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HOMOGENIZATION RESULTS FOR A NONLINEAR WAVE EQUATION IN A PERFORATED DOMAIN

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Scopul acestei lucrări îl constituie studiul comportamentului asimptotic al soluției unei ecuații a undelor într-un mediu periodic perforat. Vom considera, la microscară, o ecuație a undelor, cu surse neliniare şi condiții inițiale şi la frontiera adecvate. Ne vom concentra atenția asupra cazului în care perforațiile au o dimensiune critică şi vom demonstra că soluția acestei probleme converge, când parametrul mic ce caracterizează mărimea perforațiilor tinde către zero, către soluția unei noi probleme, care conține termeni suplimentari. Rezultatele prezentate sunt generaliză ale unor rezultate obtinute în [3], prin considerarea unor surse neliniare.

The effective behavior of the solution of a nonlinear wave equation in a periodic perforated domain is analyzed. We consider, at the microscale, a wave equation, with nonlinear sources and suitable initial and boundary conditions. We focus on the case in which the perforations are of the so-called critical size and we prove that the solution of this problem converges, as the small parameter characterizing the size of the holes tends to zero, to the solution of a new problem, containing extra zero order terms. Our paper generalizes some of the results contained in [3], by considering nonlinear sources.

Keywords: homogenization, wave equation, critical holes.

1. Introduction

The aim of this paper is to study the homogenization of a nonlinear wave equation in a periodically perforated medium. Such problems arise, for instance, in the modeling of vibrating membranes. Let Ω be an open fixed bounded set in $\mathbf{R}^n$ and let us perforate it by holes. As a result, we obtain an open set Ω^{ε}, which will be referred to as being the *perforated domain*; ε represents a small parameter related to the characteristic size of the perforations. We shall deal with the case in which the perforations are identical and periodically distributed and they are of the so-called *critical size* (see Chapter 2). If we denote by $(0,T)$ the time interval of interest, we shall consider, at the microscale, the following wave equation:

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$$\begin{cases} u_\varepsilon'' - D_f \Delta u_\varepsilon + \beta(u_\varepsilon) = f \quad \text{in} \quad \Omega^\varepsilon \times (0,T), \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega^\varepsilon \times (0,T), \\ u_\varepsilon(0) = u_\varepsilon^0,\ u_\varepsilon'(0) = u_\varepsilon^1 \quad \text{in} \quad \Omega^\varepsilon. \end{cases} \tag{1}$$

We shall take $D_f > 0$ and we shall assume that

$$u_\varepsilon^0 \in H_0^1(\Omega^\varepsilon),\ u_\varepsilon^1 \in L^2(\Omega^\varepsilon)$$

and

$$\begin{cases} \widetilde{u}_\varepsilon^0 \to u^0 \quad \text{weakly in} \quad H_0^1(\Omega), \\ \widetilde{u}_\varepsilon^1 \to u^1 \quad \text{weakly in} \quad L^2(\Omega). \end{cases} \tag{2}$$

Moreover, we assume that $f \in L^2(0,T;L^2(\Omega))$ and the function β in (1) is continuous, monotonously non-decreasing and such that $\beta(0) = 0.$ As particular important examples of such nonlinear functions, we can consider, for instance, Freundlich or Michaelis-Menten kinetics (see [5] and [7]).

The existence and uniqueness of a weak solution of (1) can be settled by using the classical theory of semilinear monotone problems (see [2], [3] and [5]). As a result, we know that there exists a unique weak solution

$$u_\varepsilon \in C^0([0,T];H_0^1(\Omega^\varepsilon)) \cap C^1([0,T];L^2(\Omega^\varepsilon)).$$

Using an homogenization procedure, we shall prove that the solution u_ε, properly extended to the whole of $\Omega,$ converges weakly to the unique solution of the following homogenized problem:

$$\begin{cases} u'' - D_f \Delta u + D_f \mu u + \beta(u) = f \quad \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T),. \\ u(0) = u^0, \quad u'(0) = u^1 \quad \text{in} \quad \Omega, \end{cases} \tag{3}$$

where the distribution μ (see Chapter 3) is generated by the special size of the perforations.

Hence, the solution of the boundary-value problem (1) converges to the solution of a new problem, associated to an operator which is the sum of a standard homogenized one and extra zero order terms coming from the geometry and the nonlinearity of the problem.

The approach we used has its origin in the famous work of D. Cioranescu and F. Murat [4]. The same technique was then used by D. Cioranescu, P. Donato, F. Murat and E. Zuazua for the case of the wave equation (see [3]). Both papers deal with the linear case. We shall generalize some of the results in [3], by considering the nonlinear term given by the function β. Let us mention that we can also treat the case of an heterogeneous medium, for which the operator $-\Delta$ is replaced by a strongly elliptic operator with highly oscillating coefficients. For details in the elliptic case, see [1] and [6]. Also, we can treat in the same manner the case in which we have also a damping term in (1).

The structure of our paper is as follows: first, let us mention that we shall just focus on the case $n \geq 3$, which will be treated explicitly. The case $n = 2$ is similar and we shall omit to treat it here. In Chapter 2, we introduce some useful notations and assumptions and we give the main result. In Chapter 3, we give the proof of the main convergence result of this paper.

2. Setting of the problem and the main result

Let Ω be a smooth bounded connected open subset of $\mathbf{R}^n$, for $n \geq 3$, and let F be another open bounded subset of $\mathbf{R}^n$, with a smooth boundary ∂F (of class C^2). We shall refer to F as being *the elementary hole*. We assume that 0 belongs to F and that F is star-shaped with respect to 0. Since F is bounded, without loss of generality, we shall assume that $\overline{F} \subset Y$, where $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ is the representative cell in $\mathbf{R}^n$. Let ε be a real parameter taking values in a sequence of positive numbers converging to zero and let us assume that the characteristic size of the perforations, r_ε, is of the order of $\varepsilon^{n/(n-2)}$. For each ε and for any integer vector $\mathbf{i} \in \mathbf{Z}^n$, we consider

$$F_{\mathbf{i}}^{\varepsilon} = \varepsilon \mathbf{i} + r(\varepsilon) F.$$

Also, let us denote by F^{ε} the set of all the holes contained in Ω, i.e.

$$F^{\varepsilon} = \bigcup \left\{ F_{\mathbf{i}}^{\varepsilon} \;\middle|\; \overline{F}_{\mathbf{i}}^{\varepsilon} \subset \Omega, \mathbf{i} \in \mathbf{Z}^{n} \right\}.$$

Set

$$\Omega^{\varepsilon} = \Omega \setminus \overline{F}^{\varepsilon}.$$

Hence, Ω^{ε} is a periodically perforated domain with holes of the size $r(\varepsilon)$. All of them have the same shape, the distance between two adjacent holes is of order ε and they do not overlap. We shall denote by S^{ε} their boundary. Also, let us remark that the holes do not intersect the fixed boundary $\partial\Omega$. Moreover, for an arbitrary function $\psi \in L^2(\Omega^{\varepsilon})$, we shall denote by $\widetilde{\psi}$ its extension by zero inside the holes.

Finally, we assume that the function β in (1) is continuous, monotonously non-decreasing and such that $\beta(0) = 0$. Also, we suppose that there exist a constant $C \geq 0$ and an exponent q such that

$$|\beta(v)| \leq C(1 + |v|^{q}). \tag{4}$$

As already mentioned, we can deal also with the case in which the function f depends in a nonlinear way of the solution u_{ε}.

The main result of this paper is the following one:

Theorem 2.1. Under our previous hypotheses, the extension by zero to the whole of Ω of the unique solution u_{ε} of the microscopic problem (1) converges weakly $*$ in $L^{\infty}(0,T;H_0^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ to the unique solution $u \in C^0([0,T];V) \cap C^1([0,T];L^2(\Omega))$ of the homogenized problem (3).

3. Preliminaries and proof of the main result

Following [3], we know that for this geometry there exist a sequence of auxiliary functions w_{ε} and a distribution μ such that the following hypotheses are satisfied:

(H.0) $\quad 0 \leq \mathrm{w}^{\varepsilon} \leq 1$ a.e. in Ω;

(H.1) $\quad \mathrm{w}^{\varepsilon} \in H^1(\Omega)$;

(H.2) $w^{\varepsilon} = 0$ on the holes F^{ε};

(H.3) $w^{\varepsilon} \to 1$ weakly in $H^1(\Omega)$;

(H.4) $\mu \in W^{-1,\infty}(\Omega)$.

Moreover, we know that for every sequence v^{ε} such that $v^{\varepsilon} = 0$ on F^{ε}, satisfying $v^{\varepsilon} \to v$; weakly in $H_0^1(\Omega)$, with $v \in H_0^1(\Omega)$, we have

(H.5) $$\left\langle -\Delta w^{\varepsilon}, v^{\varepsilon} \right\rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)} \to \langle \mu, v \rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)}.$$

Also, under our hypotheses, we know from [3] and [5] that

$$\langle \mu, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{\Omega} \left| \nabla w^{\varepsilon} \right|^2 \varphi \, dx, \tag{5}$$

for any $\varphi \in D(\Omega) = C_0^{\infty}(\Omega)$.

Proof of Theorem 2.1. Under our previous hypotheses, it is not difficult to prove (see [3] for details) that the extension by zero of the solution of (1) to the whole of Ω is bounded in $L^{\infty}(0,T; H_0^1(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega))$. Taking a subsequence, still denoted by ε, we have:

$$\begin{cases} \tilde{u}_{\varepsilon} \to u \quad \text{weakly} * \text{ in } \quad L^{\infty}(0,T; H_0^1(\Omega)), \\ \tilde{u}_{\varepsilon}' \to u' \quad \text{weakly} * \text{ in } \quad L^{\infty}(0,T; L^2(\Omega)), \end{cases} \tag{6}$$

where

$$\begin{cases} u \in C^0([0,T]; V) \cap C^1([0,T]; L^2(\Omega)), \\ \qquad V = H_0^1(\Omega) \cap L^2(\Omega; d\mu). \end{cases} \tag{7}$$

Taking, in the variational formulation of problem (1), the test function $\phi(t)\varphi(x)w^{\varepsilon}(x)$, with $\phi \in D((0,T))$ and $\varphi \in D(\Omega)$, we get:

$$\int_0^T\int_{\Omega^\varepsilon} u_\varepsilon \phi'' \varphi w^\varepsilon \, dxdt + D_f \int_0^T\int_{\Omega^\varepsilon} \nabla u_\varepsilon \nabla w^\varepsilon \phi\varphi dxdt + D_f \int_0^T\int_{\Omega^\varepsilon} \nabla u_\varepsilon \nabla\varphi w^\varepsilon \phi dxdt + \int_0^T\int_{\Omega^\varepsilon} \beta(u_\varepsilon)\phi\varphi w^\varepsilon \, dxdt = \int_0^T\int_{\Omega^\varepsilon} f\phi\varphi w^\varepsilon \, dxdt. \tag{8}$$

Extending u^{ε} by zero to the whole of Ω and using our hypotheses, we can pass to the limit, with $\varepsilon \to 0$, in all the linear terms in (8). Indeed, due to Fubini's theorem and an integration by parts, we have

$$\int_\Omega \varphi w^\varepsilon \left(\int_0^T \phi'' \tilde{u}_\varepsilon dt\right)dx + D_f \left\langle -\Delta w^\varepsilon, \varphi \int_0^T \tilde{u}^\varepsilon \phi \, dt \right\rangle_\Omega - D_f \int_\Omega \nabla w^\varepsilon \nabla\varphi \left(\int_0^T \tilde{u}^\varepsilon \phi \, dt\right)dx + D_f \int_\Omega w^\varepsilon \nabla\varphi \nabla \left(\int_0^T \tilde{u}^\varepsilon \phi \, dt\right)dx + \int_0^T\int_\Omega \beta(\tilde{u}_\varepsilon)\phi\varphi w^\varepsilon \, dxdt = \int_0^T\int_\Omega f\phi\varphi w^\varepsilon \, dxdt. \tag{9}$$

Therefore, it follows immediately that

$$\int_0^T\int_{\Omega^\varepsilon} u_\varepsilon \phi'' \varphi w^\varepsilon \, dxdt \to \int_0^T \phi'' \left(\int_\Omega u\varphi \, dx\right)dt, \tag{10}$$

$$D_f \left\langle -\Delta w^\varepsilon, \varphi \int_0^T \tilde{u}^\varepsilon \phi \, dt \right\rangle_\Omega \to D_f \int_0^T \phi \left(\int_\Omega u\varphi \, d\mu\right)dt, \tag{11}$$

$$D_f \int_\Omega w^\varepsilon \nabla\varphi \nabla \left(\int_0^T \tilde{u}^\varepsilon \phi \, dt\right)dx \to D_f \int_0^T \phi \left(\int_\Omega \nabla u \nabla\varphi \, dx\right)dt, \tag{12}$$

$$\int_0^T \int_\Omega f\,\phi\varphi\, w^\varepsilon\, dxdt \to \int_0^T \phi\,(\int_\Omega f\,\varphi\, dx)dt. \tag{13}$$

So, it remains to show how to pass to the limit in the term containing the nonlinear function β. To this end, let us remember a result in [5]. More precisely, assuming (4), one can prove that for any sequence z_ε which converges weakly in $H_0^1(\Omega)$ to z, we have

$$\beta(z_\varepsilon) \to \beta(z) \quad \text{strongly in } L^{\overline{q}}(\Omega),$$

where

$$\overline{q} = \frac{2n}{q(n-2)+n}.$$

Therefore, we have

$$\int_0^T \phi\,(\int_{\Omega^\varepsilon} \beta(u_\varepsilon)\,\varphi dx)dt \to \int_0^T \phi\,(\int_\Omega \beta(u)\,\varphi\, dx)dt. \tag{14}$$

Finally, putting together (10)-(14), we obtain

$$\begin{aligned} &\int_0^T \phi''(\int_\Omega u\varphi\, dx)dt + D_f \int_0^T \phi\,(\int_\Omega u\ \varphi\, d\mu)dt + \\ &D_f \int_0^T \phi\,(\int_\Omega \nabla u \nabla \varphi\, dx)dt + \int_0^T \phi\,(\int_\Omega \beta(u)\varphi dx)dt = \int_0^T \phi\,(\int_\Omega f\,\varphi\, dx)dt. \end{aligned} \tag{15}$$

Since $\phi \in D((0,T))$ is arbitrary, it follows immediately that

$$\begin{aligned} &\int_\Omega u''\varphi\, dx + D_f \int_\Omega u\ \varphi\, d\mu + D_f \int_\Omega \nabla u \nabla \varphi\, dx + \\ &\int_\Omega \beta(u)\,\varphi\, dx = \int_\Omega f\,\varphi\, dx \qquad \forall\ \varphi \in D(\Omega). \end{aligned} \tag{16}$$

By the density of $D(\Omega)$ in V, (16) holds true for any $\varphi \in V$. So, we obtained exactly the variational formulation of the limit problem (3).

Since, exactly like in [3], we can pass to the limit, with $\varepsilon \to 0$, in the initial conditions, we have

$$u(0) = u^0, \quad u^{'}(0) = u^1.$$

As the solution u of the limit problem (3) is uniquely determined (see [3]), the whole sequence $\widetilde{u}_\varepsilon$ converges to u and this completes the proof of Theorem 2.1.

4. Conclusions

The general question which made the object of this paper was the homogenization of a semilinear wave equation in a periodically perforated domain. In the case of a critical size of the perforations, the limit problem is given by a new operator, which contains two zero-order terms, coming from the special geometry and the nonlinearity of the problem.

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R E F E R E N C E S

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