HOMOTOPY PERTURBATION SOLUTION FOR FLOW OF A THIRD-GRADE FLUID IN HELICAL SCREW RHEOMETER

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This paper provides a theoretical study of steady flow of an incompressible third grade fluid in helical screw rheometer. The model developed in cylindrical coordinates pertains to second order nonlinear coupled differential equations that are solved using homotopy perturbation method. Expressions for velocity components in θ and $z-$direction are obtained. The volume flow rates are calculated for the azimuthal and axial components of velocity profiles by introducing the effect of flights. The results have been discussed with the help of graphs. It is noticed that extrusion process depends on the involved non-Newtonian parameter and pressure gradients.

Keywords: Helical Screw Rheometer, Homotopy perturbation method, Second order nonlinear coupled differential equations.

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1. **Introduction**

Extrusion process is widely used in food processing. Food processing is the set of methods and techniques used to transform raw ingredients into food or to transform food into other forms for consumption by humans or animals either at home or by the food processing industry. Food processing typically takes clean, harvested crops or butchered animal products and uses these to produce attractive, marketable and often long shelf-life food products. Various food items such as cookie dough, sevai, pastas, breakfast cereals, French fries, baby food, ready to eat snacks and dry pet food are most commonly manufactured using the extrusion process. The fluids used in the extrusion process are mostly non-Newtonian.

The classical Navier-Stokes equations have been proved inadequate to describe complete characteristics of non-Newtonian fluids. To study these fluids different models have been proposed [1, 2] called constitutive equations. In the literature there exist rare exact solutions for these constitutive equations of non-Newtonian fluids. This is because such equations are highly nonlinear. For the solution of such

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complicated equations, analytical techniques are mostly used to obtain the approximate solutions [6].

Literature survey show that Carley et. al.[7], Mohr and Mallouk [8], Booy[4], Squires [9], Tadmor and Klein [10], Tadmor and Gogos [11], Rauwendaal [12], analyze Newtonian and power law fluids in the geometry of Single screw extruder. Tamura et. al.[5], had done successfully, the preceding analysis in the geometry of Helical Screw Rheometer, for Newtonian and power law fluids.

Recently, non-Newtonian fluids are become of great importance due to their wide use in food industry, chemical process industry, construction engineering, power engineering, petroleum production, commercial and technological applications etc. These applications are strong motivations to study the flow of non-Newtonian fluids in Helical Screw Rheometer (HSR). In the present work for simplicity we take the fluid based on third-grade model. For this we choose the cylindrical coordinate system (r, θ, z) which seems to be a more natural choice due to the geometry of HSR. The expressions for the $v-$component and $w-$component of velocity profiles are obtained from the solution of developed second order nonlinear coupled differential equations by using homotopy perturbation method [6, 13, 14, 15, 16, 17]. Volume flow rates are calculated by introducing the effect of flights. The behavior of the velocity profiles are presented through graphs and discussed.

The paper is organized as follows. Section 2 contains the governing equations of the fluid model. In Section 3 the problem under consideration is formulated. Section 4, devoted to the description of homotopy perturbation method. Section 5, concerns with the solution of the problem. In Section 6 discussion about the behavior of the velocity profiles is given. Section 7 contains conclusion.

2. **Basic Equations**

The basic equations governing the motion of an isothermal, homogeneous and incompressible fluid are:

$$div\mathbf{V} = 0, \tag{1}$$

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{f} + div\mathbf{T}, \tag{2}$$

where ρ is the constant fluid density, $\mathbf{V}$ is the velocity vector, $\mathbf{f}$ is the body force per unit mass, $\frac{D}{Dt}$ denotes the material time derivative, and $\mathbf{T}$ is the Cauchy stress tensor expressed as

$$\mathbf{T} = -P\mathbf{I} + \mathbf{S}, \tag{3}$$

where P denotes the dynamic pressure, $\mathbf{I}$ the unit tensor and $\mathbf{S}$ denotes the extra stress tensor. For third grade fluid $\mathbf{S}$ is given by [3]

$$\mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1, \tag{4}$$

where μ is the coefficient of shear viscosity, $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2$ and β_3 are the material constants and

$$\mathbf{A}_1 = (grad\mathbf{V}) + (grad\mathbf{V})^T,$$
$$\mathbf{A}_{n+1} = \frac{D\mathbf{A}_n}{Dt} + [\mathbf{A}_n(grad\mathbf{V}) + (grad\mathbf{V})^T\mathbf{A}_n], \qquad (n = 1, 2),$$

are the first three Rivlin-Ericksen tensors.

3. Problem Formulation

Consider steady flow of an incompressible, homogeneous and isothermal third grade fluid through a Helical Screw Rheometer (HSR). The screwed channel is assumed to be bounded by the barrel and screw root surfaces and by the two sides of a helical flight as shown in Fig.1. The geometry is approximated as a shallow infinite channel, by assuming the width B of the channel large compared with the depth h i.e., $\frac{h}{B} << 1$. So that the side effects can be ignored. We choose the cylindrical coordinate system (r, θ, z) which is more suitable choice for the flow analysis in HSR. A congruent velocity distribution is assumed at parallel cross sections through the channel. We also assumed that the flow is uniform, laminar and viscosity of the fluid is constant. The outer barrel of radius r_2 is assumed to be stationary and the screw root of radius r_1 rotates with angular velocity Ω [4].

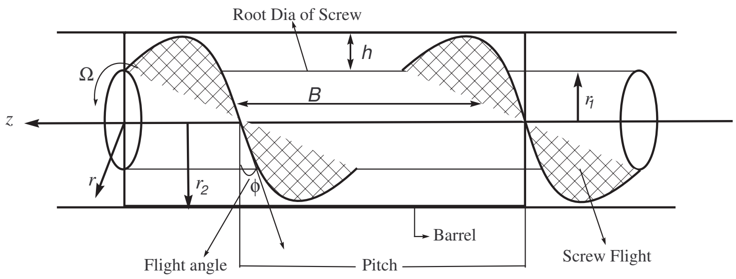

Figure 1: Geometry of the problem.

The boundary conditions are

$$\begin{array}{lllll} v = \Omega r_1, & w = 0, & \text{at} & r = r_1, \\ v = 0, & w = 0, & \text{at} & r = r_2. \end{array} \tag{5}$$

The flow is assumed fully developed in the θ and the $z-$directions so that

$$\mathbf{V} = [0, v(r), w(r)], \qquad \mathbf{S} = \mathbf{S}(r), \tag{6}$$

where v and w are azimuthal and axial velocity components, respectively. For highly viscous fluids the effect of acceleration of fluid and body forces can be ignored [4]. Equation (2) for slow flow becomes

$$0 = div\mathbf{T} = div(-P\mathbf{I} + \mathbf{S}). \tag{7}$$

The assumption $\dfrac{h}{B} << 1$ and the congruent velocity distribution at parallel cross sections, imply $\dfrac{\partial P}{\partial r} = 0$ [4].
In view of our assumption (6) equation (1) is satisfied identically and (7) in its component form results in

$$0 = \frac{1}{r}\frac{d}{dr}\left[r\left\{(2\alpha_1+\alpha_2)M\right\}\right] - \frac{\alpha_2}{r}\left(\frac{dv}{dr}-\frac{v}{r}\right)^2, \tag{8}$$

$$\frac{1}{r}\frac{\partial P}{\partial \theta} = \frac{1}{r^2}\frac{d}{dr}\left[r^2\left\{\mu+2(\beta_2+\beta_3)M\right\}\left(\frac{dv}{dr}-\frac{v}{r}\right)\right], \tag{9}$$

$$\frac{\partial P}{\partial z} = \frac{1}{r}\frac{d}{dr}\left[r\left\{\mu+2(\beta_2+\beta_3)M\right\}\frac{dw}{dr}\right]. \tag{10}$$

where $M = \left(\dfrac{dv}{dr}-\dfrac{v}{r}\right)^2 + \left(\dfrac{dw}{dr}\right)^2$. The equations (9) and (10), imply that $P = P(\theta, z)$, but the right sides of equations (9) and (10) are functions of r alone and $P \neq P(r)$, means that $\dfrac{\partial P}{\partial \theta} =$ constant and $\dfrac{\partial P}{\partial z} =$ constant. Our concentration is on azimuthal and axial flow, so only equations (9) and (10) are considered.
By introducing dimensionless parameters

$$r^* = \frac{r}{r_1},\quad z^* = \frac{z}{r_1},\quad v^* = \frac{v}{\Omega r_1},\ w^* = \frac{w}{\Omega r_1},\quad P^* = \frac{P}{\mu(\Omega)},$$

in equations (9) and (10), yield after dropping “ $*$ ”,

$$\frac{d}{dr}\left[r^2\left\{1+\beta\left(\left(\frac{dv}{dr}-\frac{v}{r}\right)^2+\left(\frac{dw}{dr}\right)^2\right)\right\}\left(\frac{dv}{dr}-\frac{v}{r}\right)\right] = rP_{,\theta}, \tag{11}$$

$$\frac{d}{dr}\left[r\left\{1+\beta\left(\left(\frac{dv}{dr}-\frac{v}{r}\right)^2+\left(\frac{dw}{dr}\right)^2\right)\right\}\frac{dw}{dr}\right] = rP_{,z}, \tag{12}$$

where $\beta = \dfrac{2(\beta_2+\beta_3)\Omega^2}{\mu}$, $P_{,\theta} = \dfrac{\partial P}{\partial \theta}$ and $P_{,z} = \dfrac{\partial P}{\partial z}$, and boundary conditions (5) become,

$$\begin{array}{llll} v = 1, & w = 0, & \text{at} & r = 1, \\ v = 0, & w = 0, & \text{at} & r = \delta, \end{array} \tag{13}$$

where $\delta = \dfrac{r_2}{r_1} > 1$.
Equations (11) and (12) are coupled second order nonlinear ordinary differential equations, the exact solution seems to be difficult. We use homotopy perturbation method (HPM) to obtain approximate solution by using the symbolic computation software Wolfram Mathematica 7.

4. Homotopy Perturbation Method

To illustrate the basic idea of this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{14}$$

with the boundary condition

$$\wp\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma, \tag{15}$$

where A is a general differential operator, $\wp$ a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω and $\frac{\partial}{\partial n}$ denotes differentiation along the normal drawn outwards from Ω. The operator A can be divide into two parts of G and N, where G is the linear operator, while N is a nonlinear one. Equation (14) can, therfore, be rewritten as:

$$G(v) + N(v) - f(r) = 0. \tag{16}$$

By the homotpy technique, we construct a homotopy as $v(r,p) : \Omega \times [0,1] \longrightarrow \Re$ which satisfies:

$$H(v,p) = (1-p)[G(v) - G(u_0)] + p[G(v) + N(v) - f(r)] = 0, \qquad p \in [0,1],\ r \in \Omega, \tag{17}$$

or

$$H(v,p) = G(v) - G(u_0) + p[G(u_0) + N(v) - f(r)] = 0, \tag{18}$$

where $p \in [0,1]$, is an embedding parameter and u_0 is an initial approximation which satisfies the boundary conditions. Now equation (18) implies

$$\begin{aligned} H(v,0) &= G(v) - G(u_0) = 0, \\ H(v,1) &= G(v) + N(v) - f(r) = 0, \end{aligned}$$

the changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $G(v)-G(u_0)$ and $G(v)+N(v)-f(r)$ are called homotopic. Here the embedding parameter $p \in [0,1]$ is introduced much more naturally, unaffected by artificial factors. So the solution of (17) can be written as a power series in p [17] :

$$v = \sum_{i=0}^{\infty} p^i v_i = v_0 + pv_1 + p^2 v_2 + \cdots \tag{19}$$

As $p \to 1$, approximate solution of (17) becomes

$$u = \lim_{p \longrightarrow 1} v = v_0 + v_1 + v_2 + \cdots \tag{20}$$

5. **Solution of the problem**

With the help of equation (18) equations (11-12) can be written in the form

$$G_1(v) - G_1(v_\theta) + pG_1(v_\theta) + p\left[\beta \frac{d}{dr}\left\{r^2\left(\frac{dv}{dr} - \frac{v}{r}\right)^3 + r^2\left(\frac{dw}{dr}\right)^2\left(\frac{dv}{dr} - \frac{v}{r}\right)\right\} - rP_{,\theta}\right] = 0, \tag{21}$$

$$G_2(w) - G_2(w_z) + pG_2(w_z) + p\left[\beta \frac{d}{dr}\left\{r\left(\frac{dv}{dr} - \frac{v}{r}\right)^2 \frac{dw}{dr} + r\left(\frac{dw}{dr}\right)^3\right\} - rP_{,z}\right] = 0. \tag{22}$$

Here $G_1 = r^2\frac{d^2}{dr^2}+r\frac{d}{dr}-1$ and $G_2 = r\frac{d^2}{dr^2}+\frac{d}{dr}$ are linear operators and let us take

$$v_\theta = \frac{\Theta_1}{r}+\Theta_2 r+\Theta_3 r\ln(r), \tag{23}$$
$$w_z = \Psi_1+\Psi_1 r^2+\Psi_2\ln(r), \tag{24}$$

as initial guess approximations where Θ_1 Θ_2, Θ_3, Ψ_1 and Ψ_2 are constant coefficients. On substituting series (19) equations (21) and (22) become

$$G_1\left(\sum_{i=0}^{\infty}p^i v_i\right)-G_1(v_\theta)+pG_1(v_\theta)+p\left[\beta\frac{d}{dr}\left\{r^2\left(\frac{d}{dr}\left(\sum_{i=0}^{\infty}p^i v_i\right)-\frac{1}{r}\sum_{i=0}^{\infty}p^i v_i\right)^3\right.\right.$$
$$\left.\left.+r^2\left(\frac{d}{dr}\left(\sum_{i=0}^{\infty}p^i w_i\right)\right)^2\left(\frac{d}{dr}\left(\sum_{i=0}^{\infty}p^i v_i\right)-\frac{1}{r}\sum_{i=0}^{\infty}p^i v_i\right)\right\}-rP_{,\theta}\right]=0, \tag{25}$$
$$G_2\left(\sum_{i=0}^{\infty}p^i w_i\right)-G_2(w_z)+pG_2(w_z)+p\left[\beta\frac{d}{dr}\left\{r\left(\frac{d}{dr}\left(\sum_{i=0}^{\infty}p^i v_i\right)\right.\right.\right.$$
$$\left.\left.\left.-\frac{1}{r}\sum_{i=0}^{\infty}p^i v_i\right)^2\frac{d}{dr}\left(\sum_{i=0}^{\infty}p^i w_i\right)+r\left(\frac{d}{dr}\left(\sum_{i=0}^{\infty}p^i w_i\right)\right)^3\right\}-rP_{,z}\right]=0, \tag{26}$$

and the boundary conditions (13) become

$$\sum_{i=0}^{\infty}p^i v_i = 1, \qquad \sum_{i=0}^{\infty}p^i w_i = 0, \qquad \text{at} \qquad r=1, \tag{27}$$
$$\sum_{i=0}^{\infty}p^i v_i = 0, \qquad \sum_{i=0}^{\infty}p^i w_i = 0, \qquad \text{at} \qquad r=\delta. \tag{28}$$

5.1. Zeroth order problem

Zeroth order linear differential equations

$$G_1(v_0)-G_1(v_\theta)=0, \tag{29}$$
$$G_2(w_0)-G_2(w_z)=0, \tag{30}$$

together with boundary conditions

$$v_0(1)=1 \qquad w_0(1)=0, \qquad v_0(\delta)=w_0(\delta)=0,$$

has the solution

$$v_0 = \frac{\Theta_1}{r}+\Theta_2 r+\Theta_3 r\ln(r), \tag{31}$$
$$w_0 = \Psi_1+\Psi_1 r^2+\Psi_2\ln(r), \tag{32}$$

where Θ_1, Θ_2, Θ_3, Ψ_1 and Ψ_2 are constant coefficients.

5.2. First order problem

First order linear differential equations

$$G_1(v_1)+G_1(v_\theta) +\beta\frac{d}{dr}\left\{r^2\left(\frac{dv_0}{dr}-\frac{v_0}{r}\right)^3+r^2\left(\frac{dw_0}{dr}\right)^2\left(\frac{dv_0}{dr}-\frac{v_0}{r}\right)\right\}-rP_{,\theta}=0, \quad (33)$$

$$G_2(w_1)+G_2(w_z)+\beta\frac{d}{dr}\left\{r\left(\frac{dv_0}{dr}-\frac{v_0}{r}\right)^2\frac{dw_0}{dr}+r\left(\frac{dw_0}{dr}\right)^3\right\}-rP_{,z}=0, \quad (34)$$

along with boundary conditions

$$v_1(1)=w_1(1)=0, \qquad v_1(\delta)=w_1(\delta)=0,$$

result in

$$v_1 = \beta\left(\frac{\Theta_4}{r^5}+\frac{\Theta_5}{r^3}+\frac{\Theta_6}{r}+\Theta_7 r+\Theta_8 r\ln(r)+\Theta_9 r^3\right), \quad (35)$$

$$w_1 = \beta\left(\frac{\Psi_3}{r^4}+\frac{\Psi_4}{r^2}+\Psi_5+\Psi_6\ln(r)+\Psi_7\ln(r)^2+\Psi_8 r^2+\Psi_9 r^4\right), \quad (36)$$

where $\Theta_i,\ \Psi_j,\ i=4,\cdots,9,\ j=3,\cdots,9$ are constant coefficients.

5.3. Second order problem

Second order linear differential equations

$$G_1(v_2)+\beta\frac{d}{dr}\left\{3r^2\left(\frac{dv_0}{dr}-\frac{v_0}{r}\right)^2\left(\frac{dv_1}{dr}-\frac{v_1}{r}\right)+2r^2\left(\frac{dw_0}{dr}\right)\left(\frac{dw_1}{dr}\right)\left(\frac{dv_0}{dr}-\frac{v_0}{r}\right)\right. \\ \left.+r^2\left(\frac{dw_0}{dr}\right)^2\left(\frac{dv_1}{dr}-\frac{v_1}{r}\right)\right\}=0, \quad (37)$$

$$G_2(w_2)+\beta\frac{d}{dr}\left\{r\left(\frac{dv_0}{dr}-\frac{v_0}{r}\right)^2\frac{dw_1}{dr}+2r\left(\frac{dv_0}{dr}-\frac{v_0}{r}\right)\left(\frac{dv_1}{dr}-\frac{v_1}{r}\right)\frac{dw_0}{dr}\right. \\ \left.+3r\left(\frac{dw_0}{dr}\right)^2\left(\frac{dw_1}{dr}\right)\right\}=0, \quad (38)$$

together with their corresponding boundary conditions

$$v_2(1)=w_2(1)=0, \qquad v_2(\delta)=w_2(\delta)=0.$$

Solving the above in conjunction with corresponding boundary conditions give

$$
\begin{aligned}
v_2 &= \beta^2 \left(\frac{\Theta_{10}}{r^9} + \frac{\Theta_{11}}{r^7} + \frac{\Theta_{12}}{r^5} + \frac{\Theta_{13}}{r^3} + \frac{\Theta_{14}}{r^3}\ln(r) + \frac{\Theta_{15}}{r} + \frac{\Theta_{16}}{r}\ln(r) \right.\\
&+ \left. \Theta_{17} r + \Theta_{18} r\ln(r) + \Theta_{19} r\ln(r)^2 + \Theta_{20} r^3 + \Theta_{21} r^5 \right), \qquad (39)\\
w_2 &= \beta^2 \left(\frac{\Psi_{10}}{r^9} + \frac{\Psi_{11}}{r^8} + \frac{\Psi_{12}}{r^7} + \frac{\Psi_{13}}{r^6} + \frac{\Psi_{14}}{r^5} + \frac{\Psi_{15}}{r^5}\ln(r) + \frac{\Psi_{16}}{r^4} + \frac{\Psi_{17}}{r^4}\ln(r) \right.\\
&+ \frac{\Psi_{18}}{r^3} + \frac{\Psi_{19}}{r^3}\ln(r) + \frac{\Psi_{20}}{r^2} + \frac{\Psi_{21}}{r^2}\ln(r) + \frac{\Psi_{22}}{r} + \frac{\Psi_{23}}{r}\ln(r) + \Psi_{24}\\
&+ \Psi_{25}\ln(r) + \Psi_{26}\ln(r)^2 + \Psi_{27}\ln(r)^3 + \Psi_{28} r + \Psi_{29} r^2\\
&+ \left. \Psi_{30} r^2\ln(r) + \Psi_{31} r^3 + \Psi_{32} r^4 + \Psi_{33} r^6 \right), \qquad (40)
\end{aligned}
$$

where $\Theta_i,\ \Psi_j,\ i = 10, \cdots, 21,\ j = 10, \cdots, 33$ are constant coefficients.

5.4. Velocity profile

5.4.1. *Velocity profile in $\theta-$direction.* Considering equations (31), (35) and (39) the HPM solution for the velocity profile in the $\theta-$direction upto second order is,

$$
\begin{aligned}
v &= (\Theta_1 + \beta\Theta_6 + \beta^2\Theta_{15})\frac{1}{r} + (\Theta_2 + \beta\Theta_7 + \beta^2\Theta_{17})r + (\Theta_3 + \beta\Theta_8 + \beta^2\Theta_{18})r\ln(r)\\
&+ (\beta\Theta_4 + \beta^2\Theta_{12})\frac{1}{r^5} + (\beta\Theta_5 + \beta^2\Theta_{13})\frac{1}{r^3} + (\beta\Theta_9 + \beta^2\Theta_{20})r^3\\
&+ \beta^2\frac{\Theta_{10}}{r^9} + \beta^2\frac{\Theta_{11}}{r^7} + \beta^2\frac{\Theta_{14}}{r^3}\ln(r) + \beta^2\frac{\Theta_{16}}{r}\ln(r)\\
&+ \beta^2\Theta_{19} r\ln(r)^2 + \beta^2\Theta_{21} r^5. \qquad (41)
\end{aligned}
$$

5.4.2. *Velocity profile in $z-$direction.* Equations (32), (36) and (40) give the HPM solution for the velocity profile in the $z-$direction upto second order as,

$$
\begin{aligned}
w &= (\Psi_1 + \beta\Psi_5 + \beta^2\Psi_{24}) + (\Psi_1 + \beta\Psi_8 + \beta^2\Psi_{29})r^2 + (\Psi_2 + \beta\Psi_6 + \beta^2 G_{25})\ln(r)\\
&+ \left(\beta\Psi_3 + \beta^2\Psi_{16}\right)\frac{1}{r^4} + (\beta\Psi_4 + \beta^2 G_{20})\frac{1}{r^2} + (\beta\Psi_7 + \beta^2\Psi_{26})\ln(r)^2\\
&+ (\beta\Psi_9 + \beta^2\Psi_{32})r^4 + \beta^2\frac{\Psi_{10}}{r^9} + \beta^2\frac{\Psi_{11}}{r^8} + \beta^2\frac{\Psi_{12}}{r^7} + \beta^2\frac{\Psi_{13}}{r^6} + \beta^2\frac{\Psi_{14}}{r^5}\\
&+ \beta^2\frac{\Psi_{15}}{r^5}\ln(r) + \beta^2\frac{\Psi_{17}}{r^4}\ln(r) + \beta^2\frac{\Psi_{18}}{r^3} + \beta^2\frac{\Psi_{19}}{r^3}\ln(r) + \beta^2\frac{\Psi_{21}}{r^2}\ln(r)\\
&+ \beta^2\frac{\Psi_{22}}{r} + \beta^2\frac{\Psi_{23}}{r}\ln(r) + \beta^2\Psi_{27}\ln(r)^3 + \beta^2\Psi_{28} r\\
&+ \beta^2\Psi_{30} r^2\ln(r) + \beta^2\Psi_{31} r^3 + \beta^2\Psi_{33} r^6. \qquad (42)
\end{aligned}
$$

5.5. Volume flow rate in θ-direction

Volume flow rate in dimensionless form is

$$
Q^*_\theta = 2\pi\delta\tan\phi\int_1^\delta v\,dr, \qquad \text{where} \qquad Q^*_\theta = \frac{Q_\theta}{\Omega r_1^3}. \qquad (43)
$$

Dropping " $*$ " we get

$$\begin{aligned} Q_\theta &= 2\pi\delta\tan\phi\Big\{(\Theta_1+\beta\Theta_6+\beta^2\Theta_{15})ln\delta+\frac{1}{2}(\Theta_2+\beta\Theta_7+\beta^2\Theta_{17})(\delta^2-1)\\ &- \frac{1}{4}(\Theta_3+\beta\Theta_8+\beta^2\Theta_{18})(\delta^2-1)+\frac{1}{2}(L_3+\beta\Theta_8+\beta^2\Theta_{18})\delta^2\ln\delta\\ &- \frac{1}{4}(\beta\Theta_4+\beta^2\Theta_{12})(\frac{1}{\delta^4}-1)-\frac{1}{2}(\beta\Theta_5+\beta^2\Theta_{13})(\frac{1}{\delta^2}-1)\\ &+ \frac{1}{4}(\beta\Theta_9+\beta^2\Theta_{20})(\delta^4-1)-\beta^2\frac{\Theta_{10}}{8}(\frac{1}{\delta^8}-1)-\beta^2\frac{\Theta_{11}}{6}(\frac{1}{\delta^6}-1)\\ &- \beta^2\frac{\Theta_{14}}{4}(\frac{1}{\delta^6}-1)-\beta^2\Theta_{14}\frac{\ln\delta}{8\delta^2}+\beta^2\frac{\Theta_{16}}{2}\ln\delta^2-\beta^2\frac{\Theta_{19}}{2}(\delta^2-1)\\ &+ \beta^2\frac{\Theta_{19}}{2}\delta^2\ln(\delta^2)+\beta^2\frac{\Theta_{21}}{6}(\delta^6-1)\Big\}. \end{aligned} \tag{44}$$

5.6. Volume flow rate in z-direction

Dimensionless volume flow rate in z-direction is

$$Q_z^* = 2\pi\int_1^\delta wrdr, \qquad \text{where} \qquad Q_z^* = \frac{Q_z}{\Omega r_1^3} \tag{45}$$

Now, dropping " $*$ " we get

$$\begin{aligned} Q_z &= 2\pi\Big\{\frac{1}{2}(\Psi_1+\beta\Psi_5+\beta^2\Psi_{24})(\delta^2-1)+\frac{1}{4}(\Psi_1+\beta\Psi_8+\beta^2\Psi_{29})(\delta^4-1)\\ &- \frac{1}{4}(\Psi_2+\beta\Psi_6+\beta^2\Psi_{25})(\delta^2-1)-\frac{1}{2}(\Psi_2+\beta\Psi_6+\beta^2\Psi_{25})\delta^2\ln\delta\\ &- \frac{1}{2}(\beta\Psi_3+\beta^2\Psi_{16})(\frac{1}{\delta^2}-1)+(\beta\Psi_4+\beta^2\Psi_{20})\ln\delta-\frac{1}{2}(\beta\Psi_7+\beta^2\Psi_{26})(\delta^2-1)\\ &+ \frac{1}{2}(\beta\Psi_7+\beta^2\Psi_{26})\delta^2\ln\delta^2+\frac{1}{6}(\beta\Psi_9+\beta^2\Psi_{32})(\delta^6-1)-\beta^2\frac{\Psi_{10}}{7}(\frac{1}{\delta^7}-1)\\ &- \beta^2\frac{\Psi_{11}}{6}(\frac{1}{\delta^6}-1)-\beta^2\frac{\Psi_{12}}{5}(\frac{1}{\delta^5}-1)-\beta^2\frac{\Psi_{13}}{4}(\frac{1}{\delta^4}-1)-\beta^2\frac{\Psi_{14}}{3}(\frac{1}{\delta^3}-1)\\ &- \beta^2\frac{\Psi_{15}}{9}(\frac{1}{\delta^3}-1)-\beta^2\frac{\Psi_{15}}{3}\frac{\ln\delta}{\delta^3}-\beta^2\frac{\Psi_{17}}{4}(\frac{1}{\delta^2}-1)-\beta^2\frac{\Psi_{17}}{2}\frac{\ln\delta}{\delta^2}\\ &- \beta^2\Psi_{18}(\frac{1}{\delta}-1)-\beta^2\Psi_{19}(\frac{1}{\delta}-1)-\beta^2\Psi_{19}\frac{\ln\delta}{\delta}+\beta^2\frac{\Psi_{21}}{2}\ln\delta^2+\beta^2\Psi_{22}(\delta-1)\\ &- \beta^2\Psi_{23}(\delta-1)+\beta^2\Psi_{23}\delta\ln\delta-\frac{3}{4}\beta^2\Psi_{27}(\delta^2-1)+\beta^2\frac{\Psi_{27}}{2}\delta^2\ln\delta^3\\ &+ \beta^2\frac{\Psi_{28}}{3}(\delta^3-1)-\beta^2\frac{\Psi_{30}}{16}(\delta^4-1)+\beta^2\frac{\Psi_{30}}{4}\delta^4\ln\delta\\ &+ \beta^2\frac{\Psi_{31}}{5}(\delta^5-1)+\beta^2\frac{\Psi_{33}}{8}(\delta^8-1)\Big\}. \end{aligned} \tag{46}$$

6. Results and Discussion

In the present work we have considered steady flow of an incompressible third grade fluid through HSR. We obtained coupled second order nonlinear ODEs. Using HPM expressions for azimuthal and axial velocity components are derived. The volume flow rates in θ and $z-$directions are also calculated. Here we discussed the effect of involved flow parameters on the velocity profiles with the help of graphical representation. Figure 2(a) is plotted for the velocity v for different values of fluid parameter β, steadily increase observed in the velocity from screw toward barrel and the velocity attains maximum values in between the channel which show shear thinning due to increases in the value of β. Figure 2(b) is sketched for the velocity profile w for different values of $\tilde{\beta}$, the velocity profile is seem to be parabolic in nature. The velocity w takes the fluid toward the exit. Figures 3(a) and 3(b) are shown for the velocity v for different values of pressure gradients $P_{,\theta}$ and $P_{,z}$ respectively, it can be seen that velocity v increases with the increase in pressure gradients. It is noticed that $P_{,z}$ resist the velocity v as graphs show the smaller magnitude of v for $P_{,z}$. Similarly figures 4(a) and 4(b) are plotted for the velocity w for different values of $P_{,\theta}$ and $P_{,z}$. With the increase in the value of $P_{,\theta}$ and $P_{,z}$, increase in the w is observed, however the effect of $P_{,\theta}$ is observed less on w which show $P_{,\theta}$ try to resist the flow in axial direction.

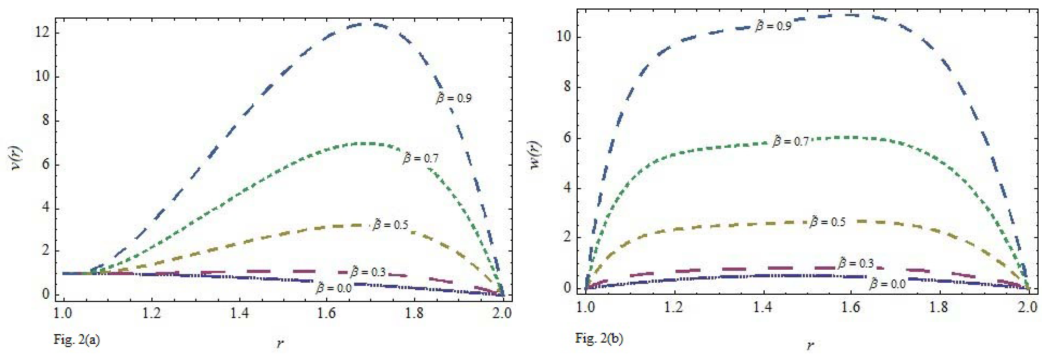

Figure 2: (a) $v(r)$ for different values of β, keeping $P_{,\theta} = -4.0$, $P_{,z} = -4.0$ and $\delta = 2$. (b) $w(r)$ for different values of β, keeping $P_{,\theta} = -4.0$, $P_{,z} = -4.0$ and $\delta = 2$.

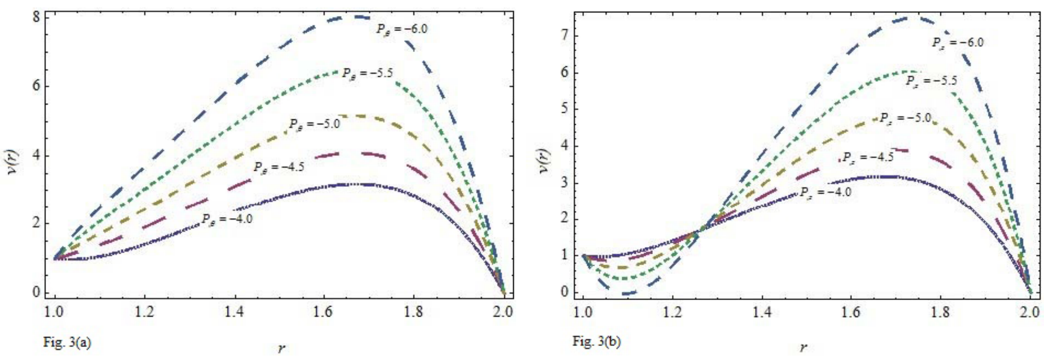

Figure 3: (a) $v(r)$ for different values of $P_{,\theta}$, keeping $\beta = 0.4$, $P_{,z} = -4.0$ and $\delta = 2$. (b) $v(r)$ for different values of $P_{,z}$, keeping $\beta = 0.4$, $P_{,\theta} = -4.0$ and $\delta = 2$.

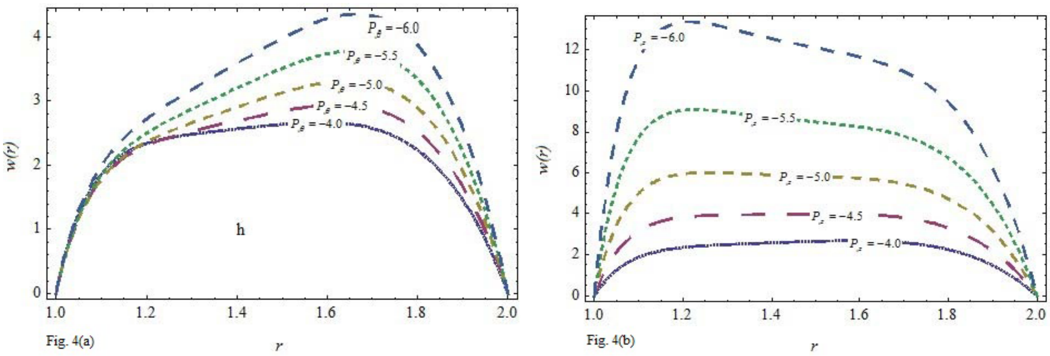

Figure 4: (a) $w(r)$ for different values of $P_{,\theta}$, keeping $\beta = 0.4$, $P_{,z} = -4.0$ and $\delta = 2$. (b) $w(r)$ for different values of $P_{,z}$, keeping $\beta = 0.4$, $P_{,\theta} = -4.0$ and $\delta = 2$.

7. conclusion

The steady flow of an isothermal, homogeneous and incompressible third-grade fluid is investigated in HSR. We choose the cylindrical coordinate system (r, θ, z) which seems to be a more natural choice due to the geometry of HSR. The model developed in cylindrical coordinates pertains to second order non linear coupled differential equations. Using HPM the analytical expressions are obtained for the flow properties i.e., velocities, volume flow rates, shear and normal stresses, the shear stresses exerted by the fluid on the screw and average velocity. Graphical discussion is given for the velocity profiles and shear stresses. It is observed that fluid velocity can be controlled with the proper choice of the values of the non-Newtonian parameter and pressure gradients.

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