\gamma(n)$, such as Euler's totient function, $\phi(n)$, or for example by von Mangoldt's function, $\Lambda(n)$. In its direct form, the factorization theorem in (2) does not accommodate a transformation of this form. However, if we change our specification of the fundamental factorization in the theorems from the previous section to allow the instance of a_k in the left-hand-side sums of (2) to be a function, $\widetilde{a}_k$, depending on $\gamma(n)$ and the Lambert series sequence, a_n, we obtain several interesting new results. The next examples where $\big(C(q), \gamma(n)\big) := \big((q;q)_\infty, \phi(n)\big), \big((q;q)_\infty, n^\alpha\big)$ for some fixed $\alpha \in \mathbb{C}$ provide the motivation for the statement of the more general theorem given in this section.

Example 3.1 (Convolutions with the Euler Totient Function)**.** *Suppose that for an arbitrary sequence, $\{a_m\}_{m \geq 1}$, we define the factorization of the Lambert series over a_n to be*

$$\sum_{n \geq 1} \frac{a_n\, q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n \geq 1} \sum_{k=1}^{n} s_{n,k}(\phi)\, \widetilde{a}_k(\phi)\, q^n,$$

where we define $s_{n,k}(\phi)$ in terms of its corresponding inverse sequence through (4) given by the divisor sum

$$s_{n,k}^{(-1)}(\phi) := \sum_{d|n} p(d-k)\, \phi(n/d).$$

Then we have an exact formula given in the following form where we note that $n = \sum_{d|n} \phi(d)$*:*

$$\widetilde{a}_k(\phi) = \sum_{d|k} a_d \cdot (k/d) = k \cdot \sum_{d|k} \frac{a_d}{d}.$$

The next theorem makes precise a generalized form of the factorization theorem variant suggested by the last two examples in the previous subsection.

Theorem 3.1 (Generalized Factorization Theorem I)**.** *Suppose that the sequence* $\{a_n\}_{n\geq 1}$ *is taken to be arbitrary and that the functions,* $C(q)$ *and* $\gamma(n)$*, are fixed. Then we have a generalized Lambert series factorization theorem expanded in the form of*

$$\sum_{n\geq 1} \frac{a_n\, q^n}{1-q^n} = \frac{1}{C(q)} \sum_{n\geq 1} \sum_{k=1}^{n} s_{n,k}(\gamma)\, \widetilde{a}_k\, q^n,$$

where $s_{n,k}(\gamma)$ *is defined through its inverse sequence by* (4) *according to the formula*

$$s_{n,k}^{(-1)}(\gamma) := \sum_{d|n} [q^{d-k}] \frac{1}{C(q)}\, \gamma(n/d),$$

and where for $\widetilde{\gamma}(n) := \sum_{d|n} \gamma(d)$ *we have that*

$$\widetilde{a}_k(\gamma) = \sum_{d|k} a_d\, \widetilde{\gamma}(n/d).$$

Proof. By the same argument justifying the matrix equation in (3) from the factorization in (2), we see that

$$\widetilde{a}_n = \sum_{k=1}^{n} s_{n,k}^{(-1)} \times [q^k] \sum_{d=1}^{k} \frac{a_d\, q^d}{1-q^d} C(q).$$

Thus for fixed $n \geq 1$ and each $1 \leq d \leq n$ we have that

$$\begin{aligned} [a_d]\widetilde{a}_n &= \sum_{k=1}^{n} s_{n,k}^{(-1)} \times \underbrace{[q^k] \frac{q^d}{1-q^d} C(q)}_{:=t_{k,d}} = \sum_{k=d}^{n} \sum_{r|n} p(r-k)\, \gamma(n/r) t_{k,d} \\ &= \sum_{r|n} \big(p(r-d)\, t_{d,d} + p(r-d-1)\, t_{d+1,d} + \cdots + p(0)\, t_{r,d}\big) \gamma(n/r) \\ &= \sum_{r|n} \sum_{i=d}^{r} p(r-i)\, t_{i,d}\, \gamma(n/r). \qquad \text{(i)} \end{aligned}$$

If we can show that the inner sum is one when $d|r$ where $d|n$ and zero otherwise, we have completed the proof of our result. We note that for $d \geq 1$ and $i \geq d-1$ we have that $t_{i,d} = [q^{i-d}]C(q)$. Then we continue expanding the inner sum in (i) as[1]

$$\begin{aligned} \sum_{i=d}^{r} p(r-t)\, t_{i,d} &= \sum_{i=0}^{r-d} p(r-d-i)\, t_{i+d,d} + \sum_{i=0}^{d-1} p(r-i)\, [q^{i-d}] \frac{C(q)}{1-q^d} \qquad \text{(ii)} \\ &= [q^{r-d}] \frac{1}{\cancel{C(q)}} \frac{q^d}{1-q^d} \cancel{C(q)} = [d|r \text{ where } r|n]_\delta\,. \end{aligned}$$

[1] <u>*Notation*</u>: *Iverson's convention* compactly specifies boolean-valued conditions and is equivalent to the *Kronecker delta function*, $\delta_{i,j}$, as $[n=k]_\delta \equiv \delta_{n,k}$. Similarly, $[\texttt{cond} = \texttt{True}]_\delta \equiv \delta_{\texttt{cond},\texttt{True}}$ in the remainder of the article.

Hence, we have from (i) and (ii) that for $1 \leq d \leq n$

$$[a_d]\widetilde{a_n} = \begin{cases} \widetilde{\gamma}(n/d) = \sum_{r|\frac{n}{d}} \gamma\left(\frac{n}{dr}\right), & \text{if } d|n; \\ 0, & \text{if } d \not| n, \end{cases}$$

which implies our formula for $\widetilde{a}_n$ stated in the theorem. Here, we notice that it is apparent from the factorization given in the first equation of the theorem that $\widetilde{a}_n = \sum_{d=1}^n \gamma_{n,d}\, a_d$ for some coefficients, $\gamma_{n,d}$, which we have just proved a formula for in the previous equation. □

Example 3.2 (New Convolution Identities from the Matrix Factorizations)**.** *The corresponding matrix factorization representation from* (3) *resulting from the theorem provides that for all* $n \geq 1$ *and fixed factorization pair parameter* $C(q)$ *we have that*

$$\widetilde{a}_n = \sum_{k=1}^n s_{n,k}^{(-1)}\, B_{k-1}, \tag{5}$$

where $\widetilde{a}_n$*,* $s_{n,k}^{(-1)}$*, and* B_m *are respectively defined as in Theorem 3.1 and Theorem 2.2. One corollary of this result (among many) provides an exact expression for the coefficients of the Lambert series over the generalized sum-of-divisors function,* $\sigma_\alpha(n)$*, for any fixed* $\alpha \in \mathbb{C}$*:*

$$\begin{aligned} [q^n] \sum_{m\geq 1} \frac{\sigma_\alpha(m)\, q^m}{1-q^m} &= \sum_{d|n} \sigma_\alpha(d) \\ &= \sum_{k=1}^n \left(\sum_{d|n} p(d-k)(n/d)^\alpha \right) \left(\sigma_0(k) + \sum_{s=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k+1}-s}{6} \right\rfloor} \sigma_0\left(k - \frac{j(3j+s)}{2}\right) \right). \end{aligned}$$

Similarly, by setting $a_n := n^\beta$ *and* $\gamma(n) := n^\alpha$ *for some fixed* $\alpha, \beta \in \mathbb{C}$*, we obtain the identity*

$$\sum_{d|n} d^\beta\, \sigma_\alpha(n/d) = \sum_{k=1}^n \left(\sum_{d|n} p(d-k)(n/d)^\alpha \right) \left(\sigma_\beta(k) + \sum_{s=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k+1}-s}{6} \right\rfloor} \sigma_\beta\left(k - \frac{j(3j+s)}{2}\right) \right).$$

If we set $\big(a_n, \gamma(n)\big) := \big(n^\beta, \phi(n)\big)$ *for a fixed* β*, we obtain the following related identity:*

$$n\, \sigma_{\beta-1}(n) = \sum_{k=1}^n \left(\sum_{d|n} p(d-k)\phi(n/d) \right) \left(\sigma_\beta(k) + \sum_{s=\pm 1} \sum_{j=1}^{\left\lfloor \frac{\sqrt{24k+1}-s}{6} \right\rfloor} \sigma_\beta\left(k - \frac{j(3j+s)}{2}\right) \right).$$

We note that these identities implicitly involving the Euler partition function $p(n)$ *correspond to the choice of the factorization pair parameter* $C(q) := (q;q)_\infty$*. We could just as easily re-phrase these expansions in terms of the partition function* $q(n)$ *where* $C(q) = 1/(-q;q)_\infty$*, or in terms of any number of other special sequences with a reciprocal generating function of* $C(q)$*.*

Example 3.3 (A Second Variation of the Theorem)**.** *Let the sequence* $\{a_n\}_{n\geq 1}$ *be fixed and suppose that the functions,* $C(q) := (q;q)_\infty$ *and* $\gamma(n) := \phi(n)$*. Then we have another construction of a generalized Lambert series factorization theorem for these parameters expanded in the form of*

$$\sum_{n\geq 1} \frac{\widetilde{a}'_n\, q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} \sum_{k=1}^n s_{n,k}(\phi)\, a_k\, q^n,$$

where $s_{n,k}(\phi)$ is defined through its inverse sequence by (4)*according to the formula*

$$s_{n,k}^{(-1)}(\phi) := \sum_{d|n} p(d-k)\,\phi(n/d).$$

We generalize the results in the preceding example by the next theorem.

Theorem 3.2 (Generalized Factorization Theorem II)**.** *Suppose that the sequence $\{a_n\}_{n\geq 1}$ is taken to be arbitrary and that the functions, $C(q)$ and $\gamma(n)$, are fixed. Then we have a generalized Lambert series factorization theorem expanded in the form of*

$$\sum_{n\geq 1} \frac{\widetilde{a}'_n\, q^n}{1-q^n} = \frac{1}{C(q)} \sum_{n\geq 1} \sum_{k=1}^{n} s_{n,k}(\gamma)\, a_k\, q^n,$$

where $s_{n,k}(\gamma)$ is defined through its inverse sequence by (4) *according to the formula*

$$s_{n,k}^{(-1)}(\gamma) := \sum_{d|n} [q^{d-k}] \frac{1}{C(q)}\, \gamma(n/d),$$

and where we have that for all $m \geq 1$

$$\sum_{d|m} \widetilde{a}'_d(\gamma) = \sum_{i=1}^{m} \sum_{j=1}^{m+1-i} a_i\, s_{m+1-j,i}\, [q^{j-1}] \frac{1}{C(q)}.$$

Proof. We equate the left-hand-side to the right-hand-side of the theorem statement to obtain the expansions

$$\begin{aligned}
\sum_{d|n} \widetilde{a}'_d &= [q^n] \sum_{n\geq 1} \frac{\widetilde{a}'_n\, q^n}{1-q^n} = \sum_{j=0}^{n} \sum_{k=1}^{n-j} s_{n-j,k}\, a_k\, [q^j] \frac{1}{C(q)} \\
&= \sum_{k=1}^{n} \sum_{j=0}^{n} s_{n-j,k}\, a_k\, [q^j] \frac{1}{C(q)} = \sum_{k=1}^{n} \sum_{j=0}^{n-k} s_{n-j,k}\, a_k\, [q^j] \frac{1}{C(q)},
\end{aligned}$$

since $s_{n-j,k}$ is zero-valued for $n-j<k$ which requires that for $s_{n,k}$ to be potentially non-zero we must have that $n-j \geq k$, or equivalently that $n-k \geq j$ as the upper bound of the inner sum with respect to j. Shifting the index of summation in the inner sum by one then leads to the identity for these Lambert series coefficients over powers of q^n. Hence we have proved the theorem. □

4. **Conclusions**

In Section 2 and Section 3 we proved several new forms of the Lambert series factorization theorem in (2) which is defined by the dependent factor pair parameters, $C(q)$ and $s_{n,k}$. The interpretation of these theorems provides a corresponding matrix factorization which effectively generalizes the known result in (3) from [3, 7]. The first theorems proved in Section 2 also lead to a number of new summation identities connecting partition functions such as $p(n)$ with sums over special multiplicative functions with well-known Lambert series expansions found in the literature [3, *cf.* §1, §3]. The generalizations of the first pair of theorems we proved later in the variations of Section 3 provide yet additional interpretations and identities between sums of the functions implicit to (1), generalized partition functions, and other special multiplicative functions of importance in number theory.

More general forms of the factorization theorems can be established as follows.

Expansions of generalized Lambert series. We seek to generalize the factorization theorem result in (2) to a corresponding form for the following generalized Lambert series expansions for fixed constants $c, d, \alpha, \beta, \gamma, \delta \in \mathbb{C}$ defined such that the series converges:

$$L_a(c, d; \alpha, \beta, \gamma, \delta) := \sum_{n\geq 1} \frac{a_n\, c^n\, q^{\alpha n+\beta}}{1 - d\, q^{\gamma n+\delta}}. \tag{6}$$

It is not difficult to show that the series coefficients of q^n in the previous Lambert series expansion are given in closed-form according to the special case formula

$$[q^n] L_a(c, d; \alpha, \gamma, \alpha, \gamma) = \sum_{\substack{\alpha m+\gamma \\ m\geq \alpha+\gamma}} c^m\, a_m\, d^{\frac{n}{\alpha m+\gamma}-1}.$$

Applications of a corresponding factorization result include new identities for the generalized Lambert series generating the sum-of-squares function, $r_2(n)$, in the form of [1, §17.10]

$$\sum_{m\geq 1} r_2(m)\, q^m = 4 \sum_{n\geq 1} \frac{(-1)^{n+1}\, q^{2n+1}}{1 - q^{2n+1}}.$$

For example, we may formulate a generalized variant of the factorization theorems in this article as

$$\sum_{n\geq 1} \frac{a_n\, c^n\, q^{2n+1}}{1 - d\, q^{2n+1}} = \frac{1}{C(q)} \sum_{n\geq 1} \sum_{k=1}^{n} s_{n,k}(d)\, c^k\, a_k\, q^n,$$

where for an arbitrary sequence, $\{a_n\}_{n\geq 1}$, we have that the series coefficients of the left-hand-side Lambert series in the previous equation are given by

$$[q^n] L(c, d; 2, 1, 2, 1) = \sum_{\substack{2m+1|n \\ m>1}} c^m\, a_m\, d^{\frac{n}{2m+1}-1}.$$

The expansions of the generalized Lambert series in (6) also allow us to approach new identities for the Lambert series generating the logarithmic derivatives of the Jacobi theta functions in the forms of [5, §20.5(ii)]

$$\begin{aligned}
\frac{\vartheta_1'(z, q)}{\vartheta_1(z, q)} &= 4 \sum_{n\geq 1} \frac{\sin(2nz)\, q^{2n}}{1 - q^{2n}} + \cot(z) \\
\frac{\vartheta_2'(z, q)}{\vartheta_2(z, q)} &= 4 \sum_{n\geq 1} \frac{(-1)^n\, \sin(2nz)\, q^{2n}}{1 - q^{2n}} - \tan(z) \\
\frac{\vartheta_3'(z, q)}{\vartheta_3(z, q)} &= 4 \sum_{n\geq 1} \frac{(-1)^n\, \sin(2nz)\, q^{n}}{1 - q^{2n}} \\
\frac{\vartheta_4'(z, q)}{\vartheta_4(z, q)} &= 4 \sum_{n\geq 1} \frac{\sin(2nz)\, q^{n}}{1 - q^{2n}}.
\end{aligned}$$

Similarly, by considering derivatives of the generalized Lambert series as in [6], we can generate higher-order cases of the derivatives of the Jacobi theta functions, including the following identities [5, §20.4(ii)]:

$$\frac{\vartheta_1'''(0,q)}{\vartheta_1'(0,q)} = -1 + 24 \sum_{n\geq 1} \frac{q^{2n}}{(1-q^{2n})^2}$$
$$\frac{\vartheta_2''(0,q)}{\vartheta_2(0,q)} = -1 - 8 \sum_{n\geq 1} \frac{q^{2n}}{(1+q^{2n})^2}$$
$$\frac{\vartheta_3''(0,q)}{\vartheta_3(0,q)} = -8 \sum_{n\geq 1} \frac{q^{2n-1}}{(1+q^{2n-1})^2}$$
$$\frac{\vartheta_4''(0,q)}{\vartheta_4(0,q)} = 8 \sum_{n\geq 1} \frac{q^{2n-1}}{(1-q^{2n-1})^2}.$$

Transformations of Lambert series. One possible transformation providing an application of the generalized factorization theorems we have already proved within this article is given by

$$\sum_{n=1}^{\infty} \frac{a_n\, q^n}{1+q^n} = \sum_{n=1}^{\infty} \frac{a_n\, q^n}{1-q^n} - 2\sum_{n=1}^{\infty} \frac{a_n\, q^{2n}}{1-q^{2n}} = \sum_{n=1}^{\infty} \frac{b_n\, q^n}{1-q^n},$$

where

$$b_n = \begin{cases} a_n, & \text{for } n \text{ odd}, \\ a_n - 2a_{n/2} & \text{for } n \text{ even}. \end{cases}$$

Likewise, the terms of the more general Lambert series

$$\sum_{n=1}^{\infty} \frac{a_n\, c^n\, q^n}{1 \pm q^n}, \qquad \max(|cq|, |q|) < 1,$$

follow from the earlier factorization theorems by substituting $a_k \mapsto c^k a_k$.

Topics for future research and investigation. The generalizations to the factorization theorems we have proved in this article suggested in this section comprise a new avenue of future research based on our new results. We anticipate that the investigation of these topics will be a fruitful source of new identities and insights to other special multiplicative functions enumerated by Lambert series generating functions of the forms defined by (6).

REFERENCES

[1] *G. H. Hardy and E. M. Wright*, *An Introduction to the Theory of Numbers*, Oxford University Press, 2008.

[2] *M. Merca*, Combinatorial interpretations of a recent convolution for the number of divisors of a positive integer, J. Number Theory, **160**(2016), 60-75.

[3] *M. Merca and M. D. Schmidt*, *Generating special arithmetic functions by Lambert series factorizations*,

[4] *M. Merca*, The Lambert series factorization theorem, Ramanujan J, **44**(2017), 417-435.

[5] *F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark*, *NIST Handbook of Mathematical Functions.* Cambridge University Press, 2010.

[6] *M. D. Schmidt*, Combinatorial sums and identities involving generalized sum-of-divisor functions with bounded divisors, Integers, **20**(2020), Paper A85, 23 p.

[7] *M. D. Schmidt*, New recurrence relations and matrix equations for arithmetic functions generated by Lambert series, Acta Arith., **181**(2017), 355-367.

[8] *N. J. A. Sloane*, The Online Encyclopedia of Integer Sequences, 2025, https://oeis.org/.