U.P.B. Sci. Bull., Series C, Vol. 73, Iss. 2, 2011 ISSN 1454-234x

FORMULAS FOR THE DISCRETE-TIME (J, J')-SPECTRAL FACTORIZATION OF A GENERAL SYSTEM

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În acest articol folosim realizări de stare descriptor pentru a obține formule constructive ale soluției factorizării (J,J')-spectrale formulate pentru o matrice rațională generală (rang arbitrar, poli şi zerouri pe cercul unitate sau la infinit). Pentru calculul factorilor am utilizat un algoritm numeric stabil bazat pe o proiecție preliminară unitară ce pune în evidență un subsistem, care îndeplineşte toate ipotezele de regularitate şi pentru care dăm soluția factorizării.

We use descriptor state–space realizations to obtain constructive formulas for the solutions of the (J, J')-spectral factorization formulated for a completely general rational matrix function (arbitrary rank, poles and zeros on the unit circle, or at infinity). For the computation of the factors we use a numerically–reliable algorithm based on a preliminary unitary projection which reveals a subsystem fulfilling all regularity assumptions and for which we actually solve the factorizations.

Keywords: discrete-time systems, spectral factorization, Riccati equation

1. Introduction

A rational matrix function with complex coefficients $\Theta(z)$ is called (J, J')–unitary if $\Theta(z)^{\#} J\Theta(z) = J'$ (where $\Theta(z)^{\#} := \Theta(\frac{1}{z})^{*}$), at every point on the unit circle at which Θ is analytic, where J and J' are two signature matrices, i.e., $J = J^{-1} = J^{*}$ (* denotes conjugate transpose). By analytic continuation, $\Theta(z)^{\#} J\Theta(z) = J'$, $\forall z \in \mathbb{C}$. If, in addition, $\Theta(z)^{*} J\Theta(z) \le J'$ for every point of analyticity of Θ in the exterior of the closed unit disk, then Θ is called (J, J')–lossless. If $J = J'$, $\Theta(z)$ is called J–unitary, and J–lossless, respectively. The normal rank of a rational matrix function $G(z)$ is its rank for almost all $z \in \mathbb{C}$.

We consider here the extension of the (J, J')-spectral factorization problem such as to become applicable to a general $p \times m$ rational matrix function with complex coefficients $G(z)$ (of arbitrary rank, with poles and zeros on the

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unit circle, possibly polynomial or improper): find a rational matrix function $\Pi(z)$ which has full row normal rank and only marginally stable zeros such that

$$G^{\#}(z)JG(z)=\Pi^{\#}(z)J'\Pi(z), \tag{1}$$

where $G(z)\Pi^{(+)}(z)$ has no poles on the unit circle. Here $\Pi^{(+)}(z)$ stands for the Moore–Penrose pseudo inverse of $\Pi(z)$.

This factorization plays an important role in optimal Hankel-norm model reduction and H^{∞} optimization, transport theory, and stochastic filtering (see for example [1,2,3,4,5]).

The spectral factorization has been investigated in several papers for a rational matrix function with full column rank and without poles/zeros on the unit circle or at infinity (see [6,7]) and the solution for a general rational matrix function was given in [8]. Considerable research has focused on the J-spectral factorization in continuous-time under various hypotheses [9,10,11,12,13] and the problem has been solved in the most general conditions in [14]. For discrete-time systems, a state-space algorithm for the J-spectral factorization has been developed for the optimal Hankel norm model reduction problem [15], where the problem is reduced to a Wiener-Hopf type spectral factorization, which is then solved by the geometric factorization principle [16]. A discrete-time version closer to our approach is [17], formulated for a stable rational matrix function. In [17], the (J,J')-spectral factorization extends the technique in [18], in which a spectral factorization algorithm associated with the discrete-time descriptor LQ regulator problem is derived.

The paper is organized as follows. In Section 2 we review briefly a couple of definitions and notations related to matrix pencils, rational matrix functions and descriptor state–space realizations of rational matrices. In Section 3 give a spectral decomposition of the system pencil associated with a descriptor realization which will be the mail tool used in the next section. Section 4 contains the main result. In Section 5 we give a numerical example for the (J,J')–spectral factorization. We draw some conclusions in Section 6. For the fluidity of the presentation, the more technical details involved and the similarities to the continuous-time case we choose to skip the proofs.

2. Preliminaries

We denote the open unit disk and the unit circle by D and $D_1(0)$, respectively, and by $D_c=\overline{\mathbb{C}}\setminus\overline{D}$ the exterior of the closed unit disk containing the infinity (the “overbar" denotes closure).

If a matrix A in $\mathbb{C}^{m\times n}$ is invertible, A^{-*} is its conjugate transpose inverse. A Hermitian matrix A satisfies $A=A^{*}$, and we denote by $A>0$ if it is in addition positive definite. A is unitary (J–unitary) if $A^*A = I$ ($A^*JA = J$). I_n is the identity matrix of size $n \times n$, and we skip the dimensions whenever they are irrelevant. By $\star$ we denote irrelevant matrix entries.

Let $A, E \in \mathbb{C}^{m\times n}$. The matrix polynomial $A - zE$ is called a *matrix pencil* (or *pencil*). The pencil is called *regular* if it is square ($m = n$) and has a non–vanishing determinant, i.e., $\det(A - zE) \not\equiv 0$. A *singular* pencil is a pencil which is not regular. The *normal rank* of the pencil – denoted $\operatorname{rank}_n(A - zE)$ – is defined as the rank of $A - zE$ for almost all $z \in \mathbb{C}$ (but a finite number of points). For an $n \times n$ regular pencil $A - zE$, the normal rank r is equal to n. If $\nu_\ell := m - r > 0$ then we say the pencil has a (nontrivial) *left singular structure*. If $\nu_r := n - r > 0$ then the pencil has a (nontrivial) *right singular structure*.

Two matrix pencils $A - zE$ and $\widetilde{A} - z\widetilde{E}$, with $A, E, \widetilde{A}, \widetilde{E} \in \mathbb{C}^{m\times n}$, are called *strictly equivalent* if there are two constant invertible matrices $Q \in \mathbb{C}^{m\times m}$, $Z \in \mathbb{C}^{n\times n}$, such that

$$Q(A - zE)Z = \widetilde{A} - z\widetilde{E}. \tag{2}$$

The equivalence relation (2) induces a canonical form (see [19]) – called the *Kronecker canonical form* – on the set of $m \times n$ pencils

$$A_{KR} - zE_{KR} = Q(A - zE)Z = \begin{bmatrix} L_{\varepsilon_1} & & & & & & & \\ & \ddots & & & & & & \\ & & L_{\varepsilon_{\nu_r}} & & & & & \\ & & & I_{n_\infty} - zE_\infty & & & & \\ & & & & A_f - zI_{n_f} & & & \\ & & & & & L^T_{\eta_1} & & \\ & & & & & & \ddots & \\ & & & & & & & L^T_{\eta_{\nu_\ell}} \end{bmatrix} \tag{3}$$

Here L_k ($k \geq 0$) denotes the bidiagonal $k \times (k+1)$ pencil

$$L_k = \begin{bmatrix} z & -1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & z & -1 \end{bmatrix},$$

and A_f and E_∞ are two square matrices in the Jordan canonical form, with E_∞ nilpotent. The finite eigenstructure of $A - zE$ is determined by the eigenvalues of

A_f, and the dimensions of the elementary infinite blocks of $I_{n_\infty} - zE_\infty$ determine the infinite eigenstructure of the pencil. The union of the finite and infinite eigenstructure of the pencil completely determines the regular part of the pencil and forms the spectrum of the pencil - $\Lambda(A - zE)$. The singular part of the pencil is defined by the right and left singular Kronecker structure: the $\varepsilon_i \times (\varepsilon_i + 1)$ blocks $L_{\varepsilon_i}, (i = 1,...,\nu_r)$, are the right elementary Kronecker blocks, and $\varepsilon_i \geq 0$ are called the right Kronecker indices; the $(\eta_j + 1) \times \eta_j$ blocks $L_{\eta_j}^T, (j = 1,...,\nu_\ell)$, are the left elementary Kronecker blocks, and $\eta_j \geq 0$ are called the left Kronecker indices. For more details on matrix pencils see Chapter 12 in [19].

For any rational matrix function $G(z)$ with coefficients in $\mathbb{C}^{p\times m}$ one can write a descriptor realization of the form (see for example [20,21])

$$G(z) = D + C(zE - A)^{-1}B =: (E, A, B, C, D), \tag{4}$$

where $A, E \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{p\times m}$, and $A - zE$ is a *regular* pencil. The dimension n of the square matrices A and E is called the *order of the realization* (4). With any realization (4) we associate two matrix pencils that play an important role in our developments: the *pole pencil* $P(z) = A - zE$ and the *system pencil* $S(z) = \begin{bmatrix} A - zE & B \\ C & D \end{bmatrix}$. The realization (4) of $G(z)$ is called *irreducible* if it satisfies the following conditions (see [21]):

$$\begin{array}{llcl}
(i) & \operatorname{rank}\begin{bmatrix} A - zE & B \end{bmatrix} & = & n, \quad \forall z \in \mathbb{C}, \\
(ii) & \operatorname{rank}\begin{bmatrix} E & B \end{bmatrix} & = & n, \\
(iii) & \operatorname{rank}\begin{bmatrix} A - zE \\ C \end{bmatrix} & = & n, \quad \forall z \in \mathbb{C}, \\
(iv) & \operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} & = & n.
\end{array} \tag{5}$$

The conditions (5) are usually known as finite and infinite controllability, and finite and infinite observability, respectively. In contrast to standard realizations, irreducibility of a descriptor realization does not automatically imply its minimality since some simple blocks of dimension 1 at infinity (so called non–dynamic modes) which are both controllable and observable might increase indefinitely the dimension of the realization while keeping its irreducibility. Starting from an arbitrary realization (4), one can compute an irreducible realization by using solely unitary transformations.

The following result taken from [22] (see Theorem 2.4) is modified to cope with *proper* (J, J')–unitary rational matrices having a descriptor realization.

Lemma 3. Let $G_c(z)$ be a proper rational matrix, having a minimal realization

$$G_c(z) := (E_x, A_x, B_x, C_x, D_x). \quad (6)$$

$G_c(z)$ is (J, J')–lossless on $D_1(0)$ if and only if there is a positive hermitian matrix X such that

$$A_x^* X A_x - E_x^* X E_x + C_x^* J C_x = 0, \quad (7)$$

$$D_x^* J C_x + B_x^* X A_x = 0, \quad (8)$$

and

$$D_x^* J D_x + B_x^* X_s B_x = J'. \quad (9)$$

3. Spectral decomposition

In this section we introduce a spectral decomposition of the system pencil with respect to the partition $\overline{\mathbb{C}} = \overline{D} \cup D_c$. The decomposition can be achieved by unitary transformations and will play a capital role in expressing our main results in the next section (for more details see [8]).

Theorem 1. Let $G(z)$ be a $p \times m$ real rational matrix given by a controllable realization (4), i.e., fulfilling (i) and (ii) in (5). Then there exist two constant unitary matrices U and Z such that

$$\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - zE & B \\ C & D \end{bmatrix} Z = \begin{bmatrix} A_{rg} - zE_{rg} & \star & \star & \star \\ 0 & A_{b\ell} - zE_{b\ell} & B_{b\ell} & B_{\ell n} - zE_{\ell n} \\ 0 & 0 & 0 & B_n \\ 0 & C_{b\ell} & D_{b\ell} & D_n \end{bmatrix} \quad (10)$$

where:

I. The pencil $A_{rg} - zE_{rg}$ has full row rank in D_c and E_{rg} has full row rank.

II. $E_{b\ell}$ and B_n are invertible, the pencil

$$\begin{bmatrix} A_{b\ell} - zE_{b\ell} & B_{b\ell} \\ C_{b\ell} & D_{b\ell} \end{bmatrix} \quad (11)$$

has full column rank in $\overline{D}$, the pencil $A_{b\ell} - zE_{b\ell}$ is regular, and the pair $\begin{bmatrix} A_{b\ell} - zE_{b\ell} & B_{b\ell} \end{bmatrix}$ is controllable.

The above theorem constructs a projection of the original system (4)

$$G_p(z) = (E_{b\ell}, A_{b\ell}, B_{b\ell}, C_{b\ell}, D_{b\ell}), \quad (12)$$

which fulfills all standard assumptions in the literature (is proper, has full column normal rank, and has no zeros on the unit circle). In the next section we will see that it is enough to solve the factorization problems for $G_p(z)$ to get the solution to the corresponding factorization for $G(z)$.

4. Main result

Once the spectral decomposition obtained, we have the coefficients of a Riccati equation, whose solution (when it exists) can be used directly to write the state-space realization of the factor $\Pi(z)$, solving the factorization problem under investigation. The following theorem gives a constructive approach for the spectral factor for a general rational matrix function. Due to the very technical details involved in the proof and to the similarities to the continuous-time case we have chosen to skip it here (for more details see [8,14]).

Theorem 2. Let $G(z)$ be a general rational matrix function given by a controllable realization (4), and let U and Z be two constant unitary matrices such that (10) holds.

I. The (J,J')–spectral factorization problem (1) has a solution if and only if the following conditions are fulfilled:
   1. The Riccati equation
$$A_{b\ell}^* X A_{b\ell} - E_{b\ell}^* X E_{b\ell} - (A_{b\ell}^* X B_{b\ell} + C_{b\ell}^* J D_{b\ell})$$
$$\times (D_{b\ell}^* J D_{b\ell} + B_{b\ell}^* X B_{b\ell})^{-1} (B_{b\ell}^* X A_{b\ell} + D_{b\ell}^* J C_{b\ell}) + C_{b\ell}^* J C_{b\ell} = 0 \qquad (13)$$
   has an invertible stabilizing solution X_s, i.e.,
$$\Lambda(A_{b\ell} + B_{b\ell} F - z E_{b\ell}) \subset \overline{D},$$
$$F := -(D_{b\ell}^* J D_{b\ell} + B_{b\ell} X_s B_{b\ell})^{-1} (B_{b\ell}^* X_s A_{b\ell} + D_{b\ell}^* J C_{b\ell}).$$
   2.
$$D_{b\ell}^* J D_{b\ell} + B_{b\ell}^* X_s B_{b\ell} = V^* J' V \qquad (14)$$
   for an appropriate constant invertible matrix V;

II. $\Pi(z)$ is given by
$$\Pi(z) := (E, A, B, \widetilde{C}, \widetilde{D}), \begin{bmatrix} \widetilde{C} & \widetilde{D} \end{bmatrix} := V \begin{bmatrix} 0 & -F & I & 0 \end{bmatrix} Z^*, \qquad (15)$$

Remark 1. In particular, the above result may be applied to a polynomial matrix $G(z)$ and provides a numerically sound state-space construction of the spectral factor. Moreover, if $G(z)$ fulfills the usual regularity assumptions (is proper, full column rank, no zeros on the unit circle), Theorem 2 recovers the well-known result in the regular case.

Remark 2. Besides solving a Riccati equation of indefinite sign instead of a positive one, the formulas for the factors resemble in detail the ones in the unitary case [8]. The main difference with respect to the continuous-time case is that we have to handle some poles and zeroes at infinity which are no longer on the boundary of the stability domain. Shortly, if $G(z)$ has a (J,J')-lossless factorization, then there is an invertible factor $R_l(z)$ which cancels in the product $R_l(z)G(z)$ all left minimal indices and the unstable zeros of $G(z)$ (see [14]), and $R_l(z)$ is in particular J–lossless. Such an invertible factor may have poles at infinity because he has to dislocate possible zeros at infinity of $G(z)$ and we cannot always write a proper realization for $R_l(z)$, but we can always write one for the inverse, $R^{-1}{}_l(z)$.

Remark 3. The existence of the stabilizing solution to the Riccati equation (13) can be checked and the equation solved by using any existing numerical algorithm that copes with indefinite sign matrix coefficients (see for example [3] and the references therein).

5. Numerical example

We exemplify the proposed approach on a simple but relevant system. For illustrative simplicity we use non–unitary transformations as well. Let

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; J' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$G(z) = \begin{bmatrix} \dfrac{1}{z-1} & 0 & -\dfrac{1}{z-1} \\ \dfrac{z}{(z+1)^2} & \dfrac{z-2}{(z+1)^2} & -\dfrac{2}{(z+1)^2} \\ -3z & z-2 & 4z-2 \end{bmatrix}.$$

$G(z)$ has a realization (6) given by

$$A - zE = \begin{bmatrix} 1-z & 0 & 0 & 0 & 0 \\ 0 & -z & -1 & 0 & 0 \\ 0 & 1 & -2-z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -z & 1 \end{bmatrix}, \; B = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -1 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C=\begin{bmatrix}2&0&0&0&0\\0&0&2&0&0\\0&0&0&0&1\end{bmatrix},\ D=\begin{bmatrix}0&0&0\\0&0&0\\0&-2&-2\end{bmatrix}$$

The structural elements of $G(z)$ are: one pole at 1 with multiplicity 1, one pole at -1 with multiplicity 2, one pole at ∞ with multiplicity 1, a zero at 2 with multiplicity 1, a zero at ∞ with multiplicity 1, one left minimal index equal to 2, one right minimal index equal to 0 and normal rank $r=2$.

With

$$U=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\end{bmatrix},Z=\begin{bmatrix}0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&1\\0&0&0&0&2&1&6&0\\-1&0&0&0&1&0&4&0\\1&0&0&0&0&0&0&0\\-1&0&0&0&1&0&3&0\end{bmatrix},$$

we get the decomposition (10) in the form

$$\begin{bmatrix}A_{rg}-zE_{rg}&B_1-zF_1&B_2-zF_2&B_3-zF_3\\0&A_{b\ell}-zE_{b\ell}&B_{b\ell}&B_{\ell n}-zF_{\ell n}\\0&0&0&B_n\\0&C_{b\ell}&D_{b\ell}&D_n\end{bmatrix}=$$

$$\begin{bmatrix}0&1-z&0&0&0&0&\frac{1}{2}&0\\0&0&-z&-1&-1&0&-3&0\\0&0&1&-2-z&\frac{1}{2}&0&2&0\\0&0&0&0&2-z&1&6&-z\\0&0&0&0&0&0&0&1\\0&2&0&0&0&0&0&0\\0&0&0&2&0&0&0&0\\0&0&0&0&0&1&0&0\end{bmatrix}$$

The Riccati equation (13) has a stabilizing positive solution.

$$X_s = \begin{bmatrix} 38.67938045306 & 6.100834940383 & -12.20892601197 & -6.796702565575 \\ 6.100834940383 & 2.324875571834 & -4.966484806355 & -1.404285085369 \\ -12.20892601197 & -4.966484806355 & 14.77157486463 & 2.862568469721 \\ -6.796702565575 & -1.404285085369 & 2.862568469721 & 1.366831275157 \end{bmatrix}$$

With

$$V = \begin{bmatrix} 0.6759496827692 & 17.78498299827 \\ 0.3001278035743 & -9.059057226919 \end{bmatrix},$$

which fulfills (14) we get

$$\Pi(z) =$$

$$\left[\begin{matrix} \dfrac{-2.027849048307z^4-3.69585650764z^3-6.035875617347z^2-2.345039123688z}{z^3+z^2-z-1} \\ \dfrac{-0.9003834107230z^4+1.897017979387z^3+7.814678265610z^2+4.714230175981z}{z^3+z^2-z-1} \end{matrix}\right.$$

$$\begin{matrix} \dfrac{0.6759496827692z^3+6.484330152537z^2+4.354953894882z+1.580816976594}{z^2+2z+1} \\ \dfrac{0.3001278035743z^3-3.952152872343z^2-6.901212718288z-3.103502091648}{z^2+2z+1} \end{matrix}$$

$$\left.\begin{matrix} \dfrac{2.703798731076z^4+9.504236977415z^3+3.906499359692z^2-0.4290977945998z-1.580816976594}{z^3+z^2-z-1} \\ \dfrac{1.20051121429z^4-6.149298655304z^3-10.76373811155z^2-0.9165195493417z+3.10350209164}{z^3+z^2-z-1} \end{matrix}\right].$$

Indeed $\Pi(z)$ fulfils (1), has full row normal rank, the same poles as $G(z)$, zeros at $0, \frac{1}{2}, 0.145898033750294$ and 0.2087121525220904, each with multiplicity 1, and one right minimal index equal to 0.

6. Conclusions

We have solved an essential factorization problem formulated for a completely general rational matrix function in discrete-time: (J,J')–spectral factorization. We have provided both closed formulas and a numerically stable algorithm. Our approach can be viewed as a *divide et impera* procedure as we isolate from the original system that subsystem which is really needed and for which we can actually solve the problems. The results put ground to the solution of the general discrete-time singular H^∞-control problem without any of the regularity assumptions.

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