ON A CONJECTURE CONCERNING RESOLVING PAIRS

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Tomescu and Imran [Graphs and Combinatorics, 2010] proposed the following conjecture : for every connected graph of order n the number of resolving pairs (i.e., pairs of vertices of G having distinct distances to all vertices of G) is bounded above by $\lfloor n^2/4 \rfloor$ and solved into affirmative this assertion for graphs with diameter two.

In this paper the conjecture is verified for bipartite graphs, graphs of order n and diameter $n-2$ and for a subclass of graphs of diameter three. It is also shown that for every integers n, k such that $n \geq 3$ and $2 \leq k \leq n-1$ there is a graph of order n and diameter k having $\lfloor n^2/4 \rfloor$ resolving pairs.

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1. Introduction

Let G be a connected graph. The distance between vertices x and y is denoted $dist(x,y)$ and the diameter of G is

$$diam(G) = \max_{x,y \in V(G)} dist(x,y).$$

$N(x)$ will denote the set of neighbors and $d(x)$ the degree of $x \in V(G)$.

A subset W of vertices of G is called a resolving set [3] or locating set [9] if every vertex in G is uniquely determined by its distances to the vertices of W. If a resolving set has minimum size then it is frequently called a metric basis [2], [5] or just a basis [1], [3] for the graph and the number of elements in a basis is the metric dimension $dim(G)$ of G [3], [5], [8], [9].
A survey of results on the metric dimension and its applications is included in [2]; also see [6].
For a pair $\{x,y\}$ of distinct vertices of G we shall denote by $RS(x,y)$ the set of vertices $z \in V(G)$ such that $dist(z,x) \neq dist(z,y)$ [10]. Such a set will be called the resolving set (or the R-set) relative to the pair $\{x,y\}$. It is clear that $x,y \subseteq RS(x,y) \subseteq V(G)$ for any pair $\{x,y\}$.

This notion implicitly appeared in [4], whose authors constructed for a given connected graph G of order n an associated bipartite graph as follows: Let V_p be the collection of all $\binom{n}{2}$ pairs of vertices in G. The associated bipartite graph has partite sets $V(G)$ and V_p. A vertex $v \in V(G)$ is joined to a vertex $s \in V_p$ if v has distinct distances to the vertices in s.

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So the neighborhood of s in this associated bipartite graph is precisely the R-set for s referred to above. Some properties of R-sets were described in [10].

A pair $\{x,y\}$ such that $RS(x,y) = V(G)$ will be called a resolving pair of G. If for a pair $\{x,y\}$ of vertices $dist(x,y)$ is even, then the middle vertex v of a shortest path between x and y has equal distances to x and to y and so $v \notin RS(x,y)$.

Lemma 1.1. [10] If $\{x,y\}$ is a resolving pair of G then $dist(x,y)$ is odd.

Theorem 1.1. [10] If G has n vertices and diameter two then the number of resolving pairs of G is bounded above by $\lfloor n^2/4 \rfloor$ and this bound is attained only for $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Also in [10] the following conjecture has been proposed :

Conjecture RP. For every connected graph G of order $n \geq 2$ the number of resolving pairs is bounded above by $\lfloor n^2/4 \rfloor$.

This conjecture is valid for graphs of diameter two, paths and cycles. Other classes of graphs verifying conjecture RP will be studied in the next sections.

2. BIPARTITE GRAPHS AND GRAPHS OF DIAMETER $n-2$

For bipartite graphs resolving pairs are easy to characterize.

Theorem 2.1. Let G be a connected bipartite graph of order n having partite sets A and B. Resolving pairs of G are precisely the pairs $\{x,y\}$ where $x \in A$ and $y \in B$. The minimum number of resolving pairs is $n-1$ and this holds if and only if G is $K_{1,n-1}$ and the maximum number equals $\lfloor n^2/4 \rfloor$ and this bound is reached if and only if

$$-1 \leq |A| - |B| \leq 1$$

Proof. If x and y belong to the same partite set then $dist(x,y) \equiv 0 \, (mod\, 2)$ and by Lemma 1.1 $\{x,y\}$ is not a resolving pair.
Otherwise, let $x \in A$ and $y \in B$. If $z \in A$ then $dist(x,z) \equiv 0 \, (mod\, 2)$ and $dist(y,z) \equiv 1 \, (mod\, 2)$; if $z \in B$ then $dist(x,z) \equiv 1 \, (mod\, 2)$ and $dist(y,z) \equiv 0 \, (mod\, 2)$.
It follows that $dist(x,z) \neq dist(y,z)$ for any $z \in V(G)$.
In a bipartite graph $\{x,y\}$ is a resolving pair if and only if $dist(x,y)$ is odd. Consequently, the number of resolving pairs of G is equal to $|A||B|$. Since $|A| + |B| = n$ the minimum value of this product is equal to $n-1$, when $\{|A|,|B|\} = \{1, n-1\}$ and the extremal graph is $K_{1,n-1}$, and the maximum value is $\lfloor n^2/4 \rfloor$, which is attained if and only if $-1 \leq |A| - |B| \leq 1$.

□

If $x \in V(G)$, by denoting $v_i(x)$ the number of vertices $y \in V(G)$ having $dist(y,x) = i$ and supposing $x \in A$, we get $|A| = \sum_{i \equiv 0 (mod\, 2)} v_i(x)$ and $|B| = \sum_{j \equiv 1 (mod\, 2)} v_j(x)$.
It follows that the condition $-1 \leq |A| - |B| \leq 1$ is equivalent to

$$-1 \leq \sum_{i \equiv 0 (mod\, 2)} v_i(x) - \sum_{j \equiv 1 (mod\, 2)} v_j(x) \leq 1 \tag{1}$$

Note that the number of resolving pairs of a bipartite graph G is also equal to

$$\sum_{i\equiv 0(mod\,2)} v_i(x) \sum_{j\equiv 1(mod\,2)} v_j(x) = |A||B|$$

and this product does not depend on the choice of the vertex x in $V(G)$.
If $G = P_n$ or G is an even cycle C_n with $n \equiv 0(mod\,2)$ then condition (1) is obviously satisfied and these graphs have a maximum number of resolving pairs, equal to $\lfloor n^2/4 \rfloor$.
Another example of an extremal bipartite graph is the n-dimensional hypercube Q_n which has 2^n vertices representing binary n-tuples $(x_1,\ldots,x_n)$ and where two vertices are adjacent if they differ in exactly one coordinate.
Q_n has partite sets $A = \{(x_1,\ldots,x_n) : \sum_{i=1}^{n} x_i \equiv 0(mod\,2)\}$ and $B = \{(y_1,\ldots,y_n) : \sum_{i=1}^{n} y_i \equiv 1(mod\,2)\}$; one has $|A| = |B| = 2^{n-1}$.

Theorem 2.2. All graphs G of order $n \geq 5$ and diameter $n-2$ have a number of resolving pairs less than or equal to $\lfloor n^2/4 \rfloor$. Equality holds if and only if: n is odd and G consists of P_{n-1} and a new vertex x adjacent to one interior vertex a of P_{n-1} or to two vertices a,b of P_{n-1} such that $dist(a,b) = 2$ or n is even and in both cases the distance between a and an endvertex of P_{n-1} is also even.

Proof. Let G be a graph of order n and diameter $n-2$. G consists of a path P_{n-1} with endvertices u and v and a pendant vertex x such that:
$a)$ $d(x) = 1$, when x is adjacent to a vertex a of P_{n-1}, different from u and v;
$b)$ $d(x) = 2$ and x is adjacent to two vertices a,b of P_{n-1} such that $dist(a,b) \in \{1,2\}$;
$c)$ $d(x) = 3$ and x is adjacent to three consecutive vertices a,b,c of P_{n-1}.
We shall consider these cases separately.
$a)$ In this case G is a tree, hence a bipartite graph.
If $v_i = v_i(u)$ denotes the number of vertices y of G with $dist(u,y) = i$, then there exists a unique index k, $2 \leq k \leq n-2$ such that $v_k = 2$ and $v_i = 1$ for every $i \neq k$.
If n is odd, then (1) becomes

$$-1 \leq v_1 + v_3 + \ldots + v_{n-2} - (1 + v_2 + v_4 + \ldots + v_{n-3}) \leq 1 \tag{2}$$

Both sums have $(n-1)/2$ terms and (2) is satisfied for any k.
It follows that all graphs consisting of P_{n-1} and another pendant vertex adjacent to any vertex $a \neq u,v$ of P_{n-1} are extremal graphs.
If n is even, (1) can be written as

$$-1 \leq v_1 + v_3 + \ldots + v_{n-3} - (1 + v_2 + v_4 + \ldots + v_{n-2}) \leq 1 \tag{3}$$

The sum with odd indices contains $n/2-1$ terms and another sum $n/2$ terms. It follows that (3) is satisfied if and only if the index k such that $v_k = 2$ is odd. This means that $dist(u,a)$ is even.
$b)$ In this case if $dist(a,b) = 1$ consider $G_1 = G - xb$.
The resolving pairs in G consisting of vertices belonging to P_{n-1} remain resolving for G_1, but in G_1 may appear new resolving pairs of vertices on P_{n-1}. Also the number of resolving pairs $\{x,t\}$ in G where $t \in P_{n-1}$ remains unchanged or increases by one and $\{x,a\}$ and $\{a,b\}$ become resolving in G_1, which is in case $a)$.
This implies that the number of resolving pairs of G is strictly less than $\lfloor n^2/4 \rfloor$.

If $dist(a,b) = 2$ then G is bipartite and numbers v_i are the same as in case a). It follows that G is extremal if n is odd or n is even and $dist(a,u)$ is even (which implies also that $dist(b,u)$ is even).
c) In this case $G_1 = G - xb$ has four new resolving pairs relatively to G, namely $\{x,a\},\{x,c\}$, $\{a,b\},\{b,c\}$ thus implying that G is not extremal since G_1 is in case b).

□

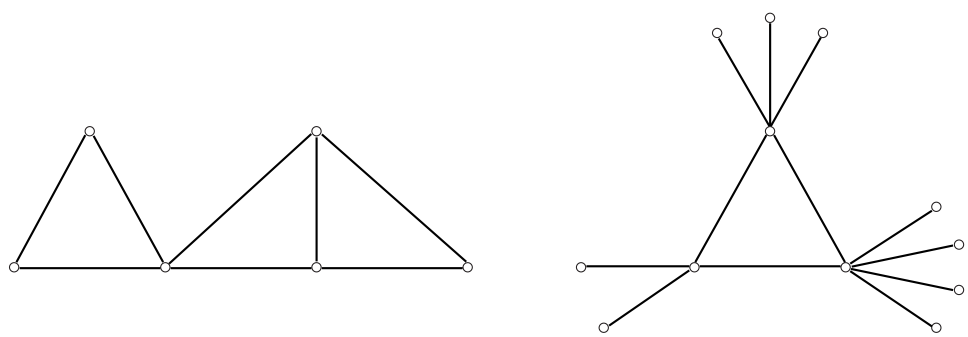
Proposition 2.1. For every integers n,k such that $n \geq 3$ and $2 \leq k \leq n-1$ there exists a connected graph G of order n and $diam(G) = k$ containing $\lfloor n^2/4 \rfloor$ resolving pairs.

Proof. For $k = 2, n-2, n-1$ we have seen that the statement is true. Let $n \geq 6$ and k be such that $3 \leq k \leq n-3$ and consider a path $P_{k+1} : u,a,b,\ldots,v$ of diameter k. We shall add $v_2 - 1$ pendant vertices adjacent to a and $v_3 - 1$ pendant vertices adjacent to b, by obtaining a caterpillar G of diameter k. Since G must have n vertices we get $v_2 + v_3 = n - k + 1$.
If k is odd (1) is equivalent to $-1 \leq v_3 - v_2 \leq 1$, which is satisfied for example by choosing $v_2 = \lceil (n-k+1)/2 \rceil$ and $v_3 = \lfloor (n-k+1)/2 \rfloor$. If k is even (1) yields $-1 \leq v_3 - v_2 - 1 \leq 1$ and we can choose $v_2 = \lceil (n-k)/2 \rceil$ and $v_3 = \lfloor (n-k/2) \rfloor + 1$.

□

3. GRAPHS OF DIAMETER THREE

If G is a graph of diameter equal to three, every resolving pair $\{x,y\}$ of G must have $dist(x,y) \in \{1,3\}$.
There exist graphs of diameter three without resolving pairs (e.g. the odd cycle C_7) or without resolving pairs at distance one or at distance three, respectively
(see graphs G_1 and G_2 from $Fig.1$).



G_1 G_2

FIGURE 1

Theorem 3.1. Let G be a connected graph of order n and diameter three containing a resolving pair $\{x,y\}$ such that:

$i)$ $dist(x,y) = 1$ and $N(x) \cup N(y) = V(G)$ or
$ii)$ $dist(x,y) = 3$ and there exists a shortest path x,u,v,y such that

$$N(x) \cup N(y) \cup N(u) \cup N(v) = V(G).$$

Then the number of resolving pairs of G is bounded above by $\lfloor n^2/4 \rfloor$ and this bound is tight.

Proof. In case $i)$ by denoting $A = N(x)$ and $B = N(y)$ we deduce that $A \cap B = \emptyset$ since $\{x,y\}$ is a resolving pair and $A \cup B = V(G)$.
Any pair of vertices from A or from B is not resolving having a common neighbor. It follows that the number of resolving pairs of G is bounded above by $|A||B| \leq \lfloor n^2/4 \rfloor$. It can be easily seen that this bound can be reached if and only if A and B are independent sets of vertices, i.e., G is bipartite, and $-1 \leq |A| - |B| \leq 1$, or $-1 \leq |N(x)| - |N(y)| \leq 1$. Since G has diameter three there exists at least a pair $\{a,b\}, a \in A$ and $b \in B$ such that $ab \notin E(G)$.
Note that this class of extremal graphs contains 4-cycles book graph $B_{4,n}$ [7] consisting of $n \geq 2$ copies of the cycle C_4 with a common edge; the copies of the cycle C_4 are called the pages of $B_{4,n}$.
$ii)$ In this case we also have $N(x) \cap N(y) = \emptyset$ and $N(u) \cap N(v) = \emptyset$; denote $A = N(x) - \{u\}$, $B = N(y) - \{v\}$, $C = N(u) - (N(x) \cup N(y) \cup \{x,v\})$, $D = N(v) - (N(x) \cup N(y) \cup \{u,y\})$ $(see\ Fig. 2)$.

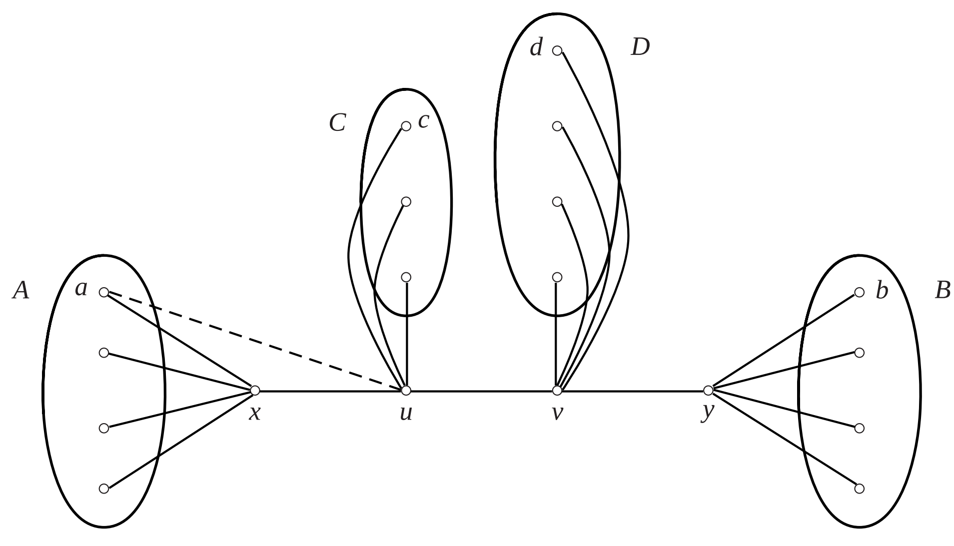


FIGURE 2

The pairs of distinct vertices from A cannot be resolving since x has equal distances to them; a similar situation occurs for pairs in B, C, D.

Suppose first that G has the following property: for any $a \in A$ we have $au \notin E(G)$ or $av \in E(G)$ and for any $b \in B$, $bv \notin E(G)$ or $bu \in E(G)$ holds.

In this case any pair $\{a,d\}$ with $a \in A$ and $d \in D$ is not resolving since $dist(u,a) = dist(u,d) = 2$ or $dist(v,a) = dist(v,d) = 1$; similarly any pair $\{b,c\}$ with $b \in B$ and $c \in C$ is not a resolving pair.

Also any pair $\{a,y\}$ with $a \in A$ is not resolving since $dist(a,u) = dist(y,u) = 2$ or $dist(a,v) = dist(y,v) = 1$ and in a similar way any pair $\{x,b\}$ is not resolving.

Another pairs which are not resolving are $\{x,c\}$ and $\{y,d\}$, where $c \in C$ and $d \in D$.
It follows that at most the following pairs of vertices can be resolving: $\{a,b\}$, $a \in A$; $b \in B$; $\{c,d\}$, $c \in C$; $d \in D$; $\{u,c\}$,$c \in C$; $\{v,d\}$,$d \in D$; $\{a,c\}$, $a \in A$; $c \in C$; $\{b,d\}$, $b \in B$; $d \in D$; $\{u,b\}$, $b \in B$; $\{v,a\}$, $a \in A$; $\{y,b\}$, $b \in B$; $\{x,a\}$, $a \in A$; $\{x,y\}$, $\{x,u\}$, $\{u,v\}$, $\{v,y\}$. By denoting $|A| = \alpha$, $|B| = \beta$, $|C| = p$ and $|D| = q$, the number of these possible resolving pairs is equal to

$$E = \alpha\beta + pq + \alpha p + \beta q + 2(\alpha + \beta + p + q) + 4 = \alpha\beta + pq + \alpha p + \beta q + 2n - 4$$

since $\alpha + \beta + p + q = n - 4$.
Substitution $\alpha = n - 4 - \beta - p - q$ yields

$$E = (p + \beta)(n - 4 - p - \beta) + 2n - 4 \leq \lfloor (n-4)^2/4 \rfloor + 2n - 4 = \lfloor n^2/4 \rfloor.$$

Suppose now that there exist subsets of vertices $A_1 \subseteq A$ and $B_1 \subseteq B$, $|A_1| = s$, $|B_1| = t$, $0 \leq s \leq \alpha$, $0 \leq t \leq \beta$, $s + t \geq 1$ such that every vertex $a \in A_1$ and every vertex $b \in B_1$ verifies $au \in E(G)$ and $av \notin E(G)$ and $bv \in E(G)$ and $bu \notin E(G)$, respectively. We will prove that the number of resolving pairs in this case is strictly less than $\lfloor n^2/4 \rfloor$.
It follows that the following modifications have been produced relatively to the case when $s = t = 0$:
All pairs $\{a,c\}$ with $a \in A_1$ and $c \in C$, $\{a,b\}$ with $a \in A_1$ and $b \in B - B_1$, $\{a,v\}$ with $a \in A_1$, $\{a,x\}$ with $a \in A_1$ and the pair $\{x,u\}$ if $s > 0$ are not resolving (they are counted as resolving ones in expression E).
All pairs $\{a,d\}$ with $a \in A_1$ and $d \in D$ and $\{a,y\}$ with $a \in A_1$ may become resolving.

A similar situation holds for the pairs containing vertices $b \in B_1$. We get that the number of resolving pairs is at most equal to

$$E_1 = E + sq + tp - s(p + \beta - t) - t(q + \alpha - s) - s - t - 1 =$$
$$(p + \beta)(n - 4 - p - \beta) + 2n - 5 + sq - s(p + \beta) + tp - t(q + \alpha) + 2st - s - t.$$

By denoting $p + \beta = k$ we deduce $q = n - 4 - k - \alpha \leq n - 4 - k - s$; $p = k - \beta \leq k - t$, which implies

$$E_1 \leq k(n - 4 - k) + 2n - 5 + s(n - 4 - k - s) - ks + kt - t^2 - t(n - 4 - k) + 2st - s - t =$$
$$(k + s)(n - 4 - k - s) + 2n - 5 + \varphi(t) - s,$$

where $\varphi(t) = -t^2 - t(n - 4 - 2k - 2s + 1)$.
Suppose that n is even and denote $\gamma = (n-4)/2 - (k+s)$. We get $(k+s)(n-4-k-s) = (n-4)^2/4 - \gamma^2$ and $\varphi(t) = -t^2 - t(2\gamma + 1)$. If $2\gamma + 1 \geq 0$ then $\varphi(t) \leq 0$ and $E_1 \leq \lfloor (n-4)^2/4 \rfloor + 2n - 5 < \lfloor n^2/4 \rfloor$. Otherwise $\gamma < -\frac{1}{2}$. The maximum value of $\varphi(t)$ is $\varphi(-\gamma - \frac{1}{2}) = \gamma^2 + \gamma - \frac{1}{4}$, which implies

$$E \leq (n-4)^2/4 + 2n - 5 + \gamma - \tfrac{1}{4} - s < \lfloor n^2/4 \rfloor.$$

A similar situation occurs for n odd by denoting $\gamma = (n-5)/2 - (k+s)$, when $(k+s)(n-4-k-s) = \lfloor (n-4)^2/4 \rfloor - \gamma^2 - \gamma$.

To see that this bound can also be reached in this case it is sufficient to consider $C = D = \emptyset$ $(p = q = 0), -1 \leq |A| - |B| \leq 1$ and any vertex $a \in A$ is adjacent to v or to a vertex $b \in B$ and any vertex $b \in B$ is adjacent to u or to a vertex $a \in A$. All these graphs are bipartite and by Theorem 2.1 the number of resolving pairs equals $\lfloor n^2/4 \rfloor$ since partite sets have $|A| + 2$ and $|B| + 2$ vertices, respectively.

□

4. **Conclusions**

All extremal graphs found in this paper are bipartite. Thus, the following conjecture seems to be plausible:

All non-bipartite graphs of order n have a number of resolving pairs less than $\lfloor n^2/4 \rfloor$. The most striking example is the odd cycle which has no resolving pair.

REFERENCES

[1] *P. S. Buczkowski, G. Chartrand, C. Poisson, P. Zhang*, On k-dimensional graphs and their bases, Periodica Math. Hung., **46(1)** (2003), 9-15.

[2] *J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood*, On the metric dimension of Cartesian products of graphs, SIAM J. Discrete Math., **21** (2007), 423-441.

[3] *G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann* , Resolvability in graphs and metric dimension of a graph, Discrete Appl. Math., **105** (2000), 99-113.

[4] *J. D. Currie, O. R. Oellermann, The metric dimension and metric independence of a graph*, J. Combin. Math. Comput., **39** (2001), 157-167.

[5] *F. Harary, R. A. Melter*, On the metric dimension of a graph, Ars Combin., **2** (1976), 191-195.

[6] *C. Hernando, M. Mora, I. M. Pelayo, C. Seara, D. R. Wood*, Extremal graph theory for metric dimension and diameter, Electron. Notes Discrete Math., **29** (2009), 339-343.

[7] *A. K. Lal, B. Bhattacharjya*, Breaking the symmetries of the book graph and the generalized Petersen graph, SIAM J. Discrete Math.,**23**, 3(2009), 1200-1216.

[8] *P. J. Slater*, Leaves of trees, Proceeding of 6th Southeastern Conference on Combinatorics, Graph Theory and Computing, Congr. Numer., **14** (1975), 549-559.

[9] *P. J. Slater*, Dominating and reference sets in graphs, J. Math. Phys. Sci., **22** (1988), 445-455.

[10] *I. Tomescu, M. Imran*, Metric dimension and R-sets of connected graphs, Graphs and Combinatorics, **27** (2011), 585-591.