U.P.B. Sci. Bull., Series A, Vol. 82, Iss. 1, 2020 ISSN 1223-7027

A FAMILY OF THREE-STAGE STOCHASTIC RUNGE-KUTTA METHODS WITH ORDER TWO AND THEIR STABILITY

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In this paper, a general form of three-stage Runge-Kutta methods is introduced for numerical solution of stochastic differential systems. The conditions that must be satisfied to methods have weak order two are obtained by comparison between the weak second-order expansion of the methods and the simplified weak order two Taylor scheme. Moreover, some particular solutions of the order conditions are given and then corresponding stochastic Runge-Kutta methods of this family are presented. The Mean-Square stability of the proposed class of methods is considered, the stability function is obtained, and the Region of MS-stability is given. Finally, the obtained methods are compared numerically in some examples.

Keywords: stochastic differential equation, Runge-Kutta method, mean square stability, weak convergence.

MSC2010: 65U05, 60H10.

1. Introduction

Stochastic differential equations (SDEs) can describe more realistic models and include stochastic effects in the model by inserting the noise term in ordinary differential equations (ODEs). There is an increasing interest in using SDEs and they have been applied to a wide range of problems, because of recent progresses in the stochastic analysis and availability of sufficiently powerful computers[2, 5, 10–12]. The exact solution of SDEs is not often available and hence, suitable numerical methods must also be introduced for solving arisen SDEs. Stochastic Runge-Kutta (SRK) methods as generalization of their analogous for ODEs are efficient instruments and has been studied by many authors[1, 3, 4, 8, 13]. The organization of this work is as follows: after some brief preliminaries, in the next section a general form of three-stage stochastic Runge-Kutta methods is presented and conditions under which this methods have weak order two, are extracted. In section 3 some stochastic Runge-Kutta methods are introduced whose parameter values are particular solutions of the obtained conditions in section 2. In section 3 the MS-stability

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of this class is studied. Section 4 is devoted to implementation issues and numerical comparisons via some examples. Finally, the results are summarized as conclusions.

2. Stochastic Runge-Kutta methods

A scalar Ito SDE is of the following form:

$$dx_t = f(t, x_t)dt + g(t, x_t)dW_t, \quad x_{t_0} = x_0 \tag{1}$$

where $f, g : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ are the drift and diffusion coefficients respectively, $\{W_t\}_{t_0 \leq t \leq T}$ represents a one-dimensional standard Wiener process and the initial value $x_0 \in \mathbb{R}$ is non-random. Moreover, suppose that coefficients f and g satisfy the conditions that ensure existence and uniqueness of solutions to (1)(see [11]). The numerical methods for solving SDEs may be strong or weak, with constant or variable step sizes, and various order of convergence. We concentrate on methods that converge in the weak sense and with order two in which the step size are assumed constant.

A numerical solution $\{x_n\}_{n=0}^{n=N}$ with step size h as an approximation of solution $\{x(t)\}_{t=0}^{t=T}$ of (1) is of weak convergence order γ if for each smooth function F, there exist a constant $K > 0$ such that [7]

$$\Big| E[F(x_N)] - E[F(x(T))] \Big| \leq Kh^{\gamma}.$$

Tocino and Vigo-Aguiar[15, 17] presented some stochastic Runge-Kutta methods of weak order two by comparing the stochastic expansion of the approximation of the method with corresponding Taylor scheme. In this work, we generalize the methods were given in [17] and we propose three-stage Runge-Kutta methods of the following form:

$$x_{n+1} = x_n + (\alpha_1 K_0 + \alpha_2 K_1 + \alpha_3 K_2)h + \Upsilon_1 S_0 + \Upsilon_2 S_1 + \Upsilon_3 S_2, \tag{2}$$

with

$$\begin{aligned}
K_0 &= f(t_n, X_n),\\
S_0 &= g(t_n, X_n),\\
K_1 &= f(t_n + \mu_0 h, X_n + \lambda_0 K_0 h + S_0 \Theta_1),\\
S_1 &= g(t_n + \bar{\mu}_0 h, X_n + \bar{\lambda}_0 K_0 h + S_0 \Phi_1),\\
K_2 &= f(t_n + \rho_0 h, X_n + \varphi_0 K_0 h + S_0 \Theta_2),\\
S_2 &= g(t_n + \bar{\rho}_0 h, X_n + \bar{\varphi}_0 K_0 h + S_0 \Phi_2),
\end{aligned} \tag{3}$$

where $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Theta_1, \Theta_2, \Phi_1$ and Φ_2 are random variables of mean-square order $\frac{1}{2}$. When $\bar{\rho}_0 = \bar{\mu}_0$, $\bar{\varphi}_0 = \bar{\lambda}_0$ and $\alpha_3 = 0$ we obtain the same form that was presented in [17]. We seek values for the constants and conditions on the random variables such that the scheme be of order two in the weak sense. To do this, as the procedure in deterministic case for construct Runge-Kutta methods [9], we must obtain the stochastic expansion of the method (2)-(3) and corresponding Taylor scheme. Then a comparison between them determines the method such that the method be of weak order two. The simplified weak order two Taylor scheme can be obtained from the order two Taylor approximation and in scalar case it is given by[15]

$$\begin{aligned} x_{n+1} &= x_n + g\Delta W_n + fh + \frac{1}{2}gg_{01}((\Delta W_n)^2 - h) \\ &+ \frac{1}{2}(g_{10} + fg_{01} + \frac{1}{2}g^2 g_{02} + gf_{01})h\Delta W_n + \frac{1}{2}(f_{10} + ff_{01} + \frac{1}{2}g^2 f_{02})h^2, \end{aligned} \tag{4}$$

where $g = g_{00} = g(t_n, x_n)$ and $g_{ij} = \frac{\partial^{i+j} g}{\partial t^i \partial x^j}$ for function $g = g(t, x)$ with $t, x \in \mathbb{R}$ and ΔW_n is any normal random variable $\mathcal{N}(0, h)$.

Now to compare the method and the simplified Taylor scheme, we need a order two truncated expansion of the method. An expression of the order two truncated expansion of process $F(t + h, x(t) + \Delta x)$ in terms of increments h and Δx can be found in [16] and it is of the following form:

$$\begin{aligned} F(t + h, x(t) + \Delta x) \overset{(2)}{\simeq} & F_{00} + F_{10}h + F_{01}\Delta x \\ & + (F_{20} + g^2 F_{12} + g^3 g_{01} F_{03} + \frac{g^4}{4} F_{04})\frac{h^2}{2} \\ & + (F_{11} + \frac{g^2}{2} F_{03})h(\Delta x) + F_{02}\frac{(\Delta x)^2}{2}. \end{aligned} \tag{5}$$

To obtain the expansion of the method we need also to employ 2-equivalent processes. Two processes $\{y_t\}$ and $\{z_t\}$ are said 2-equivalent in the weak sense and are denoted by $y_t \overset{(2)}{\simeq} z_t$, if they have same weak order two Itô-Taylor expansion at each point.

Consider the SDE $dx(t) = dW(t)$ with $t(0) = 0$. Then for process $F(t, W(t)) = W^3(t)$ we have

$$\begin{aligned} & F(t + h, W(t) + \Delta W)\Big|_{t=0} = (\Delta W)^3 \\ & \overset{(2)}{\simeq} \Big(W^3(t) + (0)h + 3W^2(t)\Delta W + (0)\frac{h^2}{2} + \frac{1}{2}(6h\Delta W) + 6W(t)\frac{(\Delta W)^2}{2}\Big)\Big|_{t=0} \\ & = 3h\Delta W, \end{aligned}$$

which state

$$(\Delta W)^3 \overset{(2)}{\simeq} 3h\Delta W \tag{6}$$

Also, for process $F(t, W(t)) = tW^2(t)$ we have

$$\begin{aligned} & F(t + h, W(t) + \Delta W)\Big|_{t=0} = h(\Delta W)^2 \\ & \overset{(2)}{\simeq} \Big(tW^2(t) + hW^2(t) + 2tW(t)\Delta W + h^2 + 2hW(t)\Delta W + t(\Delta W)^2\Big)\Big|_{t=0} \\ & = h^2, \end{aligned}$$

and therefore

$$h(\Delta W)^2 \overset{(2)}{\simeq} h^2 \tag{7}$$

As a same way the following 2-equivalents can be obtained

$$\frac{(\Delta W)^4}{h} \overset{(2)}{\simeq} 6(\Delta W)^2 - 3h, \tag{8}$$

$$\frac{(\Delta W)^4}{\sqrt{h}} \overset{(2)}{\simeq} 6\sqrt{h}(\Delta W)^2 - 3h^{\frac{3}{2}}, \tag{9}$$

$$\frac{(\Delta W)^5}{h} \overset{(2)}{\simeq} 15h\Delta W, \tag{10}$$

$$\frac{(\Delta W)^6}{h^{\frac{3}{2}}} \overset{(2)}{\simeq} 45\sqrt{h}(\Delta W)^2 - 30h^{\frac{3}{2}}. \tag{11}$$

By using truncated expansion (5) and 2-equivalence $h^3 \overset{(2)}{\simeq} 0$, from the deterministic part of (2) we obtain

$$\begin{aligned}
(\alpha_1 K_0 + \alpha_2 K_1 + \alpha_3 K_2)h &= \alpha_1 f(t_n, x_n)h + \alpha_2 f(t_n + \mu_0 h, x_n + \lambda_0 K_0 h + S_0\Theta_1)h \\
&\quad + \alpha_3 f(t_n + \rho_0 h, x_n + \varphi_0 K_0 h + S_0\Theta_2)h \\
&\overset{(2)}{\simeq} (\alpha_1 + \alpha_2 + \alpha_3) fh + (\alpha_2\mu_0 + \alpha_3\rho_0) f_{10} h^2 + (\alpha_2\lambda_0 + \alpha_3\varphi_0) f_{01} f h^2 \\
&\quad + \alpha_2 f_{01} g\Theta_1 h + \alpha_3 f_{01} g\Theta_2 h + \frac{1}{2}\alpha_2 f_{02} g^2\Theta_1^2 h + \frac{1}{2}\alpha_3 f_{02} g^2 \Theta_2^2 h.
\end{aligned} \tag{12}$$

Besides, the stochastic part of (2) can also be written as follows:

$$\begin{aligned}
&\Upsilon_1 S_0 + \Upsilon_2 S_1 + \Upsilon_3 S_2 \\
&= g\Upsilon_1 + g(t_n + \bar{\mu}_0 h, x_n + \bar{\lambda}_0 fh + g\Phi_1)\Upsilon_2 + g(t_n + \bar{\rho}_0 h, x_n + \bar{\varphi}_0 K_0 h + S_0\Phi_2)\Upsilon_3 \\
&\overset{(2)}{\simeq} (\Upsilon_1 + \Upsilon_2 + \Upsilon_3)g + \bar{\mu}_0 g_{10}\Upsilon_2 h + \bar{\rho}_0 g_{10}\Upsilon_3 h + \bar{\lambda}_0 g_{01} f\Upsilon_2 h + \bar{\varphi}_0 g_{01} f \Upsilon_3 h \\
&+ g_{01} g\Phi_1\Upsilon_2 + g_{01} g\Phi_2\Upsilon_3 + (g_{11} + \frac{g^2}{2} g_{03})\bar{\mu}_0 g\Phi_1\Upsilon_2 h + (g_{11} + \frac{g^2}{2} g_{03})\bar{\rho}_0 g\Phi_2\Upsilon_3 h \\
&+ \bar{\lambda}_0 g_{02} f g\Phi_1\Upsilon_2 h + \bar{\varphi}_0 g_{02} f g \Phi_2\Upsilon_3 h + \frac{1}{2} g_{02} g^2\Phi_1^2\Upsilon_2 + \frac{1}{2} g_{02} g^2 \Phi_2^2 \Upsilon_3.
\end{aligned} \tag{13}$$

Then by substituting 2-equivalences (12) and (13) in (2), we can easily see that the Runge-Kutta method (2)-(3) and the simplified Taylor scheme (4) are 2-equivalent if the following relations hold:

$$\begin{aligned}
\alpha_1 + \alpha_2 + \alpha_3 &= 1, \\
\mu_0\alpha_2 + \rho_0\alpha_3 &= \frac{1}{2}, \\
\lambda_0\alpha_2 + \varphi_0\alpha_3 &= \frac{1}{2},
\end{aligned} \tag{14}$$

$$\begin{aligned}(\alpha_2\Theta_1+\alpha_3\Theta_2)h &\overset{(2)}{\simeq} \frac{1}{2}h\Delta W_n,\\ (\alpha_2\Theta_1^2+\alpha_3\Theta_2^2)h &\overset{(2)}{\simeq} \frac{1}{2}h^2,\end{aligned} \tag{15}$$

$$\begin{aligned}\Upsilon_1+\Upsilon_2+\Upsilon_3 &\overset{(2)}{\simeq} \Delta W_n,\\ (\bar{\mu}_0\Upsilon_2+\bar{\rho}_0\Upsilon_3)h &\overset{(2)}{\simeq} \frac{1}{2}h\Delta W_n,\\ (\bar{\lambda}_0\Upsilon_2+\bar{\varphi}_0\Upsilon_3)h &\overset{(2)}{\simeq} \frac{1}{2}h\Delta W_n,\\ \Phi_1\Upsilon_2+\Phi_2\Upsilon_3 &\overset{(2)}{\simeq} \frac{1}{2}((\Delta W_n)^2-h),\\ \Phi_1^2\Upsilon_2+\Phi_2^2\Upsilon_3 &\overset{(2)}{\simeq} \frac{1}{2}h\Delta W_n.\end{aligned} \tag{16}$$

$$\begin{aligned}(\bar{\mu}_0\Phi_1\Upsilon_2+\bar{\rho}_0\Phi_2\Upsilon_3)h &\overset{(2)}{\simeq} 0,\\ (\bar{\lambda}_0\Phi_1\Upsilon_2+\bar{\varphi}_0\Phi_2\Upsilon_3)h &\overset{(2)}{\simeq} 0.\end{aligned} \tag{17}$$

Therefore, preceding discussions conclude the following theorem:

Theorem 2.1. *Suppose that f and g in SDE (1) are sufficiently differentiable and satisfy conditions for the existence and uniqueness of solutions. Then the stochastic Runge-Kutta method* (2)-(3) *has order two in the weak sense if* (14)-(17) *hold.*

3. Some particular stochastic Runge-Kutta methods

In order to methods (2)-(3) be applicable, we must determine some particular solutions of systems (15)-(17). First consider system (15) and take $\Theta_1=\nu_1\Delta W_n$ and $\Theta_2=\delta_1\Delta W_n$. Then by 2-equivalence (6) this system becomes as follows:

$$\begin{aligned}\alpha_2\nu_1+\alpha_3\delta_1 &= \frac{1}{2},\\ \alpha_2\nu_1^2+\alpha_3\delta_1^2 &= \frac{1}{2}.\end{aligned} \tag{18}$$

Solving systems (14) and (18) we get three class of solutions as follows:

Case 1. The four-parameter solution

$$\begin{aligned}
\mu_0 &= \frac{(2\alpha_3\rho_0 - 1)(-1 + 2\alpha_3\delta_1^2)}{1 - 4\alpha_3\delta_1 + 4\alpha_3^2\delta_1^2},\\
\nu_1 &= \frac{-1 + 2\alpha_3\delta_1^2}{-1 + 2\alpha_3\delta_1},\\
\alpha_1 &= \frac{1}{2}(\frac{-1 - 4\alpha_3\delta_1 + 2\alpha_3 + 4\alpha_3\delta_1^2}{-1 + 2\alpha_3\delta_1^2}),\\
\alpha_2 &= -\frac{1}{2}(\frac{1 - 4\alpha_3\delta_1 + 4\alpha_3^2\delta_1^2}{-1 + 2\alpha_3\delta_1^2}),\\
\lambda_0 &= \frac{(-1 + 2\alpha_3\phi_0)(-1 + 2\alpha_3\delta_1^2)}{1 - 4\alpha_3\delta_1 + 4\alpha_3^2\delta_1^2}.
\end{aligned} \tag{19}$$

Case 2. The three-parameter solution

$$\begin{array}{lll}
\mu_0 = -\frac{1}{2}(\frac{\rho_0 - 1}{\alpha_2}), & \nu_1 = 0, & \alpha_1 = -\alpha_2 + \frac{1}{2},\\
\alpha_3 = \frac{1}{2}, & \delta_1 = 1, & \lambda_0 = -\frac{1}{2}(\frac{-1+\varphi_0}{\alpha_2}).
\end{array} \tag{20}$$

Case 3. The parameter free solution

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = 0, \quad \alpha_3 = \frac{1}{2}, \quad \delta_1 = 1, \quad \varphi_0 = 1, \quad \rho_0 = 1. \tag{21}$$

Now we consider the systems (16) and (17) using linear combinations of variables ΔW_n, $\sqrt{h}$ and $\frac{(\Delta W_n)^2}{\sqrt{h}}$ as follows:

$$\begin{aligned}
\Upsilon_1 &= \gamma_1\Delta W_n + \gamma_2\sqrt{h} + \gamma_3\frac{(\Delta W_n)^2}{\sqrt{h}},\\
\Upsilon_2 &= \lambda_1\Delta W_n + \lambda_2\sqrt{h} + \lambda_3\frac{(\Delta W_n)^2}{\sqrt{h}},\\
\Upsilon_3 &= \mu_1\Delta W_n + \mu_2\sqrt{h} + \mu_3\frac{(\Delta W_n)^2}{\sqrt{h}},\\
\Phi_1 &= \beta_1\Delta W_n + \beta_2\sqrt{h} + \beta_3\frac{(\Delta W_n)^2}{\sqrt{h}},\\
\Phi_2 &= \eta_1\Delta W_n + \eta_2\sqrt{h} + \eta_3\frac{(\Delta W_n)^2}{\sqrt{h}}.
\end{aligned} \tag{22}$$

With these random variables, if we take $\bar{\mu}_0 = \bar{\rho}_0$ and $\bar{\lambda}_0 = \bar{\varphi}_0$ in (3), then the order condition $\Phi_1\Upsilon_2 + \Phi_2\Upsilon_3 \overset{(2)}{\simeq} \frac{1}{2}((\Delta W_n)^2 - h)$ in system (16) yields to $(\Phi_1\Upsilon_2 + \Phi_2\Upsilon_3)h \overset{(2)}{\simeq} \frac{1}{2}((h\Delta W_n)^2 - h^2)$. By 2-equivalent $h(\Delta W)^2 \overset{(2)}{\simeq} h^2$ in (7) we get $(\Phi_1\Upsilon_2 + \Phi_2\Upsilon_3)h \overset{(2)}{\simeq} 0$ and thus system (17) is omitted. Using equivalences (6) and (8)-(11) it can be seen that system (16) has the following one parameter solution [17]:

$$\begin{aligned}
&\beta_3 = \eta_3 = 0, \quad \bar{\mu}_0 = \bar{\lambda}_0 = 1, \quad \gamma_1 = \frac{1}{2}, \quad \gamma_2 = \gamma_3 = 0, \quad \mu_1 = \lambda_1 = \frac{1}{4}, \\
&\mu_2 = -\lambda_2 = \frac{1-48\mu_3^2}{32\mu_3}, \quad \lambda_3 = -\mu_3, \quad \beta_1 = \eta_1 = 1 + \frac{32\mu_3^2}{1-48\mu_3^2}, \\
&\beta_2 = -\eta_2 = \frac{8\mu_3}{1-48\mu_3^2},
\end{aligned} \tag{23}$$

where $\mu_3 \neq 0$ and $\mu_3 \neq \frac{1}{4\sqrt{3}}$.

Now, supposing $\mu_3 = -\frac{1}{4}$ in solution (23) of system (16), we state some particular solutions of systems (14)-(17) and we obtain following examples of second-order stochastic Runge-Kutta method (2)-(3):

(I) A solution of system (19) in Case 1 is

$$\begin{aligned}
\alpha_1 &= \frac{1}{6}, \\
\alpha_2 &= \frac{4}{6}, \\
\alpha_3 &= \frac{1}{6}, \\
\nu_1 &= \frac{3}{5} \mp \frac{\sqrt{6}}{10}, \\
\delta_1 &= \frac{3}{5} \pm \frac{2\sqrt{6}}{5}, \\
\mu_0 &= -\frac{1}{4}\rho_0 + \frac{3}{4}, \\
\lambda_0 &= -\frac{1}{4}\phi_0 + \frac{3}{4}.
\end{aligned} \tag{24}$$

Therefore, if we take $\rho_0 = \phi_0 = 1$, we have $\mu_0 = \lambda_0 = \frac{1}{2}$ and we obtain the following method, which we call it "*SRK1 method*":

$$\begin{aligned}
x_{n+1} = x_n + \frac{1}{6}(K_0 + 4K_1 + K_2)h + \frac{1}{4}(2S_0 + S_1 + S_2)\Delta W_n \\
+\frac{1}{4}(S_2 - S_1)(\sqrt{h} - \frac{(\Delta W_n)^2}{\sqrt{h}}),
\end{aligned}$$

with

$$\begin{aligned}
K_0 &= f(t_n, x_n),\\
S_0 &= g(t_n, x_n),\\
K_1 &= f(t_n + \frac{1}{2}h, x_n + \frac{1}{2}K_0 h + (\frac{3}{5} \mp \frac{\sqrt{6}}{10})\Delta W_n S_0),\\
S_1 &= g(t_n + h, x_n + K_0 h + \sqrt{h} S_0),\\
K_2 &= f(t_n + h, x_n + K_0 h + \frac{3}{5} \pm \frac{2\sqrt{6}}{5}\Delta W_n S_0),\\
S_2 &= g(t_n + h, x_n + K_0 h - \sqrt{h} S_0).
\end{aligned}$$

(II) Let $\alpha_2 = \frac{1}{4}$ and $\rho_0 = \varphi_0 = \frac{3}{4}$ in system (20) in case 2. Then we get "*SRK2 method*" as follows:

$$\begin{aligned}
x_{n+1} = x_n + \frac{1}{4}(K_0 + K_1 + 2K_2)h + \frac{1}{4}(2S_0 + S_1 + S_2)\Delta W_n\\
+\frac{1}{4}(S_2 - S_1)(\sqrt{h} - \frac{(\Delta W_n)^2}{\sqrt{h}}),
\end{aligned}$$

with

$$\begin{aligned}
K_0 &= f(t_n, x_n),\\
S_0 &= g(t_n, x_n),\\
K_1 &= f(t_n + \frac{1}{2}h, x_n + \frac{1}{2}K_0 h),\\
S_1 &= g(t_n + h, x_n + K_0 h + \sqrt{h} S_0),\\
K_2 &= f(t_n + \frac{3}{4}h, x_n + \frac{3}{4}K_0 h + \Delta W_n S_0),\\
S_2 &= g(t_n + h, x_n + K_0 h - \sqrt{h} S_0).
\end{aligned}$$

(III) Parameter values in case 3 (system (21)) yield to the weak second-order "*Platen method*"[7].

4. Mean-Square stability

In this section we consider MS-stability of the test equation for SDEs first, and then we consider numerically MS-stability of proposed methods for SDEs. Consider the Ito scalar linear test equation as follows

$$dx(t) = \alpha x(t)dt + \beta x(t)dW(t),\ t > 0,\ \alpha, \beta \in \mathbb{C}, \tag{25}$$

with nonrandom initial condition $x(t_0) = x_0 \in \mathbb{R}, x_0 \neq 0$. Exact solution of (25) is given by

$$x(t) = x_0 \exp\{(\alpha - \frac{1}{2}\beta^2)t + \beta W(t)\}.$$

It can be shown [14] that the solution of the test equation (25) is MS-stable if and only if $2\Re(\alpha) + |\beta|^2 < 0$.

Definition 4.1. *Numerical solution $\{x_n\}_{n\in\mathbb{N}}$ for solving test equation (25) generated by a scheme with equidistant step size is MS-stable if*

$$\lim_{n\to\infty} E[|x_n|^2] = 0.$$

Applying second-order RK schemes given in previous section on the test equation (25) we obtain the following difference equation

$$x_{n+1} = x_n\Big(1 + \alpha(\alpha_1 + \alpha_2 + \alpha_3)h + \alpha^2(\alpha_2\lambda_0 + \alpha_3\phi_0)h^2 + \beta\alpha h(\alpha_2\Theta_1 + \alpha_3\Theta_2) + \beta(\Upsilon_1 + \Upsilon_2 + \Upsilon_3) + \beta\alpha h(\bar{\lambda}_0\Upsilon_2 + \bar{\varphi}_0\Upsilon_3) + \beta^2(\Phi_1\Upsilon_2 + \Phi_2\Upsilon_3)\Big). \tag{26}$$

Using definitions of variables in (22) we have

$$\Upsilon_1 + \Upsilon_2 + \Upsilon_3 = (\gamma_1 + \lambda_1 + \mu_1)\Delta W_n + (\gamma_2 + \lambda_2 + \mu_2)\sqrt{h} + (\gamma_3 + \lambda_3 + \mu_3)\frac{\Delta W_n}{\sqrt{h}},$$

$$\bar{\lambda}_0\Upsilon_2 + \bar{\varphi}_0\Upsilon_3 = (\bar{\lambda}_0\lambda_1 + \bar{\varphi}_0\mu_1)\Delta W_n + (\bar{\lambda}_0\lambda_2 + \bar{\varphi}_0\mu_2)\sqrt{h} + (\bar{\lambda}_0\lambda_3 + \bar{\varphi}_0\mu_3)\frac{\Delta W_n}{\sqrt{h}},$$

$$\Phi_1\Upsilon_2 + \Phi_2\Upsilon_3 = (\beta_1\lambda_1 + \eta_1\mu_1)(\Delta W_n)^2 + (\beta_2\lambda_2 + \eta_2\mu_2)h + (\beta_3\lambda_3 + \eta_3\mu_3)\frac{(\Delta W_n)^4}{h} + (\beta_1\lambda_2 + \beta_2\lambda_1 + \eta_1\mu_2 + \eta_2\mu_1)\sqrt{h}\Delta W_n + (\beta_2\lambda_3 + \beta_3\lambda_2 + \eta_2\mu_3 + \eta_3\mu_2)(\Delta W_n)^2 + (\beta_1\lambda_3 + \beta_3\lambda_1 + \eta_1\mu_3 + \eta_3\mu_1)\frac{(\Delta W_n)^3}{\sqrt{h}},$$

and then by parameter values in solution (23) of system (16) with assumption $\bar{\lambda}_0 = \bar{\varphi}_0$ we get

$$\begin{aligned} \Upsilon_1 + \Upsilon_2 + \Upsilon_3 &= \Delta W_n, \\ \bar{\lambda}_0\Upsilon_2 + \bar{\varphi}_0\Upsilon_3 &= \tfrac{1}{2}\Delta W_n, \\ \Phi_1\Upsilon_2 + \Phi_2\Upsilon_3 &= \tfrac{1}{2}(\Delta W_n)^2 - \tfrac{1}{2}h. \end{aligned} \tag{27}$$

In addition, considering stochastic variables $\Theta_1 = \nu_1\Delta W_n$ and $\Theta = \delta_1\Delta W_n$ and using first equation in system (18), we obtain

$$\alpha_2\Theta_1 + \alpha_3\Theta_2 = \frac{1}{2}\Delta W_n. \tag{28}$$

Now by employing system (14) and inserting (27) and (28) in (26),

$$x_{n+1} = x_n\Big(1 + \alpha h + \frac{1}{2}\alpha^2h^2 + \beta\Delta W_n(1 + \alpha h) + \frac{1}{2}\beta^2((\Delta W_n)^2 - h)\Big). \tag{29}$$

Let us denote $y_n = \|x_n\|^2 = E|x_n|^2$. Taking the square of the mean square norm of (29), by this property of moments of the Wiener process ΔW_n, that state $E[(\Delta W_n)^n] = 0$ if n is an odd number and $E[(\Delta W_n)^n] = (n-1)(n-3)(n-5)\dots 1$ if n is an even number, the following one-step difference equation is obtained:

$$y_{n+1} = y_n(|1 + \alpha h + \frac{1}{2}\alpha^2h^2|^2 + |\beta|^2h|1 + \alpha h|^2 + \frac{1}{2}|\beta|^4h^2). \tag{30}$$

If we take $\Delta = \alpha h$ and $k = -\beta^2/\alpha$, then $h\beta^2 = -k\Delta$ and therefore, the one-step difference equation (30) becomes of the form $y_{n+1} = P(\Delta, k)y_n$, where

$$P(\Delta, k) = |1 + \Delta + \frac{1}{2}\Delta^2|^2 + |1 + \Delta|^2|\Delta k| + \frac{1}{2}|\Delta k|^2. \qquad (31)$$

The region determined by $\mathcal{R} = \{(\Delta, k) \in \mathbb{R}^2 : |P(\Delta, k)| < 1\}$ is called the region of MS-stability of the scheme.
Notice that the stability function $P(\Delta, k)$ is same with the stability function of Platan method [7] and with the stability function of family A in [14] with parameter $\alpha_2 = \frac{1}{2}$ and hence MS-stability region of this class of schemes is similar to MS-stability region of these methods. Fig 1 shows the region stability of this class of weak order 2 stochastic RK methods.

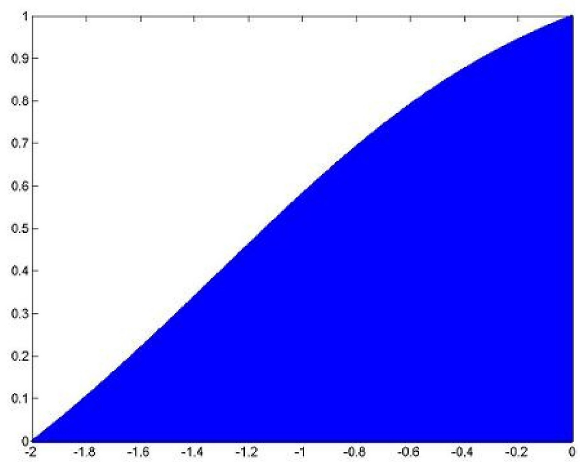

FIGURE 1. MS-stability region.

5. Numerical experiments

In this section we employ numerical examples to examine and illustrate stochastic Runge-Kutta methods developed in this article. Each example is solved by a method for 5,000 independent sample paths with step size h and then mean of these solutions is considered as the weak numerical solution[6, 7]. Error and standard deviation of each method is calculated as follows:

$$\epsilon = \Big|E[x_N] - E[x(T)]\Big|,$$

$$\sigma = \sqrt{E[x_N^2] - E^2[x_N]},$$

where $x(T)$ and x_N are the exact solution and numerical solution at the end time T, respectively. To simulate the increments ΔW_n we use a normal distribution $\mathcal{N}(0, h)$.

Example 5.1. *Consider non-linear SDE*

$$dx(t) = (\frac{1}{3}x(t)^{1/3} + 6x(t)^{2/3})dt + x(t)^{2/3}dW(t), \; x(0) = 1, \tag{32}$$

with exact solution $x(t) = (2t + 1 + \frac{W(t)}{3})^3$.
We compare SRK1 method and SRK2 method with Platen method. The first moment of the solutions will be approximated at $T = 1$ *whose exact solution is* $E[x(1)] = 28$. *The errors and standard deviations of the employed methods has been summarized in Table 1. Moreover, the speed of the methods are compared*

TABLE 1. Errors and standard deviations in the approximation of $E[x(1)]$ in Example 5.1.

	Platen		*SRK1*		*SRK2*	
h	*err*	*s. d*	*err*	*s. d*	*err*	*s. d*
2^{-1}	*5.874*	*6.11*	*5.197*	*6.26*	*5.291*	*6.24*
2^{-2}	*2.182*	*7.99*	*1.845*	*8.09*	*1.888*	*8.08*
2^{-3}	*0.772*	*8.71*	*0.649*	*8.75*	*0.664*	*8.75*
2^{-4}	*0.229*	*9.23*	*0.191*	*9.24*	*0.196*	*9.24*
2^{-5}	*0.029*	*9.17*	*0.019*	*9.17*	*0.020*	*9.17*

and time of computations are summarized in Table 2 as CPU-time.

TABLE 2. CPU-time of evaluations to approximate $E[x(1)]$ in Example 5.1.

h	*Platen*	*SRK1*	*SRK2*
2^{-1}	*0.078125*	*0.062500*	*0.062500*
2^{-2}	*0.109375*	*0.140625*	*0.125000*
2^{-3}	*0.171875*	*0.296875*	*0.281250*
2^{-4}	*0.359375*	*0.453125*	*0.453125*
2^{-5}	*0.718750*	*0.921875*	*0.921875*

Example 5.2. *Consider the following system of SDEs*

$$\begin{aligned} dx_1(t) &= 0.5a^2 x_1(t)dt + ax_2(t)dW(t), \\ dx_2(t) &= 0.5a^2 x_2(t)dt + ax_1(t)dW(t). \end{aligned} \tag{33}$$

The exact solution of system (33) *is*

$$x(t) = \begin{pmatrix} \cosh\Big(aW(t) + \cosh^{-1}(x_1(0))\Big) \\ \sinh\Big(aW(t) + \sinh^{-1}(x_2(0))\Big) \end{pmatrix} \tag{34}$$

We estimate first moments of components x_1 *and* x_2 *at* $T = 1$ *employing Euler-Maruyama method and SRK1 method. Errors of approximations are given in Table 3 for* $a = 1$.

TABLE 3. Errors in approximation of first moments $E[x_1(1)]$ and $E[x_2(1)]$ in Example 5.2.

	Euler		*SRK1*	
h	$E[x_1(1)]$	$E[x_2(1)]$	$E[x_1(1)]$	$E[x_2(1)]$
10^{-1}	1.638840	1.110225	0.030163	0.042791
10^{-2}	0.896515	1.701824	0.003467	0.005086
10^{-3}	0.869014	1.691193	0.000365	0.000539

Example 5.3. *In this example we consider the stability of schemes SRK1 and SRK2 by applying the difference equation* (29) *with equidistant step size* h *to the linear test SDEs*

$$dx(t) = \alpha x(t)dt + \beta x(t)dW(t), \; x(0) = 1$$

with the following values for α *and* β*:*
(i) $\alpha = -100$*,* $\beta = 10$ *and* $h = 0.005$*,*
(ii) $\alpha = -100$*,* $\beta = 10$ *and* $h = 0.01$*,*
(iii) $\alpha = -120$*,* $\beta = 11$ *and* $h = 0.01$*.*
In cases (i)-(iii) we have respectively $P(\Delta, k) = 0.640625 < 1$*,* $P(\Delta, k) = 0.75 < 1$ *and* $P(\Delta, k) = 1.05085 > 1$ *and so for examples (i) and (ii) schemes are MS-stable and for example (iii) they are MS-unstable. Running a MATLAB code with 10,000 independent trials, the results, summarized in Table 4, confirm the above analysis.*

TABLE 4. Values of $\|x(t)\|^2 = E|x(t)|^2$ using RK scheme for test examples (i)-(iii).

	$\|x(t)\|^2 = E\|x(t)\|^2$		
t	*(i)*	*(ii)*	*(iii)*
0.01	*0.637739*	*0.791478*	*1.013640*
0.02	*0.233158*	*0.600301*	*1.351227*
0.03	*0.064782*	*0.333338*	*0.894226*
0.04	*0.019702*	*0.215174*	*0.551664*
0.05	*0.004635*	*0.072973*	*1.542861*
0.06	*0.000846*	*0.093904*	*1.250331*
0.07	*0.000875*	*0.029113*	*0.270396*
0.08	*0.000139*	*0.002788*	*1.161399*
0.09	*0.000018*	*0.001060*	*3.229528*
0.10	*0.000003*	*0.000079*	*3.321441*

6. **Conclusions**

In this paper, we studied a family of three-stage stochastic Runge-Kutta methods. The conditions that must satisfy in order to the methods have weak order two were obtained. The SRK1 method and the SRK2 method were obtained as particular solutions of the conditions. It was shown that the Platen method can be also concluded as a particular solution of this family. Numerical simulations show that errors resulted by SRK1 and SRK2 methods are less than Platen method for each step size. Moreover, CPU-times for these methods have less values in large step sizes but not in small step sizes due to an extra stage in these methods rather than Platen method. In addition, the mean-square stability of some particular schemes of this family was studied and the stability function and the region of stability were given. The region of stability is same as the region of Platen's method. Also, the stability of the obtained stochastic schemes were numerically compared for various values of parameters and step sizes.

REFERENCES

[1] M. Abukhaled and E. Allen, *A class of second-order runge-kutta methods for numerical solution of stochastic differential equations*, Stochastic Analysis and Applications **16** (1998), 977–991.

[2] E. Allen, *Modeling with Itô stochastic differential equations*, vol. 22, Springer Science & Business Media, 2007.

[3] K. Burrage and P. Burrage, *High strong order explicit runge-kutta methods for stochastic ordinary differential equations*, Applied Numerical Mathematics **22** (1996), 81–101.

[4] K. Debrabant and A. Rößler, *Classification of stochastic runge–kutta methods for the weak approximation of stochastic differential equations*, Mathematics and Computers in Simulation **77** (2008), 408–420.

[5] R. Farnoosh and M. Parsamanesh, *Stochastic differential equation systems for an SIS epidemic model with vaccination and immigration*, Communications in Statistics-Theory and Methods **46** (2017), 8723–8736.

[6] D. J. Higham, *An algorithmic introduction to numerical simulation of stochastic differential equations*, SIAM review **43** (2001), 525–546.

[7] P. E. Kloeden, E. Platen and H. Schurz, *Numerical solution of SDE through computer experiments*, Springer Science & Business Media, 2012.

[8] Y. Komori, *Weak second-order stochastic runge–kutta methods for non-commutative stochastic differential equations*, Journal of computational and applied mathematics **206** (2007), 158–173.

[9] J. D. Lambert, *Numerical methods for ordinary differential systems: the initial value problem*, John Wiley & Sons, Inc., 1991.

[10] X. Mao, *Stochastic differential equations and applications*, Elsevier, 2007.

[11] B. Øksendal, *Stochastic differential equations*, in: *Stochastic differential equations*, Springer, 2003, pp. 65–84.

[12] S. Rong, *Theory of stochastic differential equations with jumps and applications: mathematical and analytical techniques with applications to engineering*, Springer Science & Business Media, 2006.

[13] A. Rößler, *Runge–kutta methods for itô stochastic differential equations with scalar noise*, BIT Numerical Mathematics **46** (2006), 97–110.

[14] A. Tocino, *Mean-square stability of second-order runge–kutta methods for stochastic differential equations*, Journal of computational and applied mathematics **175** (2005), 355–367.

[15] A. Tocino and R. Ardanuy, *Runge–kutta methods for numerical solution of stochastic differential equations*, Journal of Computational and Applied Mathematics **138** (2002), 219–241.

[16] A. Tocino and R. Ardanuy, *Truncated itô-taylor expansions*, Stochastic analysis and applications **20** (2002), 427–443.

[17] A. Tocino and J. Vigo-Aguiar, *Weak second order conditions for stochastic runge–kutta methods*, SIAM Journal on Scientific Computing **24** (2002), 507–523.