CREDIBILITY FORMULAS ALLOWING EFFECTS LIKE INFLATION

Virginia ATANASIU[1]

În acest articol, vom obţine, la fel ca şi în cazul modelelor clasice de credibilitate, o soluţie de credibilitate sub forma unei combinaţii liniare a estimării individuale (bazată pe datele unui stat particular) cu estimarea colectivă (bazată pe datele agregate SUA).

In this article we will obtain, just like in the case of classical credibility model, a credibility solution in the form of a linear combination of the individual estimate (based on the data of a particular state) and the collective estimate (based on aggregate USA data).

Keywords: linearized regression credibility premium, the structural parameters, unbiased estimators.

Mathematics Subject Classifications: 62P05.

1. Introduction

The article contains a description of the credibility formulas, allowing effects like inflation.

In **Section 2** we give the simple model, which involves only one isolated contract.

The simple model, involving only one contract, contains the basics of all further regression credibility models.

In this section we will give the assumption of the simple regression credibility models and the optimal linearized regression credibility premium is derived.

Section 3 describes the classical model. In the classical model, a portfolio of contracts is studied.

Just as in **Section 2** we will derive the best linearized regression credibility premium for this model and we will provide some useful estimators for the structure parameters.

2. The simple model allowing effects like inflation

In the simple regression credibility model, we consider one contract with unknown and fixed risk parameter θ, during a period of t (≥ 2) years. The yearly

[1] Lecturer Dr., Mathematics Department, Academy of Economic Studies, Bucharest, Romania

claim amounts are denoted by $X_1,...,X_t$. Suppose $X_1,...,X_t$ are random variables with finite variance. The contract is a random vector consisting of a random structure parameter θ and observations $X_1,...,X_t$. Therefore, the contract is equal to $(\theta, \underline{X}')$, where $\underline{X}' = (X_1,...,X_t)$. For this model we want to estimate the net premium: $\mu(\theta) = E(X_j \mid \theta)$, $j = \overline{1,t}$ for a contract with risk parameter θ.

Remark 2.1

In the credibility models, the pure net risk premium of the contract with risk parameter θ is defined as:

$$\mu(\theta) = E(X_j \mid \theta), \forall j = \overline{1,t} \qquad (2.1).$$

Instead of assuming time independence in the net risk premium (2.1) one could assume that the conditional expectation of the claims on a contract changes in time, as follows:

$$\mu_j(\theta) = E(X_j \mid \theta) = \underline{Y}'_j \underline{b}(\theta), \forall j = \overline{1,t} \qquad (2.2),$$

where the design vector $\underline{Y}_j$ is known ($\underline{Y}_j$ is a column vector of length q, the non-random (q×1) vector $\underline{Y}_j$ is known) and where the components of $\underset{\sim}{b}(\theta)$ are the unknown regression constants ($\underset{\sim}{b}(\theta)$ is a column vector of length q).

Remark 2.2

Because of inflation we are not willing to assume that $E(X_j \mid \theta)$ is independent of j. Instead we make the regression assumption $E(X_j \mid \theta) = \underline{Y}'_j \underline{b}(\theta)$.

When estimating the vector $\underline{\beta}$ from the initial regression hypothesis $E(X_j) = \underline{Y}'_j \underline{\beta}$ the actuary found great differences. He then assumed that to each of the states there was related an unknown random risk parameter θ containing the risk characteristics of the state, and that's from different states were independent and identically distributed. Again considering one particular state, we assume that $E(X_j \mid \theta) = \underline{Y}'_j \underline{b}(\theta)$, with $E[\underline{b}(\theta)] = \underline{\beta}$.

After these motivating introductory remarks, we state the model assumptions in more detail.

Let $\underline{X} = (X_1,...,X_t)'$ be an observed random $(t \times 1)$ vector and θ an unknown random risk parameter. We assume that:

(H₁) $E(\underline{X} \mid \theta) = \underline{Y}\underline{b}(\theta)$.

It is assumed that the matrices:

(H2) $\underline{\Lambda} = Cov[\underline{b}(\theta)], (\underline{\Lambda} = \underline{\Lambda}^{(q\times q)})$,

(H3) $\underline{\Phi} = E[Cov(\underline{X} \mid \theta)], (\underline{\Phi} = \underline{\Phi}^{(t\times t)})$

are positive definite. We finally introduce:

(H4) $E[\underline{b}(\theta)] = \underline{\beta}$.

Let $\tilde{\mu}_j$ be the credibility estimator of $\mu_j(\theta)$ based on $\underline{X}$. (Theory: A linear estimator $\dot{X}_{t+1} = a_0 + \sum_{j=1}^{t} a_j X_j$ of X_{t+1} is a credibility estimator if and only if it satisfies the normal equations:

$$E\left(\dot{X}_{t+1}\right) = E(X_{t+1}) \quad (2.3),$$

$$Cov\left(\dot{X}_{t+1}, X_j\right) = Cov(X_{t+1}, X_j),\ j = \overline{1,t} \quad (2.4)).$$

The optimal choice of $\tilde{\mu}_j$ is determined in the following **application**:

Application 2.3: The credibility estimator $\tilde{\mu}_j$ is given by:

$$\tilde{\mu}_j = \underline{Y}'_j \left[\underline{Z}\hat{\underline{b}} + (\underline{I} - \underline{Z})\underline{\beta} \right] \quad (2.5),$$

with:

$$\hat{\underline{b}} = \left(\underline{Y}'\underline{\Phi}^{-1}\underline{Y}\right)^{-1} \underline{Y}'\underline{\Phi}^{-1}\underline{X} \quad (2.6),$$

$$\underline{Z} = \underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y}\left(\underline{I} + \underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y}\right)^{-1} \quad (2.7),$$

where $\underline{I}$ denotes the $q \times q$ identity matrix $\left(\hat{\underline{b}} = \hat{\underline{b}}^{(q\times 1)}; \underline{z} = \underline{z}^{(q\times q)}\right)$ for some fixed j.

Proof. Let: $\tilde{\mu}_j = \gamma_0 + \underline{\gamma}'\underline{X}$, where γ_0 is a scalar constant, and $\underline{\gamma}$ is a constant $(t \times 1)$ vector. We write the normal equations (2.4) as: $Cov\left(\tilde{\mu}_j, \underline{X}'\right) = Cov[\mu_j(\theta), \underline{X}']$, that is:

$$\underline{\gamma}' Cov(\underline{X}) = Cov[\mu_j(\theta), \underline{X}'] \quad (2.8).$$

We have:

$Cov(\underline{X}) = E[Cov(\underline{X} \mid \theta)] + Cov[E(\underline{X} \mid \theta)] = \underline{\Phi} + Cov[\underline{Y}\underline{b}(\theta)] = \underline{\Phi} + Cov[\underline{Y}\underline{b}(\theta), (\underline{Y}\underline{b}(\theta))']$
$= \underline{\Phi} + Cov[\underline{Y}\underline{b}(\theta), \underline{b}(\theta)'\underline{Y}'] = \underline{\Phi} + \underline{Y}Cov[\underline{b}(\theta)]\underline{Y}' = \underline{\Phi} + \underline{Y}\underline{\Lambda}\underline{Y}'$ and $Cov(\mu_j(\theta), \underline{X}') =$

$= E[\mu_j(\theta)\underline{X}'] - E[\mu_j(\theta)]E(\underline{X}') = E\{E[\mu_j(\theta)\underline{X}' \mid \theta]\} - E[\mu_j(\theta)]E[E(\underline{X}' \mid \theta)] =$
$= E[\mu_j(\theta)E(\underline{X}' \mid \theta)] - E[\mu_j(\theta)]E[E(\underline{X}' \mid \theta)] = Cov[\mu_j(\theta), E(\underline{X}' \mid \theta)] = Cov[\underline{Y}_j^{'} \cdot$
$\cdot \underline{b}(\theta), (\underline{Y}\underline{b}(\theta))'] = Cov[\underline{Y}_j^{'}\underline{b}(\theta), \underline{b}(\theta)'\underline{Y}'] = \underline{Y}_j^{'}Cov[\underline{b}(\theta)]\underline{Y}' = \underline{Y}_j^{'}\underline{\Lambda}\underline{Y}'$, and insertion in (2.8) gives: $\underline{\gamma}'(\underline{\Phi} + \underline{Y}\underline{\Lambda}\underline{Y}') = \underline{Y}_j^{'}\underline{\Lambda}\underline{Y}'$, that is:
$\underline{\gamma} = \underline{Y}_j^{'}\underline{\Lambda}\underline{Y}'(\underline{\Phi} + \underline{Y}\underline{\Lambda}\underline{Y}')^{-1} = \underline{Y}_j^{'}\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}(\underline{I} + \underline{Y}\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1})^{-1}$ (because: $\underline{\Phi}^{-1}(\underline{I} + \underline{Y}\underline{\Lambda}\underline{Y}' \cdot$
$\cdot \underline{\Phi}^{-1})^{-1} = [(\underline{I} + \underline{Y}\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1})\underline{\Phi}]^{-1} = (\underline{\Phi} + \underline{Y}\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{\Phi})^{-1} = (\underline{\Phi} + \underline{Y}\underline{\Lambda}\underline{Y}')^{-1}$). For the development of an expression for $\tilde{\mu}_j$, we shall need the following lemma: "Let $\underline{A}$ be an $(r \times s)$ matrix and $\underline{B}$ an $(s \times r)$ matrix. Then:
$(\underline{I} + \underline{A}\underline{B})^{-1} = \underline{I} - \underline{A}(\underline{I} + \underline{B}\underline{A})^{-1}\underline{B}$,
if the displayed inverses exist". This lemma now gives: $\underline{\gamma}' = \underline{Y}_j^{'}\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}[\underline{I} - \underline{Y} \cdot$
$\cdot (\underline{I} + \underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y})^{-1}\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}] = \underline{Y}_j^{'}[\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1} - \underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y}(\underline{I} + \underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y})^{-1}\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}] =$
$= \underline{Y}_j^{'}[\underline{I} - \underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y}(\underline{I} + \underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y})^{-1}]\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1} = \underline{Y}_j^{'}(\underline{I} + \underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y})^{-1}\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}$. This gives: $\underline{\gamma}'\underline{X} = \underline{Y}_j^{'}(\underline{I} + \underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y})^{-1}\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{X} = \underline{Y}_j^{'}(\underline{I} + \underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y})^{-1}\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y}\hat{\underline{b}}$, with $\hat{\underline{b}}$ given by (2.6), and we get:
$\underline{\gamma}'\underline{X} = \underline{Y}_j^{'}[\underline{I} - (\underline{I} + \underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y})^{-1}]\hat{\underline{b}} = \underline{Y}_j^{'}\underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y}(\underline{I} + \underline{\Lambda}\underline{Y}'\underline{\Phi}^{-1}\underline{Y})^{-1}\hat{\underline{b}}$,
that is:

$$\underline{\gamma}'\underline{X} = \underline{Y}_j^{'}\underline{Z}\hat{\underline{b}} \qquad (2.9),$$

with $\underline{Z}$ given by (2.7). Now (2.5) follows from (2.9) and (2.3). Indeed: $E\left(\tilde{\mu}_j\right) =$
$= E[\mu_j(\theta)] \overset{(2.2)}{\Leftrightarrow} E\left(\gamma_0 + \underline{Y}_j^{'}\underline{Z}\hat{\underline{b}}\right) = E[\underline{Y}_j^{'}\underline{b}(\theta)] \Leftrightarrow E(\gamma_0) + E\left(\underline{Y}_j^{'}\underline{Z}\hat{\underline{b}}\right) = \underline{Y}_j^{'}E[\underline{b}(\theta)] \Leftrightarrow$
$\overset{(H_4)}{\Leftrightarrow} \gamma_0 + \underline{Y}_j^{'}\underline{Z}E\left(\hat{\underline{b}}\right) = \underline{Y}_j^{'}\underline{\beta} \overset{(2.10)}{\Leftrightarrow} \gamma_0 + \underline{Y}_j^{'}\underline{Z}\underline{\beta} = \underline{Y}_j^{'}\underline{\beta} \Leftrightarrow \gamma_0 = \underline{Y}_j^{'}(\underline{I} - \underline{Z})\underline{\beta} \qquad (2.11),$
where: $E\left(\hat{\underline{b}}\right) \overset{(2.6)}{=} (\underline{Y}'\underline{\Phi}^{-1}\underline{Y})^{-1}\underline{Y}'\underline{\Phi}^{-1}E(\underline{X}) = (\underline{Y}'\underline{\Phi}^{-1}\underline{Y})^{-1}\underline{Y}'\underline{\Phi}^{-1}E[E(\underline{X} \mid \theta)] \overset{(H_1)}{=}$
$\overset{(H_1)}{=} (\underline{Y}\underline{\Phi}^{-1}\underline{Y})^{-1}\underline{Y}'\underline{\Phi}^{-1}E[\underline{Y}\underline{b}(\theta)] = (\underline{Y}'\underline{\Phi}^{-1}\underline{Y})^{-1}\underline{Y}'\underline{\Phi}^{-1}\underline{Y}E[\underline{b}(\theta)] \overset{(H_4)}{=} (\underline{Y}'\underline{\Phi}^{-1}\underline{Y})^{-1}(\underline{Y}'\underline{\Phi}^{-1}\underline{Y})$
$\cdot \underline{\beta} = \underline{\beta} \qquad (2.10).$

So: $\tilde{\mu}_j = \gamma_0 + \underline{\gamma}'\underline{X} \underset{(2.9)}{\overset{(2.11)}{=}} \underline{Y}_j^{'}(\underline{I}-\underline{Z})\underline{\beta} + \underline{Y}_j^{'}\underline{Z}\hat{\underline{b}} = \underline{Y}_j^{'}\left[\underline{Z}\hat{\underline{b}} + (\underline{I}-\underline{Z})\underline{\beta}\right]$ □

Remark 2.4

By the credibility estimator of a vector we shall mean the vector of the credibility estimator of each element of the vector to be estimated. Let $\tilde{\underline{\mu}}$ be the credibility estimator of the vector $\underline{\mu}(\theta) = \underline{Y}\underline{b}(\theta)$, where the non-random $(t \times q)$ matrix $\underline{Y}$ is known. Then we clearly have:

$$\tilde{\underline{\mu}} = \underline{Y}\left[\underline{Z}\hat{\underline{b}} + (\underline{I}-\underline{Z})\underline{\beta}\right] \qquad (2.12).$$

An interesting special case is $\underline{Y} = \underline{I}$, where we get:

$$\tilde{\underline{b}} = \underline{Z}\hat{\underline{b}} + (\underline{I}-\underline{Z})\underline{\beta} \qquad (2.13),$$

as the credibility estimator of $\underline{b}(\theta)$.

We introduce the following **definition**:

Definition 2.5:

Let $r(\theta)$ be a real-valued function of θ and $\hat{r}$ an estimator of $r(\theta)$. We shall say that $\hat{r}$ is a θ**-unbiased estimator** of $r(\theta)$, if $E\left(\hat{r} \mid \theta\right) = r(\theta)$ a.s.

By the requirement of θ-unbiasedness for $\mu_j(\theta)$, we obtain the following **application**:

Application 2.6*:* The best linear θ-unbiased estimator of $\mu_j(\theta)$ based on $\underline{X}$ is:

$$\hat{\mu}_j = \underline{Y}_j^{'}\hat{\underline{b}} \qquad (2.14),$$

for any fixed $j = \overline{1,t}$, where $\hat{\underline{b}}$ is defined by (2.6).

Proof. Let $\dot{\mu}_j = g_0 + \underline{g}'\underline{X}$ be a linear θ-unbiased estimator of $\mu_j(\theta)$. Then: $E\left(\dot{\mu}_j \mid \theta\right) = \mu_j(\theta)$ a.s., that is (see (2.2)): $E\left(g_0 + \underline{g}'\underline{X} \mid \theta\right) = \underline{Y}_j^{'}\underline{b}(\theta)$ a.s., or:

$g_0 + \underline{g}' E(\underline{X} \mid \theta) = \underline{Y}_j^{'} \underline{b}(\theta)$ a.s., or (see (**H₁**)): $g_0 + \underline{g}' \underline{Y} \underline{b}(\theta) = \underline{Y}_j^{'} \underline{b}(\theta)$ a.s., that is: $g_0 = \left(\underline{Y}_j^{'} - \underline{g}' \underline{Y}\right) \underline{b}(\theta)$ a.s. (2.15),

and thus: $0 = Cov\left[g_0, \underline{b}(\theta)'\right] = Cov\left[\left(\underline{Y}_j^{'} - \underline{g}' \underline{Y}\right)\underline{b}(\theta), \underline{b}(\theta)'\right] = \left(\underline{Y}_j^{'} - \underline{g}' \underline{Y}\right) Cov\left[\underline{b}(\theta),\right.$ $\left., \underline{b}(\theta)'\right] = \left(\underline{Y}_j^{'} - \underline{g}' \underline{Y}\right) Cov\left[\underline{b}(\theta)\right] \overset{(H_2)}{=} \left(\underline{Y}_j^{'} - \underline{g}' \underline{Y}\right) \underline{\Lambda}$. As $\underline{\Lambda}$ was assumed to be positive definite, we must have: $\underline{Y}_j^{'} = \underline{g}' \underline{Y} \Leftrightarrow \left(a_1, \ldots, a_q\right) = \left(g_1, \ldots, g_t\right) \begin{pmatrix} Y_{11} & \cdots & Y_{1q} \\ \vdots & \vdots & \vdots \\ Y_{t1} & \cdots & Y_{tq} \end{pmatrix} \Leftrightarrow$

$$\Leftrightarrow \left(a_1, \ldots, a_q\right) = \left(g_1 Y_{11} + \ldots + g_t Y_{t1}, \ldots, g_1 Y_{1q} + \ldots + g_t Y_{tq}\right) \Leftrightarrow a_i = \sum_{j'=1}^{t} g_{j'} Y_{j'i}, i = \overline{1,q} \Leftrightarrow$$

$$\Leftrightarrow a_i - \sum_{j'=1}^{t} g_{j'} Y_{j'i} = 0, i = \overline{1,q} \qquad (2.16),$$

where: $\underline{Y}_j = \left(a_1, \ldots, a_q\right)'$ and $\underline{Y} = \left(Y_{ij'}\right)_{\substack{i=\overline{1,t} \\ j'=\overline{1,q}}}$. Inserting this ($\underline{Y}_j^{'} = \underline{g}' \underline{Y}$) in (2.15) gives: $g_0 = 0$. So: $\dot{\mu}_j = \underline{g}' \underline{X} = \sum_{j'=1}^{t} g_{j'} X_{j'}$, where $\underline{g} = \left(g_1, g_2, \ldots, g_t\right)'$. Thus we have to minimize $E\left\{\left[\mu_j(\theta) - \underline{g}' \underline{X}\right]^2\right\}$ under the constraint (2.16). Let:

$$Q = E\left\{\left[\mu_j(\theta) - \underline{g}' \underline{X}\right]^2\right\} + 2\sum_{i=1}^{q} \psi_i \left(a_i - \sum_{j'=1}^{t} g_{j'} Y_{j'i}\right) = E\left\{\left[\mu_j(\theta) - \sum_{j'=1}^{t} g_{j'} X_{j'}\right]^2\right\} +$$

$+ 2\sum_{i=1}^{q} \psi_i \left(a_i - \sum_{j'=1}^{t} g_{j'} Y_{j'i}\right)$, where $\psi_1, \ldots, \psi_q$ are Lagrange multipliers. By the necessary extreme conditions $\frac{\partial Q}{\partial g_k} = 0, k = \overline{1,t}$ we get: $E\left[\left(\mu_j(\theta) - \sum_{j'=1}^{t} g_{j'} X_{j'}\right) X_k\right] + \sum_{i=1}^{q} \psi_i Y_{ki} = 0, k = \overline{1,t}$ and, by the constraint, $Cov\left[\left(\mu_j(\theta) - \sum_{j'=1}^{t} g_{j'} X_{j'}\right), X_k\right] + \sum_{i=1}^{q} \psi_i Y_{ki} = 0, k = \overline{1,t}$, or in vector form: $Cov\left[\underline{X}, \mu_j(\theta) - \underline{X}' \underline{g}\right] + \underline{Y} \underline{\psi} = 0$, with $\underset{\sim}{\psi} = \left(\psi_1, \ldots, \psi_q\right)'$. We get: $Cov\left[\underline{X}, \mu_j(\theta)\right] - Cov\left[\underline{X}, \underline{X}' \underline{g}\right] + \underline{Y} \underline{\psi} = 0$, or, by using the proof, the notations of

Application 2.3, one obtains: $\underline{Y}\Lambda\underline{a} - (\Phi + \underline{Y}\Lambda\underline{Y}')\underline{g} + \underline{Y}\underline{\psi} = 0$ and, once more using the constraint: $\Phi\underline{g} = \underline{Y}\underline{\psi}$, one obtains: $\underline{g} = \Phi^{-1}\underline{Y}\underline{\psi}$. Whence:

$$\underline{g}' = \underline{\psi}'\underline{Y}'\Phi^{-1} \qquad (2.17),$$

and thus: $\underline{g}'\underline{Y} = \underline{\psi}'\underline{Y}'\Phi^{-1}\underline{Y}$. The constraint now gives: $\underline{Y}_j' \overset{(2.16)}{=} \underline{g}'\underline{Y} = \underline{\psi}'\underline{Y}'\Phi^{-1}\underline{Y}$, that is: $\underline{\psi}' = \underline{Y}_j'\left(\underline{Y}'\Phi^{-1}\underline{Y}\right)^{-1}$. Insertion in (2.17) gives: $\underset{\sim}{g}' = \underset{\sim}{Y}_j'\left(\underset{\sim}{Y}'\Phi^{-1}\underset{\sim}{Y}\right)^{-1}\underset{\sim}{Y}'\Phi^{-1}$, and we get: $\hat{\mu}_j = \underline{g}'\underline{X} = \underline{Y}_j'\left(\underline{Y}'\Phi^{-1}\underline{Y}\right)^{-1}\underline{Y}'\Phi^{-1}\underline{X} \overset{(2.6)}{=} \underline{Y}_j'\hat{\underline{b}}$. This proves the application. □

Remark 2.7

As by the credibility estimators, we directly transfer the results of *Application 2.6*, to estimation of vectors and get that $\hat{\underline{\mu}} = \underline{Y}\hat{\underline{b}}$ is the best linear θ-unbiased estimator of $\underline{\mu}(\theta) = \underline{Y}\underline{b}(\theta)$ and as a special case, that $\hat{\underline{b}}$ is the best linear θ-unbiased estimator of $\underline{b}(\theta)$. This last result gives the following **interpretation** of (2.13).

Application 2.8*:* The credibility estimator $\hat{\underline{b}}$ of $\underset{\sim}{b}(\theta)$ is a weighted mean of the best linear θ-unbiased estimator $\hat{\underline{b}}$ of $\underset{\sim}{b}(\theta)$ and the expectation $\underline{\beta}$ of $\underset{\sim}{b}(\theta)$.

3. The classical model allowing effects like inflation

In this section we will introduce the classical regression model, which consists of a portfolio of k contracts, satisfying the constraints of the simple model.

The contract indexed j is a random vector consisting of a random structure θ_j and observations $X_{j1},...,X_{jt}$. Therefore the contract indexed j is equal to $\left(\theta_j, \underline{X}_j'\right)$, where $\underline{X}_j' = \left(X_{j1},...,X_{jt}\right)$ and $j = \overline{1,k}$ (the variables describing the j^{th} contract are $\left(\theta_j, \underline{X}_j'\right), j = \overline{1,k}$). Just as in **Section 2** we will derive the best linearized regression credibility estimators for this model.

Instead of assuming time independence in the net risk premium:

$$\mu(\theta_j) = E\left(X_{jq} \mid \theta_j\right), j = \overline{1,k}, q = \overline{1,t} \qquad (3.1)$$

one could assume that the conditional expectation of the claims on a contract changes in time, as follows:

$$\mu_q(\theta_j) = E(X_{jq} \mid \theta_j) = y_{jq}\beta(\theta_j),\ j = \overline{1,k}, q = \overline{1,t} \qquad (3.2),$$

with y_{jq} assumed to be known and $\beta(\cdot)$ assumed to be unknown..

Consequence of the hypothesis (3.2):

$$\underline{\mu}^{(t,1)}(\theta_j) = E(\underline{X}_j \mid \theta_j) = x^{(t,n)}\underline{\beta}^{(n,1)}(\theta_j),\ j = \overline{1,k} \qquad (3.3),$$

where $x^{(t,n)}$ is a matrix given in advance, the so-called design matrix, and where the $\underline{\beta}(\theta_j)$ are the unknown regression constants. Again one assumes that for each contract the risk parameters $\underline{\beta}(\theta_j)$ are the same functions of different realizations of the structure parameter.

For some fixed design matrix $x^{(t,n)}$ of full rank n ($n < t$), and a fixed weight matrix $v_j^{(t,t)}$, the hypotheses of the classical model are:

(**H$_1$**) The contracts $(\theta_j, \underline{X}_j^{'})$ are independent; the variables $\theta_1,...,\theta_k$ are independent and identically distributed.

(**H$_2$**) $E(\underline{X}_j^{(t,1)} \mid \theta_j) = x^{(t,n)}\underline{\beta}^{(n,1)}(\theta_j), j = \overline{1,k}$, where $\underline{\beta}$ is an unknown regression vector;

$Cov(\underline{X}_j^{(t,1)} \mid \theta_j) = \sigma^2(\theta_j)v_j^{(t,t)}$, where: $\sigma^2(\theta_j) = Var(X_{jr} \mid \theta_j), \forall r = \overline{1,t}$ and $v_j = v_j^{(t,t)}$ is a known non-random weight $(t \times t)$ matrix, with rank $v_j = t, j = \overline{1,k}$.

We introduce the structural parameters, which are natural extensions of those in the Bühlmann-Straub model. We have:

$$s^2 = E[\sigma^2(\theta_j)] \qquad (3.4)$$

$$a = a^{(n,n)} = Cov[\underline{\beta}(\theta_j)] \qquad (3.5)$$

$$\underline{b} = \underline{b}^{(n,1)} = E[\underline{\beta}(\theta_j)] \qquad (3.6),$$

where $j = \overline{1,k}$.

After the credibility result based on these structural parameters is obtained, one has to construct estimates for these parameters. We use the following notations:

$c_j = c_j^{(t,t)} = Cov(\underline{X}_j)$

$u_j = u_j^{(n,n)} = (x' v_j^{-1} x)^{-1}$

$z_j = z_j^{(n,n)} = a(a + s^2 u_j)^{-1}$ = [the resulting credibility factor for contract j], $j = \overline{1,k}$.

We can now derive the regression credibility results for the estimates of the parameters in the linear model. Multiplying this vector of the estimates by the design matrix provides us with the credibility estimate for $\underline{\mu}(\theta_j)$, see (3.3).

Theorem 3.1: (***Linearized regression credibility premium***)

The best linearized estimate of $E\left[\underline{\beta}^{(n,1)}(\theta_j)\,|\,\underline{X}_j\right]$ is given by:

$$\underline{M}_j = z_j^{(n,n)}\underline{B}_j^{(n,n)} + \left(I - z_j^{(n,n)}\right)\underline{b}^{(n,1)} \qquad (3.7),$$

and the best linearized estimate of $E\left[x^{(t,n)}\underline{\beta}^{(n,1)}(\theta_j)\,|\,\underline{X}_j\right]$ is given by:

$$x^{(t,n)}\underline{M}_j = x^{(t,n)}\left[z_j^{(n,n)}\underline{B}_j^{(n,n)} + \left(I - z_j^{(n,n)}\right)\underline{b}^{(n,1)}\right] \qquad (3.8),$$

where $\underline{B}_j$ is the classical result for the regression vector, namely the GLS–estimator for $\underline{\beta}(\theta_j)$ [if $c_j = s^2 v_j + xax'$], $j = \overline{1,k}$, then the vector $\underline{B}_j$ minimizing the weighted distance to the observations $\underline{X}_j, d(\underline{B}_j) = (\underline{X}_j - x\underline{B}_j)' v_j^{-1}(\underline{X}_j - x\underline{B}_j)$, can be written as:

$\underline{B}_j = (x'v_j^{-1}x)^{-1}x'v_j^{-1}\underline{X}_j = u_j x'v_j^{-1}\underline{X}_j = (x'c_j^{-1}x)^{-1}x'c_j^{-1}\underline{X}_j$, in case.

For the proof see [11].

Remark 3.2

From (3.7) we see that the credibility estimates for the parameters of the linear model are given as the matrix version of a convex mixture of the classical regression result $\underline{B}_j$ and the collective result $\underline{b}$.

Theorem 3.1 concerns a special contract j. By the assumption, the structural parameters $a, \underline{b}$ and s^2 do not depend on j. So if there are more contracts, these parameters can be estimated.

Every vector $\underline{B}_j$ gives an unbiased estimator of $\underline{b}$. Consequently, so does every linear combination of type $\sum_j \alpha_j \underline{B}_j$, where the vector of matrices $\left(\alpha_j^{(n,n)}\right)_{j=\overline{1,k}}$, is such that:

$$\sum_{j=1}^{k} \alpha_j^{(n,n)} = I^{(n,n)} \qquad (3.9).$$

The optimal choice of $\alpha_j^{(n,n)}$ is determined in the following **application**:

Theorem 3.3: (***Estimation of the parameters*** $\underline{b}$ ***in the regression credibility model***)

The optimal solution to the problem:

$$\underset{\underline{\alpha}}{Min}\, d(\underline{\alpha}) \qquad (3.10),$$

where: $d(\underline{\alpha}) = \left\| \underline{b} - \sum_j \alpha_j \underline{B}_j \right\|_P^2 \overset{def.}{=} E\left[\left(\underline{b} - \sum_j \alpha_j \underline{B}_j\right)' P \left(\underline{b} - \sum_j \alpha_j \underline{B}_j\right)\right] =$ (the distance from $\left(\sum_j \alpha_j \underline{B}_j\right)$ to the parameters $\underline{b}$), $P = P^{(n,n)}$ a given positive definite matrix, with the vector of matrices $\underline{\alpha} = (\alpha_j)_{j=\overline{1,k}}$ satisfying (3.9), is:

$$\hat{\underline{b}}^{(n,1)} = Z^{-1} \sum_{j=1}^{k} z_j \underline{B}_j \qquad (3.11),$$

where: $Z = \sum_{j=1}^{k} z_j$, and: z_j is defined as: $z_j = a(a + s^2 u_j)^{-1}, j = \overline{1,k}$.

For the proof see [11].

Theorem 3.4: (***Unbiased estimator for*** s^2 ***foe each contract group***)

In case the number of observations t_j in the j^{th} contract is larger than the number of regression constants n, the following is an unbiased estimator of s^2:

$$\hat{s}_j^2 = \frac{1}{t_j - n} \left(\underline{X}_j - x_j \underline{B}_j\right)' v_j^{-1} \left(\underline{X}_j - x_j \underline{B}_j\right) \qquad (3.12).$$

For the proof see [11].

Corollary 3.5: (***Unbiased estimator for*** s^2 ***in the regression model***)

Let K denote the number of contracts j, with $t_j > n$. Then $E\left(\hat{s}^2\right) = s^2$, if:

$$\hat{s}^2 = \frac{1}{K} \sum_{j; t_j > n} \hat{s}_j^2 \qquad (3.13).$$

For a, we give an unbiased pseudo-estimator, defined in terms of itself, so it can only be computed iteratively:

Theorem 3.6: (***Pseudo-estimator for a***)

The following random variable has expected value a:

$$\hat{a} = \frac{1}{k-1}\sum_j z_j\left(\underline{B}_j - \hat{\underline{b}}\right)\left(\underline{B}_j - \hat{\underline{b}}\right)' \quad (3.14).$$

For the proof see [11].

Remark 3.7

Another unbiased estimator for a is the following:

$$\hat{a} = 1/\left(w_{\cdot}^2 - \sum w_j^2\right)\left\{\frac{1}{2}\sum_{i,j} w_i w_j \left(\underline{B}_i - \underline{B}_j\right)\left(\underline{B}_i - \underline{B}_j\right)' - \hat{s}^2 \sum_{j=1}^{k} w_j \left(w_{\cdot} - w_j\right) u_j\right\}, \quad (3.15),$$

where w_j is the volume of the risk for the j^{th} contract, $j = \overline{1,k}$ and $w_{\cdot} = \sum_{j=1}^{k} w_j$.

Observation 3.8

This estimator is a statistic; it is not a pseudo-estimator. Still, the reason to prefer (3.14) is that this estimator can easily be generalized to multi-level hierarchical models. In any case, the unbiasedness of the credibility premium disappears even if one takes (3.15) to estimate a .

4. Conclusions

The article contains a credibility solution in the form of a linear combination of the individual estimate (based on the data of a particular state) and the collective estimate (based on aggregate USA data). This idea is worked out in regression credibility theory.

In case there is an increase (for instance by inflation) of the results on a portfolio, the risk premium could be considered to be a linear function in time of the type $\beta_0(\theta) + t\beta_1(\theta)$. Then two parameters $\beta_0(\theta)$ and $\beta_1(\theta)$ must be estimated from the observed variables. This kind of problem is named regression credibility. This model arises in cases where the risk premium depends on time, e.g. by inflation. One can assume that a linear effect on the risk premium is a good approximation to the real growth, as is also the case in time series analysis.

These regression models can be generalized to get credibility models for general regression models, where the risk is characterized by outcomes of other related variables.

This paper contains a description of the regression model allowing for effects like inflation. If there is an effect of inflation, it is contained in the claim figures, so one should use estimates based on these figures instead of external data. This can be done using regression model.

In this article the regression credibility result for the estimate of the parameters in the linear model is derived. After the credibility result based on the structural parameters is obtained, one has to construct estimates for these parameters.

The mathematical theory provides the means to calculate useful estimators for the structure parameters.

The property of unbiasedness of these estimators is very appealing and very attractive from the practical point of view.

The fact that it is based on complicated mathematics, involving conditional expectations and conditional covariances, needs not bother the user more than it does when he applies statistical tools like discriminant analysis, scoring models, SAS and GLIM.

In this paper we mixed all the elements from the reference for the credibility theory into some examples, which leads to the credibility formulas allowing effects like inflation and which can be applied in order to solve a number of practical insurance problems. The goal of these examples is to illustrate how practical problems can be translated into problems that can be handled through the credibility formulas allowing effects like inflation.

R E F E R E N C E S

[1] *V. Atanasiu*, Useful applications of the credibility theory, Metalurgia International (journal ISI), no. 4 special issue, **vol. XIV**(2009), 2009, pp. 22-28.
[2] *V. .Atanasiu*, Techniques for estimating the premiums for the risks of the insurance companies in Romania, Metalurgia International (journal ISI), no. 7, **vol. XIV** (2009), 2009, pp. 61-66.
[3] *V. .Atanasiu*, The calculations of credibility in the hierarchical model with two-levels, Metalurgia International (journal ISI), no. 4 special issue, **vol. XIV**(2009), 2009, pp. 118-123.
[4] *V. Atanasiu*, The semi-linear credibility model, U.P.B. Sci. Bull., Series A, No. 2, **Vol. 69**, 2007, pp. 49-60.
[5] *V. Atanasiu*, Estimation of the structural parameters in the general semi-linear credibility model, U.P.B. Sci. Bull., Series A, No. 4, **Vol. 69**, 2007, pp. 31-42.
[6] *V. Atanasiu*, Semi-linear credibility, Economic Computation and Economic Cybernetics Studies and Research (journal ISI), Number 3-4/2007, **Volume 41**, 2007, pp. 213-224.
[7] *C. D. Daykin, T. Pentikäinen, M. Pesonen,* Practical Risk Theory for Actuaries**,** Chapman & Hall, 1993.
[8] *F. De Vylder,* Optimal semilinear credibility**,** Mitteilungen der VSVM, 76, 1976, pp. 17-40.
[9] *F. De Vylder & M. J. Goovaerts,* Semilinear credibility with several approximating functions, Insurance: Mathematics and Economics, no. 4, 1985, pp. 155-162. (Zbl.No. 0167-6687).
[10] *H. U. Gerber,* Credibility for Esscher premiums**,** Mitteilungen der VSVM, 80, no. 3, 1980, pp. 307-312.
[11] *M. J. Goovaerts, R.. Kaas*, *A. E. Van Heerwaarden, T. Bauwelinckx,* Effective Actuarial Methods, **vol. 3**, Elsevier Science Publishers B.V., 1990, pp. 175-187.
[12] *R. V. Hogg* & *S .A Klugman,,* Loss distributions**,** John Wiley and Sons, New York, 1984.
[13] *B. Sundt,* An Introduction to Non-Life Insurance Mathematics, volume of the "Mannheim Series", 1984, pp. 22-54.