AN EFFICIENT SPECTRAL METHOD FOR HIGH-ORDER NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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In this study a numerical method is presented to solve high-order nonlinear Volterra-Fredholm integro-differential equations under the mixed conditions. In the proposed method, orthogonal Legendre polynomials and their properties are used to approximate the solution of nonlinear integro-differential equation and reduce it to a nonlinear system of algebraic equations. The accuracy estimation of the method is given and the efficiency of the method is illustrated through some numerical examples.

Keywords: non-linear integro-differential equation; Legendre polynomials; operational matrix; function approximation; accuracy estimation.

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1. Introduction

The main feature of spectral methods is to take various orthogonal systems of infinitely differentiable global functions as trial functions [1]. Different trial functions lead to different spectral approximations; for instance, trigonometric polynomials for periodic problems, Legendre, Chebyshev and Jacobi polynomials for non-periodic problems, Laguerre polynomials for problems on half line and Hermite polynomials for problems on the whole line. In particular, classical orthogonal polynomials, such as Legendre and Chebyshev polynomials, have played important roles in spectral methods for differential and integral equations [2, 3, 4]. Therefore, such polynomials can be applied to approximate the solution of nonlinear integro-differential equations and convert them into a nonlinear system of algebraic equations.

In this paper, Lgendre polynomials method has been developed to approximate the solution of the high-order nonlinear Volterra-Fredholm integro-differential equations of the form

$$\sum_{j=0}^{m}\mu_j(x)y^{(j)}(x)-\lambda_1\int_a^x k_1(x,t)[y(t)]^p\,dt-\lambda_2\int_a^b k_2(x,t)[y(t)]^q\,dt=f(x), \qquad (1)$$

$$a\le x\le b,$$

with the mixed conditions

$$\sum_{j=0}^{m-1}\left[a_{ij}y^{(j)}(a)+b_{ij}y^{(j)}(b)+c_{ij}y^{(j)}(\zeta)\right]=\beta_i,\quad i=0,1,\dots,m-1,\quad a\le\zeta\le b, \qquad (2)$$

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where the functions $\mu_j(x)(j = 0, 1, \ldots, m)$, $f(x)$, $k_1(x,t)$, $k_2(x,t)$ are known, $y(x)$ is the unknown function to be found and $\beta_i(i = 0, 1, \ldots, m-1)$, a, b, λ_1, λ_2 are constants and p, q are nonnegative integers. Equations of this type appear in many applications. For example, it occurs in solving problems arising in biological, physical, and engineering problems [5, 6].

From the numerical point of view, several methods have been presented for solving integro-differential equations such as the successive approximation method, the Adomian decomposition method, the Chebyshev, Legendre and Taylor collocation methods, wavelet-Galerkin method, the block-pulse functions (BPFs) method and the triangular functions (TFs) method [5-14].

In this paper we are concerned with the direct solution technique to expand the unknown function $y(x)$ in Eqs. (1) and (2) as Legendre polynomials series with unknown coefficients. The unknown coefficients are then determined based on the properties of the Legendre polynomials and some operational matrices.
Most scholars researching Legendre polynomials method [14-18] only mentioned that how it could be utilized to solve the integral equations or systems. They have really neglected an important question, how large the rank n representing the order of Legendre polynomials should be on earth to yield more accurate numerical solutions. We propose that the available optimal value of n can minimize the errors of the numerical solutions.

The detailed approach is demonstrated in following sections and validated through several numerical results.

2. **Properties of Legendre polynomials**

There are several ways to define the Legendre polynomials, and in fact, they are equivalent in some sense. In practice, the manipulation of different usages will depend on our purpose and convenience [19]. Mathematically, Legendre polynomials are solutions to Legendre's differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right] + \lambda P_n(x) = 0$$

where the eigenvalue λ equals $n(n+1)$. Also the Legendre polynomials are given by the following expression, known as Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \qquad n \geq 0,\ -1 \leq x \leq 1.$$

We note that $P_n(x)$ is the nth derivative of a polynomial of degree $2n$ and hence it is a polynomial of degree n.
Alternatively, the Legendre polynomials are given by the following iteration:

$$\begin{aligned}
&P_0(x) = 1,\\
&P_1(x) = x,\\
&P_n(x) = \tfrac{2n-1}{n} x P_{n-1}(x) - \tfrac{n-1}{n} P_{n-2}(x), \qquad n \geq 2,\ -1 \leq x \leq 1.
\end{aligned}$$

The recurrence relation between the derivatives of Legendre polynomials is also given by [20]:

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \quad n \geq 1.$$

2.1. The expansion of a function

Definition 2.1. *Let $f(x)$ be a Rieman integrable function defined on $[-1,1]$, then the associated infinite series*

$$\sum_{j=0}^{\infty} c_j P_j(x) \tag{3}$$

is called the Legendre expansion of $f(x)$, where the coefficients c_j are determined by [17]

$$c_j = \frac{2j+1}{2}\int_{-1}^{1} f(x)P_j(x)dx, \quad j = 0,1,2,\dots . \tag{4}$$

In the following theorem, we indicate that the Legendre expansion of a function $f(x)$, with bounded second derivative, converges uniformly to $f(x)$.

Theorem 2.1. *If a continuous function $f(x)$ defined on $[-1,1]$, has bounded second derivative, say $|f''(x)| \le M$, then the Legendre expansion of the function converges uniformly to the function.*

Proof. From Eq. (4),

$$c_j = \frac{2j+1}{2}\int_{-1}^{1} f(x)P_j(x)dx.$$

Now, let $u = f(x)$ and $dv = (2j+1)P_j(x)dx$. We have

$$dv = \Big[P'_{j+1}(x) - P'_{j-1}(x)\Big]dx = d(P_{j+1}(x) - P_{j-1}(x))$$

Consequently, using integration by parts two times, we obtain:

$$\begin{aligned}
c_j &= \frac{1}{2}f(x)(P_{j+1}(x) - P_{j-1}(x))\Big|_{-1}^{1} - \frac{1}{2}\int_{-1}^{1} f'(x)(P_{j+1}(x) - P_{j-1}(x))dx \\
&= -\frac{1}{2}\int_{-1}^{1} f'(x)(P_{j+1}(x) - P_{j-1}(x))dx \\
&= -\frac{1}{2}f'(x)\Big[\frac{P_{j+2}(x) - P_j(x)}{2j+3} - \frac{P_j(x) - P_{j-2}(x)}{2j-1}\Big]_{-1}^{1} \\
&\qquad + \frac{1}{2}\int_{-1}^{1} f''(x)\Big[\frac{P_{j+2}(x) - P_j(x)}{2j+3} - \frac{P_j(x) - P_{j-2}(x)}{2j-1}\Big]dx. \\
&= \frac{1}{2}\int_{-1}^{1} f''(x)\Big[\frac{P_{j+2}(x) - P_j(x)}{2j+3} - \frac{P_j(x) - P_{j-2}(x)}{2j-1}\Big]dx.
\end{aligned}$$

Consider

$$\Big|\int_{-1}^{1} f''(x)\Big[\frac{P_{j+2}(x) - P_j(x)}{2j+3} - \frac{P_j(x) - P_{j-2}(x)}{2j-1}\Big]dx\Big|^2$$

$$= \Big|\int_{-1}^{1} f''(x)\Big[\frac{(2j-1)P_{j+2}(x) - 2(2j+1)P_j(x) + (2j+3)P_{j-2}(x)}{(2j+3)(2j-1)}\Big]dx\Big|^2$$

$$\leq \int_{-1}^{1} \Big| f''(x) \Big|^2 dx$$

$$\times \int_{-1}^{1} \Big| \frac{(2j-1)P_{j+2}(x) - 2(2j+1)P_j(x) + (2j+3)P_{j-2}(x)}{(2j+3)(2j-1)} \Big|^2 dx$$

$$< 2M^2 \int_{-1}^{1} \frac{(2j-1)^2 P_{j+2}^2(x) + (4j+2)^2 P_j^2(x) + (2j+3)^2 P_{j-2}^2(x)}{(2j+3)^2(2j-1)^2} dx$$

$$= \frac{2M^2}{(2j+3)^2(2j-1)^2} \Big[\frac{2(2j-1)^2}{2j+5} + \frac{2(4j+2)^2}{2j+1} + \frac{2(2j+3)^2}{2j-3} \Big]$$

$$\leq \frac{2M^2}{(2j+3)^2(2j-1)^2} \Big[\frac{2(2j+3)^2}{2j-3} + \frac{8(2j+3)^2}{2j-3} + \frac{2(2j+3)^2}{2j-3} \Big]$$

$$= \frac{24M^2}{(2j-3)(2j-1)^2}$$

Thus, we get

$$\Big| \int_{-1}^{1} f''(x) \Big[\frac{P_{j+2}(x) - P_j(x)}{2j+3} - \frac{P_j(x) - P_{j-2}(x)}{2j-1} \Big] dx \Big| < \frac{\sqrt{24}M}{(2j-1)\sqrt{2j-3}}.$$

Therefore, we have

$$|c_j| < \frac{\sqrt{6}M}{(2j-1)\sqrt{2j-3}},$$

$$\leq \frac{\sqrt{6}M}{(2j-3)^{\frac{3}{2}}}. \tag{5}$$

Hence, the series $\sum_{j=0}^{\infty} c_j$ is absolute convergent, it follows that $\sum_{j=0}^{\infty} c_j P_j(x)$ converges to the function $f(x)$ uniformly. □

If the infinite series in (3) is truncated, then it can be written as

$$f(x) \approx \sum_{j=0}^{n} c_j P_j(x) = \mathrm{C}^T \mathrm{P}(x) \tag{6}$$

where C and $\mathrm{P}(x)$ are $(n+1) \times 1$ vectors given by

$$\mathrm{C} = [c_0, c_1, c_2, \ldots, c_n]^T, \tag{7}$$

and

$$\mathrm{P}(x) = [P_0(x), P_1(x), P_2(x), \ldots, P_n(x)]^T. \tag{8}$$

Similarly the kernel function, $k(x,t)$, may be estimated as :

$$k(x,t) \approx \mathrm{P}^T(x)\, \mathrm{K}\, \mathrm{P}(t),$$

where K is a $(n+1) \times (n+1)$ matrix, with

$$\mathrm{K}_{ij} = \frac{\big\langle P_i(x), \langle k(x,t), P_j(t) \rangle \big\rangle}{\big\langle P_i(x), P_i(x) \big\rangle \big\langle P_j(t), P_j(t) \big\rangle}, \tag{9}$$

where $\langle .,. \rangle$ denotes the inner product in $L^2[-1,1]$. It can be shown that

$$|\mathrm{K}_{ij}| \leq \frac{6M'}{(2i-3)^{\frac{3}{2}}(2j-3)^{\frac{3}{2}}}, \tag{10}$$

which indicates that when i or $j \to \infty$, it follows $|\mathrm{K}_{ij}| \to 0$, which concludes the sparsity of the coefficients matrix K. M' is the bound for mixed fourth partial derivative of $k(x,t)$, say

$$\left|\frac{\partial^4 k(x,t)}{\partial x^2 \partial t^2}\right| \leq M'.$$

Also, for a positive integer p, $[y(x)]^p$ may be approximated by Legendre series as:

$$[y(x)]^p = [\mathrm{Y}^T\mathrm{P}(x)]^p = \mathrm{Y}_p^{*T}\mathrm{P}(x), \tag{11}$$

where Y_p^* is a column vector, whose elements are nonlinear combinations of the elements of the vector Y. Y_p^* is called the operational vector of the pth power of the function $y(x)$. For the Legendre Polynomials with $n = 4$ the operational vector of second power of $y(x)$ is computed as follows:

$$\mathbf{Y}_2^* = \begin{pmatrix} y_0^2 + \frac{y_1^2}{3} + \frac{y_2^2}{5} + \frac{y_3^2}{7} + \frac{y_4^2}{9} \\ 2y_0y_1 + \frac{4y_1y_2}{5} + \frac{18y_2y_3}{35} + \frac{8y_3y_4}{21} \\ \frac{2y_1^2}{3} + 2y_0y_2 + \frac{2y_2^2}{7} + \frac{6y_1y_3}{7} + \frac{4y_3^2}{21} + \frac{4y_2y_4}{7} + \frac{100y_4^2}{693} \\ \frac{6y_1y_2}{5} + 2y_0y_3 + \frac{8y_2y_3}{15} + \frac{8y_1y_4}{9} + \frac{4y_3y_4}{11} \\ \frac{18y_2^2}{35} + \frac{8y_1y_3}{7} + \frac{18y_3^2}{77} + 2y_0y_4 + \frac{40y_2y_4}{77} + \frac{162y_4^2}{1001} \end{pmatrix}.$$

2.2. Accuracy estimation

Theorem 2.2. *Let $f(x)$ be a continuous function defined on $[-1,1]$, with bounded second derivative $|f''(x)|$ bounded by M, then we have the following accuracy estimation:*

$$\sigma_n \leq 2\sqrt{3}M\left(\sum_{j=n+1}^{\infty}\frac{1}{(2j-3)^4}\right)^{\frac{1}{2}}, \tag{12}$$

where

$$\sigma_n = \left(\int_{-1}^{1}\Big[f(x) - \sum_{j=0}^{n} c_jP_j(x)\Big]^2 dx\right)^{\frac{1}{2}}.$$

Proof.

$$\begin{aligned}\sigma_n^2 &= \int_{-1}^{1}\Big[f(x) - \sum_{j=0}^{n} c_jP_j(x)\Big]^2 dx \\ &= \int_{-1}^{1}\Big[\sum_{j=0}^{\infty} c_jP_j(x) - \sum_{j=0}^{n} c_jP_j(x)\Big]^2 dx \\ &= \int_{-1}^{1}\Big[\sum_{j=n+1}^{\infty} c_jP_j(x)\Big]^2 dx \\ &= \int_{-1}^{1}\sum_{j=n+1}^{\infty} c_j^2P_j^2(x)dx\end{aligned}$$

$$= \sum_{j=n+1}^{\infty} c_j^2 \int_{-1}^{1} P_j^2(x)dx$$
$$= \sum_{j=n+1}^{\infty} c_j^2 \frac{2}{2j+1}$$

It follows from Eq. (5),

$$\sigma_n^2 \le \sum_{j=n+1}^{\infty} \frac{6M^2}{(2j-3)^3} \frac{2}{2j+1}$$
$$\le (12M^2) \sum_{j=n+1}^{\infty} \frac{1}{(2j-3)^4}$$

Then one has

$$\sigma_n \le (2\sqrt{3}M) \left(\sum_{j=n+1}^{\infty} \frac{1}{(2j-3)^4} \right)^{\frac{1}{2}}.$$

□

2.3. Operational matrix of derivative

The differentiation of the vector $\mathrm{P}(x)$ in (8) can be expressed as

$$\mathrm{P}'(x) \approx \mathrm{D}\,\mathrm{P}(x), \tag{13}$$

where D is the $(n+1) \times (n+1)$ operational matrix of derivatives for Legendre polynomials vector as follows

$$\mathrm{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 5 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 3 & 0 & 7 & \cdots & 2n-3 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 0 & 2n-1 & 0 \end{pmatrix} for\, odd\, n,$$

$$\mathrm{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 3 & 0 & 7 & \ddots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 2n-3 & 0 & 0 \\ 0 & 3 & 0 & 7 & \cdots & 0 & 2n-1 & 0 \end{pmatrix} for\, even\, n.$$

By using equation (13), we have

$$y'(x) \approx \mathrm{Y}^T \mathrm{P}'(x) = \mathrm{Y}^T \mathrm{D}\,\mathrm{P}(x), \tag{14}$$

and

$$y^{(k)}(x) \approx \mathrm{Y}^T \mathrm{D}^k \, \mathrm{P}(x), \tag{15}$$

where k is the order of derivatives of $y(x)$.

3. Solving nonlinear integro-differential equations

In this section, we consider the nonlinear Volterra-Fredholm integro-differential equation of the form in (1) with (2). We suppose without loss of generality that the interval of integration in (1) is $[-1, 1]$, which is the domain of the Legendre polynomials, since any finite interval $[a, b]$ can be transformed to interval $[-1, 1]$ by linear maps [21]. If the integrals are bounded in the range $[0, 1]$, then solution can be obtained by means of the shifted Legendre polynomials $P_j^*(x)$.
Approximating functions $y(x), [y(x)]^p$ and $k_1(x,t), k_2(x,t)$ by Legendre polynomials, as described in section 2, we obtain

$$\begin{aligned} y(x) &\approx \mathrm{P}^T(x)\mathrm{Y}, &(16)\\ y^p(x) &\approx \mathrm{P}^T(x)\mathrm{Y}_p^*, &(17)\\ k_1(x,t) &\approx \mathrm{P}^T(x)\,\mathrm{K}_1\,\mathrm{P}(t), &(18)\\ k_2(x,t) &\approx \mathrm{P}^T(x)\,\mathrm{K}_2\,\mathrm{P}(t), &(19)\end{aligned}$$

where Y is the unknown vector and Y_p^* is a column vector which can be expressed as a nonlinear function of the vector Y.
To approximate the integral parts of Eq. (1), from Eqs. (16-19), we get

$$\begin{aligned} \int_{-1}^{x} k_1(x,t)[y(t)]^p \, dt &\approx \int_{-1}^{x} \mathrm{P}^T(x)\mathrm{K}_1\mathrm{P}(t)\mathrm{P}^T(t)\mathrm{Y}_p^* \, dt \\ &\approx \mathrm{P}^T(x)\mathrm{K}_1 \left(\int_{-1}^{x} \mathrm{P}(t)\mathrm{P}^T(t)dt \right) \mathrm{Y}_p^* \end{aligned} \tag{20}$$

and

$$\begin{aligned} \int_{-1}^{1} k_2(x,t)[y(t)]^q \, dt &\approx \int_{-1}^{1} \mathrm{P}^T(x)\mathrm{K}_2\mathrm{P}(t)\mathrm{P}^T(t)\mathrm{Y}_q^* \, dt \\ &\approx \mathrm{P}^T(x)\mathrm{K}_2 \left(\int_{-1}^{1} \mathrm{P}(t)\mathrm{P}^T(t)dt \right) \mathrm{Y}_q^* \end{aligned} \tag{21}$$

Here we have to simplify $\int_{-1}^{x} \mathrm{P}(t)\mathrm{P}^T(t)dt$.

From previous researches [18, 22], we assume a $(n+1) \times (n+1)$ square matrix $\mathrm{Z}(x)$ whose elements z_{ij} are, which can be easily calculated based on a given x:

$$\mathrm{z}_{ij} = \int_{-1}^{x} P_i(t)P_j(t)dt. \tag{22}$$

Also to approximate the differential part of Eq. (1), we use Eqs. (13-16)

$$\sum_{j=0}^{m} \mu_j(x) y^{(j)}(x) \approx \sum_{j=0}^{m} \mu_j(x)\mathrm{Y}^T\mathrm{D}^j\mathrm{P}(x). \tag{23}$$

Substituting Eqs. (15-23) into Eq. (1) we get

$$\sum_{j=0}^{m} \mu_j(x)\mathrm{Y}^T\mathrm{D}^j\mathrm{P}(x) - \lambda_1\mathrm{P}^T(x)\mathrm{K}_1\mathrm{Z}(x)\mathrm{Y}_p^* - \lambda_2\mathrm{P}^T(x)\mathrm{K}_2\mathrm{Z}(1)\mathrm{Y}_q^* = f(x). \tag{24}$$

Also using Eqs. (2) and (15) we have

$$\sum_{j=0}^{m-1} \left[a_{ij}\mathrm{Y}^T\mathrm{D}^j\mathrm{P}(-1) + b_{ij}\mathrm{Y}^T\mathrm{D}^j\mathrm{P}(1) + c_{ij}\mathrm{Y}^T\mathrm{D}^j\mathrm{P}(\zeta)\right] = \beta_i, \tag{25}$$

where $-1 \leq \zeta \leq 1$ and $i = 0, 1, \ldots, m-1$. Eq. (25) gives m linear equations $(m < n)$.
Since the total unknowns for vector Y in (16) is $(n+1)$, we collocate equation (24) in $(n-m+1)$ points $x_i = \cos(i\pi/n)$ in the interval $[-1, 1]$:

$$\sum_{j=0}^{m} \mu_j(x_i)\mathrm{Y}^T\mathrm{D}^j\mathrm{P}(x_i) - \lambda_1\mathrm{P}^T(x_i)\mathrm{K}_1\mathrm{Z}(x_i)\mathrm{Y}_p^* - \lambda_2\mathrm{P}^T(x_i)\mathrm{K}_2\mathrm{Z}(1)\mathrm{Y}_q^* = f(x_i). \tag{26}$$

The resulting equations (25) and (26) generate a system of $(n+1)$ nonlinear equations which can be solved by numerical methods such as Newton's iterative method.

4. Numerical examples

In this section, we applied the method presented in this paper for solving high-order nonlinear Volterra-Fredholm integro-differential equation and solved some examples. These problems have been previously solved in different references [5, 9-13]. The computed errors σ_n are defined by

$$\sigma_n = \left\{\int_{-1}^{1} e_n^2(x)dx\right\}^{1/2} \simeq \left\{\frac{1}{n}\sum_{j=0}^{n} e_n^2(x_j)\right\}^{1/2}, \tag{27}$$

where $e_n(x) = y(x) - \mathrm{P}^T(x)\mathrm{Y}$ and $x_j = \cos(j\pi/(n+1))$.
All calculations are performed using Mathematica 7.
Example 1. Consider the second-order Fredholm integro-differential equation

$$x^2y''(x) + 50xy'(x) - 35y(x) = \frac{1 - e^{(x+1)}}{x+1} + (x^2 + 50x - 35)e^x + \int_0^1 e^{xt}y(t)\,dt, \tag{28}$$

with $y(0) = 1,\ y(1) = e$. If we solve (28) for $y(x)$ directly, the analytic solution can be shown to be $y(x) = e^x$. The comparison among the numerical results of presented method, together with the results of [5] obtained by wavelet collocation and wavelet Galerkin methods for $x \in [0, 1]$ is shown in Table 1 for $n = 6$ and $n = 8$, which confirms that the Legendre polynomials method in Section 3 gives almost the same solution of the analytic method. We make a simulation and display of the computed errors σ_n in Table 2 and Fig. 1 for different values of n. Better approximation is expected by choosing $n = 8$, which we get $\sigma_n = 1.15705 \times 10^{-9}$.

TABLE 1. The absolute relative errors for Example 1.

x_i	Presented method		Wavelet Col.	Wavelet Gal.
	$n = 6$	$n = 8$	$M = 6$ [5]	$M = 6$ [5]
0.000	0.0	0.0	0.0	0.0
0.125	9.0×10^{-09}	1.5×10^{-09}	2.6×10^{-2}	2.7×10^{-4}
0.250	2.8×10^{-08}	5.6×10^{-10}	1.6×10^{-2}	3.1×10^{-5}
0.375	8.5×10^{-09}	5.2×10^{-10}	9.3×10^{-3}	2.6×10^{-4}
0.500	6.1×10^{-08}	1.2×10^{-09}	5.2×10^{-3}	4.3×10^{-4}
0.625	1.7×10^{-08}	1.3×10^{-10}	2.5×10^{-3}	5.6×10^{-4}
0.750	5.6×10^{-08}	4.6×10^{-10}	1.0×10^{-3}	6.6×10^{-4}
0.875	1.6×10^{-08}	8.5×10^{-10}	2.4×10^{-4}	7.2×10^{-4}
0.100	0.0	0.0	0.0	0.0

TABLE 2. The computed errors σ_n for $n = 5, 6, \ldots, 10$.

n	The computed errors σ_n
5	2.13449×10^{-6}
6	6.87366×10^{-8}
7	1.97861×10^{-9}
8	1.15705×10^{-9}
9	1.60484×10^{-8}
10	2.04909×10^{-6}

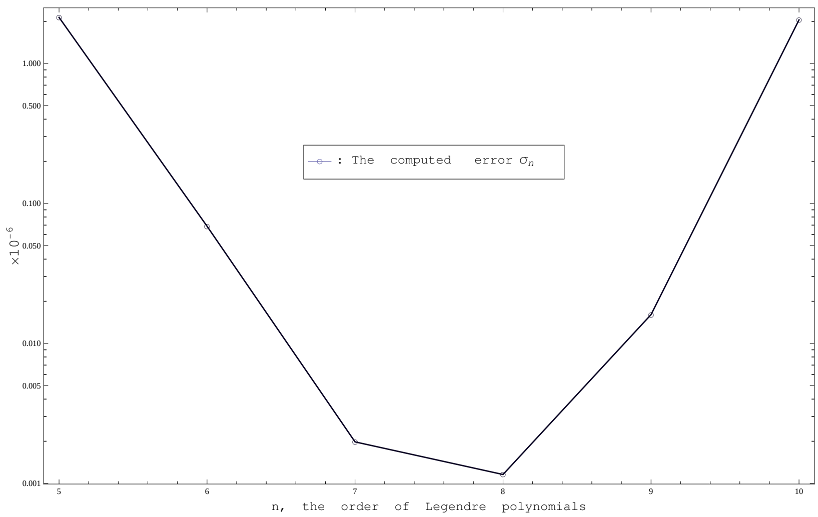

FIGURE 1. The computed error σ_n for $n = 5, 6, \ldots, 10$.

Example 2. As the second example, consider the following second-order Volterra integro-differential equation [9, 11]

$$y''(x) + xy(x) = f(x) + \int_0^x x^2 e^t y(t)\, dt,$$

TABLE 3. Numerical results for Example 2.

x_i	Exact solution	Presented method $n = 8$	Taylor solution $N = 7$
0.0	1	1	1
0.2	0.98006658	0.98006658	0.98006658
0.4	0.92106099	0.92106099	0.92106099
0.6	0.82533561	0.82533561	0.82533562
0.8	0.69670671	0.69670671	0.69670674
1.0	0.54030231	0.54030231	0.54030258

TABLE 4. The computed errors σ_n for $n = 6, 7, \ldots, 11$.

n	The computed errors σ_n
6	1.26748×10^{-07}
7	1.00238×10^{-08}
8	1.44142×10^{-10}
9	1.49066×10^{-10}
10	2.52808×10^{-09}
11	8.65550×10^{-07}

with conditions

$$y(0) = 1, \;\; y'(0) = 0,$$

where

$$f(x) = x\cos(x) - \frac{1}{2}(e^x(\cos(x) + \sin(x)) - 1)x^2 - \cos(x).$$

The analytic solution of this problem is $y(x) = \cos(x)$. The comparison among the Legendre polynomials solution, Taylor solution and the analytic solution for $x \in [0, 1]$ is shown in Table 3 for $n = 8$, which confirms that the Legendre polynomials method in Section 3 gives almost the same solution of the analytic method. Better approximation is expected by choosing $n = 8$, which we get $\sigma_n = 1.44142 \times 10^{-10}$. We make a simulation and display of the computed errors σ_n in Table 4 and Fig. 2 for different values of n.

Example 3. Consider the following nonlinear Volterra-Fredholm integro-differential equation [9, 13]

$$y'(x) + 2xy(x) = f(x) + \int_0^x (x + t)y^3(t)dt + \int_0^1 (x - t)y(t)dt,$$

with the initial condition $y(0) = 1$, where

$$f(x) = x\cos(x) - \frac{1}{2}(e^x(\cos(x) + \sin(x)) - 1)x^2 - \cos(x),$$

and the exact solution $y(x) = e^x$. The comparison among the presented method, triangular functions (TFs) method [13] and Taylor polynomials method [9] together with the exact solution for $x \in [0, 1]$, is shown in Table 5, which confirms that the Legendre polynomials method in Section 3 gives almost the same solution of the analytic method.

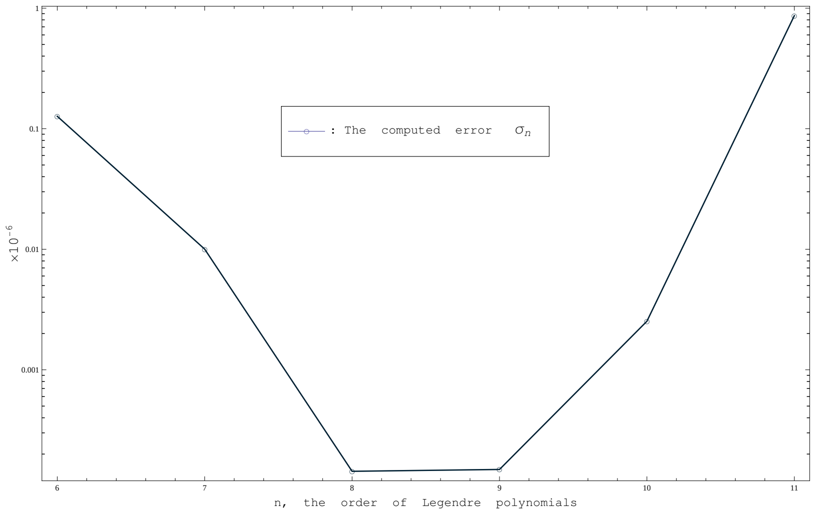

FIGURE 2. The computed error σ_n for $n = 6, 7, \ldots, 11$.

TABLE 5. Numerical results for Example 3.

x_i	Exact solution	Presented method ($n = 7$)	TFs method ($m = 32$)	Taylor method ($N = 7$)
0.0	1.000000	1.000000	1.000000	1.000000
0.1	1.105171	1.105171	1.105223	—
0.2	1.221403	1.221403	1.221494	1.221403
0.3	1.349859	1.349859	1.349971	—
0.4	1.491825	1.491825	1.491933	1.491825
0.5	1.648721	1.648721	1.648795	—
0.6	1.822119	1.822119	1.822484	1.822120
0.7	2.013753	2.013753	2.014465	—
0.8	2.225541	2.225541	2.226719	2.225542
0.9	2.459603	2.459603	2.461507	—
1.0	2.718282	2.718282	2.721505	2.718281

Example 4. As the final example, consider the following nonlinear Volterra-Fredholm integro-differential equation [11]

$$y'''(x) + y(x) = f(x) + \int_{-1}^{x} y^2(t)\, dt + \int_{-1}^{1} (x^2 t + x t^2) y^2(t)\, dt, \tag{29}$$

with the conditions

$$y(0) = -1,$$
$$y'(0) = y''(0) = 0,$$

where

$$f(x) = \frac{47}{14} - \frac{17}{9}x + \frac{4}{5}x^2 + x^3 + \frac{1}{2}x^4 - \frac{1}{7}x^7.$$

If we apply the Legendre polynomials method in Section 3 to solve Eq. (29), by taking $n = 6$, we obtain

$$y(x) = \mathrm{Y}^T \mathrm{P}(x), \qquad -1 \le x \le 1, \tag{30}$$

where the collocation points are $x_i = \{-\frac{\sqrt{3}}{2}, 0, \pm\frac{1}{2}\}$, and the nonlinear algebraic systems (25) and (26) yield to the solution

$$\mathrm{Y} = \Big\{ -1, \frac{3}{5}, 0, \frac{2}{5}, 0, 0, 0 \Big\}^T.$$

By substituting in (30) we get $y(x) = x^3 - 1$, which is the analytic solution of this problem.

5. Concluding remarks

In summary, a collocation method based on Legendre polynomials for the nonlinear Volterra-Fredholm integro-differential equations is presented. The uniform convergence analysis and the error estimate are also studied. Moreover the numerical results and L_2 error norm are presented. The efficiency of the method is verified by making comparison with other methods such as wavelet and Taylor polynomial methods. It has been shown that the obtained results are in excellent agreement with the exact solution. This method can be easily extended and applied to general nonlinear Volterra and Fredholm integro-differential equations of arbitrary order and systems of integro-differential equations with suitable initial conditions.

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