STABILITY IN p-TH MOMENT FOR UNCERTAIN SINGULAR SYSTEMS

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The uncertain singular system is a type of singular system that is disrupted by the canonical Liu process and is described as a multidimensional uncertain differential equation. Although stability in mean, stability in measure, and almost sure stability have been investigated for uncertain singular systems, these three types of stability may not apply in all cases. This paper aims to introduce the concept of stability in p-th moment for uncertain singular systems as a supplementary type of stability. A stability theorem is also presented for uncertain singular systems that are stable in p-th moment. Furthermore, this paper discusses the relationships between stability in measure and stability in p-th moment, between stability in p_1-th moment and stability in p_2-th moment for uncertain singular systems. An example is provided to demonstrate the effectiveness of our results.

Keywords: uncertainty theory; uncertain singular systems; p-th moment stability; the relationships between stabilities

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1. Introduction

Singular systems, such as descriptor systems, implicit systems, and generalized state-space systems, are modeled by differential-algebraic equations. These systems [1, 2, 3] have been extensively researched over the past few decades due to their ability to describe various natural phenomena in physical systems, including economics [4], demography [5], and microelectronic circuits [6], among others [7, 8, 9, 10]. However, when analyzing these systems, uncertain or stochastic disturbances may occur, which require the stability analysis of singular systems to be considered. Unlike stochastic singular systems [11], uncertain singular systems are a type of multidimensional uncertain differential equation that are disturbed by uncertain processes associated with belief degrees. This type of uncertainty associated with belief degrees is a distinct type of indeterminate phenomenon that can be described using uncertainty theory, which was introduced as the opposite of probability theory by Liu in 2007 [12] and updated in 2015 [13]. Today, uncertainty theory is widely applied in various fields,such as uncertain variational inequalities [14, 15], uncertain systems [16, 17] and so on.

In 2008, Liu [18] proposed the concept of uncertain differential equations, which are driven by the canonical Liu process. Later in 2009, Liu [19] claimed that the Liu process is an uncertain process with stationary and independent normal uncertain increments. In 2010, Chen and Liu [20] provided an analytic solution for a linear uncertain differential equation. Following this, Yao and Chen [21, 22] developed a numerical method to obtain the uncertainty distributions of the solution to an uncertain differential equation. Besides

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theoretical research, uncertain differential equations have numerous applications in dynamical systems, one of which is in uncertain singular systems, as explored in this paper and previous works [23, 24, 25, 26].

Considering various kinds of uncertain factors disturbances which may occur in uncertain dynamical system, Yao [27] proposed multi-dimensional uncertain differential equation (MUDE) and proved existence and uniqueness of its solution, which depends on the initial value, and stability is required to obtain a stable solution. Stability analysis [28] for an uncertain differential system is fundamental and important. Su et al. [29] presented the concept of stability for the multidimensional uncertain differential equation in the sense of uncertain measure, and subsequent studies investigated stable in p-th moment [30], stability in mean [31], almost sure stability [32]. Tao and Zhu [34] investigated attractivity and stability of uncertain differential systems in 2015 and considered other types of stabilities and attractivity in optimistic value for dynamical systems with uncertainty in 2016, providing a solid foundation for studying uncertain singular systems. In 2017, Su et al. [35, 36] investigated three types of stabilities for an uncertain singular system. However, these cannot be applied to all cases, so this paper aims to supplement the existing research by presenting a concept of stability in p-th moment for uncertain singular systems.

The paper is organized as follows: Section 2 will provide a review of some basic concepts, lemmas, and theorems. Section 3 presents the concept of p-th moment stability and proves the stability theorem. In Section 4, we provide an example to demonstrate the effectiveness of the results. Finally, a brief summary will be given in Section 5.

2. Preliminaries

Definition 2.1. [14] *Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}dx - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\}dx$$

provided that at least one of the two integrals is finite.

Definition 2.2. [14] *Suppose that C_k is a canonical Liu process, f and g are continuous functions, k is the time. Given an initial value U_0, the uncertain differential equation*

$$dU_k = f(k, U_k)dk + g(k, U_k)dC_k$$

is called an uncertain differential equation with an initial value U_0.

Theorem 2.1. [18] *Let U_k and U_k^α be the solution and α-path of the uncertain differential equation*

$$dU_k = f(k, U_k)dk + g(k, U_k)dC_k$$

Then

$$\mathcal{M}\{U_k \leq U_k^\alpha, \forall k\} = \alpha, \mathcal{M}\{U_k > U_k^\alpha, \forall k\} = 1 - \alpha.$$

Theorem 2.2. [28] *Let C_k be a Liu process on uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then there exists an uncertain variable K such that $K(\gamma)$ is a Lipschitz constant of the sample path $C_k(\gamma)$ for each γ,*

$$\lim_{x \to +\infty} \mathcal{M}\{\gamma \in \Gamma | K(\gamma) \leq x\} = 1$$

and

$$\mathcal{M}\{\gamma \in \Gamma | K(\gamma) \leq x\} \geq 2\Phi(x) - 1.$$

Theorem 2.3. [20] *Suppose that C_k is a canonical Liu process, and U_k is an integrable uncertain process on $[a,b]$. Then the inequality*

$$|\int_a^b U_k(\gamma)dC_k| \leq K_\gamma \int_a^b |U_k(\gamma)|dk$$

holds, where K_γ is the Lipschitz constant of the sample path $U_k(\gamma)$.

Definition 2.3. [33] *Suppose that C_k is an n-dimensional canonical Liu process,$f(k,\mathbf{u})$ is a vector-valued function from $T \times R^n$ to R^m, and $g(k,\mathbf{u})$ is a matrix-valued function from $T \times R^n$ to the set of $m \times n$ matrices. Then*

$$d\mathbf{U}_k = f(k,\mathbf{U}_k)dk + g(k,\mathbf{U}_k)dC_k$$

is called an m-dimensional uncertain differential equation (MUDE)driven by an n-dimensional canonical Liu process. An m-dimensional uncertain process that satisfies this equation identically at each time k is called a solution of the MUDE.

Su etal. [29] studied the stability concerning the MUDE.

Theorem 2.4. [29] *Suppose the MUDE*

$$d\mathbf{U}_k = f(k,\mathbf{U}_k)dk + g(k,\mathbf{U}_k)dC_k$$

has a unique solution for each initial value. If the coefficients $f(k,\mathbf{u})$ and $g(k,\mathbf{u})$ satisfies

$$\begin{aligned}&| f(k,\mathbf{u}) - f(k,\mathbf{v}) | + | g(k,\mathbf{u}) - g(k,\mathbf{v}) | \\ &\leq L_k \mid \mathbf{u} - \mathbf{v} \mid, \quad \forall \mathbf{u},\mathbf{v} \in R^m, k \geq 0\end{aligned}$$

for some positive functions L_k with

$$\int_0^{+\infty} L_k dk < +\infty,$$

then the MUDE is stable.

Definition 2.4. [21]*Let α be a real number with $0 < \alpha < 1$. An UDE*

$$dU_k = g_1(U_k,k)dk + g_2(U_k,k)dC_k$$

is said to have an α-path U_k^α if it solves the corresponding ODE

$$dU_k^\alpha = g_1(U_k^\alpha,k)dk + \mid g_2(U_k^\alpha,k) \mid \Phi^{-1}(\alpha)dk$$

where $\Phi^{-1}(\alpha)$is the inverse standard normal uncertainty distribution, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha}.$$

Theorem 2.5. [28] *Let C_k be a canonical Liu process. Then there exists an uncertain variable K such that for each γ, K_γ is a Lipschitz constant of the sample path $C_k(\gamma)$, and*

$$\mathcal{M}\{K \leq x\} \geq 2\left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}}\right)\right)^{-1} - 1.$$

Theorem 2.6. [28]*Let C_k be a canonical Liu process. Then there exists a nonnegative uncertain variable K such that K_γ is a Lipschitz constant of the sample path $C_k(\gamma)$ for each γ, and*

$$\lim_{x \to +\infty} \mathcal{M}\{K \leq x\} = 1.$$

The uncertain singular system can be written as the following MUDE:

$$\begin{cases} \mathfrak{F}d\mathbf{U}_k = g(k)\mathfrak{A}\mathbf{U}_k dk + h(k)\mathfrak{B}\mathbf{U}_k dC_k, \\ \mathbf{U}\mid_{k=0} = \mathbf{U}_0, k \geq 0. \end{cases} \tag{1}$$

where the system is represented by a state vector $\mathbf{U}_k \in R^n$. The functions $g(k)$ and $h(k)$, which are bounded and defined for $k \geq 0$, are associated with the system. Additionally, there are known coefficient matrices $\mathfrak{A} \in R^{n\times n}$ and $\mathfrak{B} \in R^{n\times n}$ that are respectively related to the state vector $\mathbf{U}_k$. $\mathfrak{F}$ is a known (singular) matrix with rank $(\mathfrak{F}) = q \leq n$, and $deg(det(z\mathfrak{F} - \mathfrak{A})) = r$ where z is a complex variable. C_k is a canonical Liu process defined on uncertainty space, representing the noise of the system. Throughout this paper, for a matrix $\mathfrak{A} = [a_{ij}]_{n\times n}$ and a vector $\mathbf{U} = (u_1, u_2, \ldots, u_n)^T$, we define

$$\|\mathfrak{A}\| = \sum_{i,j=1}^{n} \mid a_{ij} \mid, \|\mathbf{U}\| = \sum_{i=1}^{n} \mid u_i \mid .$$

Definition 2.5. [11] *(i)$(\mathfrak{F},\mathfrak{A})$ is said to be regular if $det(z\mathfrak{F} - \mathfrak{A})$ is not identically zero. (ii)$(\mathfrak{F},\mathfrak{A})$ is said to be impulse-free if $deg(det(z\mathfrak{F} - \mathfrak{A})) = rank(\mathfrak{F})$.*

Lemma 2.1. [36]*If $(\mathfrak{F},\mathfrak{A})$ is regular, impulse-free and $rank[\mathfrak{F},\mathfrak{B}] = rank(\mathfrak{F})$, there exist a pair of nonsingular matrices $\mathfrak{P} \in R^{n\times n}$ and $\mathfrak{Q} \in R^{n\times n}$ for the triplet $(\mathfrak{F},\mathfrak{A},\mathfrak{B})$ such that the following conditions are satisfied:*

$$\mathfrak{P}\mathfrak{F}\mathfrak{Q} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathfrak{P}\mathfrak{A}\mathfrak{Q} = \begin{bmatrix} \mathfrak{A}_1 & 0 \\ 0 & I_{n-r} \end{bmatrix},$$
$$\mathfrak{P}\mathfrak{B}\mathfrak{Q} = \begin{bmatrix} \mathfrak{B}_1 & \mathfrak{B}_2 \\ 0 & 0 \end{bmatrix},$$
$$\textit{where } \mathfrak{A}_1 \in R^{r\times r}, \mathfrak{B}_1 \in R^{r\times r}, \mathfrak{B}_2 \in R^{r\times n-r}.$$

Let $\begin{bmatrix} \mathbf{U}_{1,k} \\ \mathbf{U}_{2,k} \end{bmatrix} = \mathfrak{Q}^{-1}\mathbf{U}_k$, where $\mathbf{U}_{1,k} \in R^r$ and $\mathbf{U}_{2,k} \in R^{n-r}$. The system (1) is equivalent to

$$\begin{cases} d\mathbf{U}_{1,k} = g(k)\mathfrak{A}_1\mathbf{U}_{1,k}dk \\ \qquad\qquad + h(k)[\mathfrak{B}_1\mathbf{U}_{1,k} + \mathfrak{B}_2\mathbf{U}_{2,k}]dC_k, \\ 0 = g(k)\mathbf{U}_{2,k}dk, \end{cases}$$

or

$$\begin{cases} d\mathbf{U}_{1,k} = g(k)\mathfrak{A}_1\mathbf{U}_{1,k}dk + h(k)\mathfrak{B}_1\mathbf{U}_{1,k}dC_k, \\ 0 = \mathbf{U}_{2,k}, \end{cases}$$

for all $k \geq 0$.

Lemma 2.2. [35]*System (1) has a unique solution if $(\mathfrak{F},\mathfrak{A})$ is regular, impulse-free and $rank[\mathfrak{F},\mathfrak{B}] = rank(\mathfrak{F})$. Moreover, the solution is sample-continuous.*

Definition 2.6. [36] *A MUDE is said to be stable in measure if for any solutions $\mathbf{U}_k$ and $\mathbf{V}_k$ with initial values $\mathbf{U}_0$ and $\mathbf{V}_0$, respectively, we have*

$$\lim_{\|\mathbf{U}_0 - \mathbf{V}_0\|\to 0} \mathcal{M}\{\|\mathbf{U}_k - \mathbf{V}_k\| > \varepsilon\} = 0, \forall k \geq 0$$

for any given number $\varepsilon > 0$.

Theorem 2.7. [36] *System (1) is stable in measure if $(\mathfrak{F},\mathfrak{A})$ is regular and impulse-free, $rank[\mathfrak{F},\mathfrak{B}] = rank(\mathfrak{F})$ and the bounded functions $g(k)$ and $h(k)$ are both integrable on $[0,+\infty)$.*

Lemma 2.3. [36] *If a vector function*

$$f(k) = (f_1(k), f_2(k), \ldots, f_n(k))^T$$

is derivable on $k \in (0,+\infty)$, then $\|f(k)\|$ is derivable almost everywhere for $k > 0$.

3. **Stability Theorem**

Inspired by the work in Sheng et al. [30], we now consider stability in p-th moment of uncertain singular system (1).

Definition 3.1. *A MUDE is said to be stable in p-th moment if for any solutions* $\mathbf{U}_k$ *and* $\mathbf{V}_k$ *with initial values* $\mathbf{U}_0$ *and* $\mathbf{V}_0$, *respectively, we have*

$$\lim_{\|\mathbf{U}_0-\mathbf{V}_0\|\to 0} E[\sup_{k\geq 0}\|\mathbf{U}_k-\mathbf{V}_k\|^p]=0. \tag{2}$$

Theorem 3.1. *System* (1) *is stable in p-th moment if* $(\mathfrak{F},\mathfrak{A})$ *is regular and impulse-free,rank*$[\mathfrak{F},\mathfrak{B}]=$ *rank*$(\mathfrak{F})$, *the bounded function* $g(k)$ *is integrable on* $[0,+\infty)$ *and*

$$\int_0^{+\infty} h(s)ds<\frac{\pi}{\sqrt{3}p\|\mathfrak{B}_1\|}.$$

Proof By applying Lemma 2.2, we establish the existence and uniqueness of the solution. Let

$$\begin{bmatrix}\mathbf{U}_{1,k}\\ \mathbf{U}_{2,k}\end{bmatrix}=\mathfrak{Q}^{-1}\mathbf{U}_k,$$

where $\mathbf{U}_{1,k}\in R^r$ and $\mathbf{U}_{2,k}\in R^{n-r}$. For each γ, $\mathbf{U}_{1,k}(\gamma)-\mathbf{V}_{1,k}(\gamma)$ is differentiable on $(0,+\infty)$, then by Lemma 2.3 we know that $\|\mathbf{U}_{1,k}(\gamma)-\mathbf{V}_{1,k}(\gamma)\|$ is differentiable almost everywhere for $k>0$. Denote

$$\mathfrak{A}_\gamma=\{k\in(0,+\infty)\mid\|\mathbf{U}_{1,k}(\gamma)-\mathbf{V}_{1,k}(\gamma)\|\}$$

is differentiable, and

$$\mathfrak{A}'_\gamma=(0,+\infty)-\mathfrak{A}_\gamma.$$

It is evident that we obtain

$$m\mathfrak{A}'_\gamma=0.$$

We can then express it as

$$\mathbf{U}_{1,k}(\gamma)=(u_1(\gamma),u_2(\gamma),\dots,u_r(\gamma))^T$$

and

$$\mathbf{V}_{1,k}(\gamma)=(v_1(\gamma),v_2(\gamma),\dots,v_r(\gamma))^T.$$

Hence, for any $k\in\mathfrak{A}_\gamma$, we can always apply Lemma 2.1 to conclude that

$$\begin{aligned}
&d\|\mathbf{U}_{1,k}(\gamma)-\mathbf{V}_{1,k}(\gamma)\|\\
&=d\sum_{i=1}^r\pm(u_i(\gamma)-v_i(\gamma))\\
&=\sum_{i=1}^r\pm(du_i(\gamma)-dv_i(\gamma))\\
&\leq\sum_{i=1}^r|(du_i(\gamma)-dv_i(\gamma))|\\
&=\|d\mathbf{U}_{1,k}(\gamma)-d\mathbf{V}_{1,k}(\gamma)\|\\
&=\|g(k)\mathfrak{A}_1(\mathbf{U}_{1,k}(\gamma)-\mathbf{V}_{1,k}(\gamma))\|dk+\|h(k)\mathfrak{B}_1(\mathbf{U}_{1,k}(\gamma)-\mathbf{V}_{1,k}(\gamma))dC_k(\gamma)\|\\
&\leq g(k)\|\mathfrak{A}_1(\mathbf{U}_{1,k}(\gamma)-\mathbf{V}_{1,k}(\gamma))\|dk+K_\gamma h(k)\|\mathfrak{B}_1(\mathbf{U}_{1,k}(\gamma)-\mathbf{V}_{1,k}(\gamma))\|dk\\
&\leq g(k)\|\mathfrak{A}_1\|\cdot\|\mathbf{U}_{1,k}(\gamma)-\mathbf{V}_{1,k}(\gamma)\|dk+K_\gamma\ h(k)|\mathfrak{B}_1\|\cdot\|\mathbf{U}_{1,k}(\gamma)-\mathbf{V}_{1,k}(\gamma)\|dk,
\end{aligned}$$

where, K_γ denotes the Lipschitz constant of the sample path $C_k(\gamma)$ as stated in Theorem 2.5. This implies that

$$\begin{aligned}&\|\mathbf{U}_{1,k}(\gamma)-\mathbf{V}_{1,k}(\gamma)\|\\&\quad\le \|\mathbf{U}_{1,0}-\mathbf{V}_{1,0}\|\exp\left(\|\mathfrak{A}_1\|\int_0^k g(s)ds\right)\cdot\exp\left(K_\gamma\|\mathfrak{B}_1\|\int_0^k h(s)ds\right).\end{aligned}$$

Let us consider any $k\in\mathfrak{A}'_\gamma$, to begin with, we can choose $k_1\in((1-\frac{1}{2})k,k)$, such that $k_1\in\mathfrak{A}_\gamma$; Furthermore, we can select $k_2\in((1-\frac{1}{2^2})k,k)-\{k_1\}$ such that $k_2\in\mathfrak{A}_\gamma$. In a similar manner, for any $n\in N_+$, we can choose $k_n\in((1-\frac{1}{2^n})k,k)-\{k_1,k_2,\ldots,k_{n-1}\}$ such that $k_n\in\mathfrak{A}_\gamma$. It is clear that $k_n\to k$, as $n\to+\infty$. For any $n\in N_+$, $k_n\in\mathfrak{A}_\gamma$, building on our earlier discussion, we can now establish the following inequality:

$$\begin{aligned}&\|\mathbf{U}_{1,k_n}(\gamma)-\mathbf{V}_{1,k_n}(\gamma)\|\\&\quad\le \|\mathbf{U}_{1,0}-\mathbf{V}_{1,0}\|\exp\left(\|\mathfrak{A}_1\|\int_0^{k_n} g(s)ds\right)\cdot\exp\left(K_\gamma\|\mathfrak{B}_1\|\int_0^{k_n} h(s)ds\right)\\&\quad\le \|\mathbf{U}_{1,0}-\mathbf{V}_{1,0}\|\exp\left(\|\mathfrak{A}_1\|\int_0^{k} g(s)ds\right)\cdot\exp\left(K_\gamma\|\mathfrak{B}_1\|\int_0^{k} h(s)ds\right).\end{aligned}$$

Let $n\to+\infty$, since Lemma 2.2 guarantees the sample-continuity of $\mathbf{U}_k$, we can conclude that

$$\begin{aligned}&\|\mathbf{U}_{1,k}(\gamma)-\mathbf{V}_{1,k}(\gamma)\|\\&\quad\le \|\mathbf{U}_{1,0}-\mathbf{V}_{1,0}\|\exp\left(\|\mathfrak{A}_1\|\int_0^k g(s)ds\right)\cdot\exp\left(K_\gamma\|\mathfrak{B}_1\|\int_0^k h(s)ds\right).\end{aligned}$$

In summary, for any $k\in(0,+\infty)$ and considering the arbitrariness of γ, we can always deduce that

$$\begin{aligned}&\|\mathbf{U}_{1,k}-\mathbf{V}_{1,k}\|\\&\quad\le \|\mathbf{U}_{1,0}-\mathbf{V}_{1,0}\|\exp\left(\|\mathfrak{A}_1\|\int_0^k g(s)ds\right)\cdot\exp\left(K\|\mathfrak{B}_1\|\int_0^k h(s)ds\right)\\&\quad\le \|\mathbf{U}_{1,0}-\mathbf{V}_{1,0}\|\exp\left(\|\mathfrak{A}_1\|\int_0^{+\infty} g(s)ds\right)\cdot\exp\left(K\|\mathfrak{B}_1\|\int_0^{+\infty} h(s)ds\right)\end{aligned}\tag{3}$$

almost surely, where K is a nonnegative uncertain variable such that

$$\mathcal{M}\{K\ge x\}=1-\mathcal{M}\{K<x\}\le 2\left(1+\exp\left(\frac{\pi x}{\sqrt{3}}\right)\right)^{-1}$$

according to Theorem 2.5. Hence we know

$$\begin{aligned}&\sup_{k>0}\|\mathbf{U}_{1,k}-\mathbf{V}_{1,k}\|\\&\quad\le \|\mathbf{U}_{1,0}-\mathbf{V}_{1,0}\|\exp\left(\|\mathfrak{A}_1\|\int_0^{+\infty} g(s)ds\right)\cdot\exp\left(K\|\mathfrak{B}_1\|\int_0^{+\infty} h(s)ds\right)\end{aligned}\tag{4}$$

almost surely. Taking p-th moment on both sides, we have

$$\begin{aligned}&E\Big[\sup_{k>0}\|\mathbf{U}_{1,k}-\mathbf{V}_{1,k}\|^p\Big]\\&\quad\le \|\mathbf{U}_{1,0}-\mathbf{V}_{1,0}\|^p\exp\left(p\|\mathfrak{A}_1\|\int_0^{+\infty} g(s)ds\right)\cdot E\left[\exp\left(pK\|\mathfrak{B}_1\|\int_0^{+\infty} h(s)ds\right)\right].\end{aligned}\tag{5}$$

Given that the bounded function $g(k)$ is integrable over the interval $[0, +\infty)$, it follows that we can clearly deduce that

$$\exp\left(p\int_0^{+\infty} g(s)ds\right) < +\infty.$$

For the expected value

$$E\left[\exp\left(pK\|\mathfrak{B}_1\|\int_0^{+\infty} h(s)ds\right)\right],$$

we denote

$$\varrho = \int_0^{+\infty} h(s)ds < \frac{\pi}{\sqrt{3}p\|\mathfrak{B}_1\|}.$$

By applying Definition 2.1 of expected value and utilizing Theorem 2.5, we can conclude that

$$\begin{aligned}
&E\left[\exp\left(pK\cdot\|\mathfrak{B}_1\|\varrho\right)\right]\\
&\quad= \int_0^{+\infty} \mathcal{M}\left\{\exp\left(\|\mathfrak{B}_1\|\varrho\cdot pK\right)\geq x\right\}dx\\
&\quad= \int_0^{+\infty} \mathcal{M}\left\{K \geq \frac{\ln x}{\|\mathfrak{B}_1\|\varrho\cdot p}\right\}dx\\
&\quad\leq 2\int_0^{+\infty} \left(1+\exp\left(\frac{\pi\ln x}{\sqrt{3}\|\mathfrak{B}_1\|\varrho\cdot p}\right)\right)^{-1}dx\\
&\quad= 2\int_0^{+\infty} \left(1+x^{\frac{\pi}{\sqrt{3}\|\mathfrak{B}_1\|\varrho\cdot p}}\right)^{-1}dx < +\infty.
\end{aligned}$$

Therefore, we can derive at

$$\lim_{\|\mathbf{U}_{1,0}-\mathbf{V}_{1,0}\|\to 0} E\left[\sup_{k>0}\|\mathbf{U}_{1,k}-\mathbf{V}_{1,k}\|^p\right] = 0.$$

Since Lemma 2.2 proves that $\mathbf{U}_{2,k} = 0$ and $\mathbf{V}_{2,k} = 0$ hold for all $k > 0$, we can conclude that

$$\begin{aligned}
&E\left[\sup_{k>0}\|\mathbf{U}_k-\mathbf{V}_k\|^p\right]\\
&\quad= E\left[\sup_{k>0}\left\|\mathfrak{Q}\begin{bmatrix}\mathbf{U}_{1,k}\\ \mathbf{U}_{2,k}\end{bmatrix} - \mathfrak{Q}\begin{bmatrix}\mathbf{V}_{1,k}\\ \mathbf{V}_{2,k}\end{bmatrix}\right\|^p\right]\\
&\quad= E\left[\sup_{k>0}\|\mathfrak{Q}_1(\mathbf{U}_{1,k}-\mathbf{V}_{1,k})\|^p\right]\\
&\quad\leq E\left[\|\mathfrak{Q}_1\|^p\cdot\sup_{k>0}\|(\mathbf{U}_{1,k}-\mathbf{V}_{1,k})\|^p\right]\to 0,
\end{aligned}$$

as $\|\mathbf{U}_{1,0}-\mathbf{V}_{1,0}\|\to 0$, where

$$\mathfrak{Q} = \begin{bmatrix}\mathfrak{Q}_1 & \mathfrak{Q}_2\end{bmatrix}$$

and $\mathfrak{Q}_1 \in R^{n\times r}$. Namely,

$$\lim_{\|\mathbf{U}_{1,0}-\mathbf{V}_{1,0}\|\to 0} E\left[\sup_{k>0}\|\mathbf{U}_k-\mathbf{V}_k\|^p\right] = 0.$$

It is readily evident that $\|\mathbf{U}_{1,0}-\mathbf{V}_{1,0}\|\to 0$, as $\|\mathbf{U}_0-\mathbf{V}_0\|\to 0$, Thus, we obtain

$$\lim_{\|\mathbf{U}_0-\mathbf{V}_0\|\to 0} E\left[\sup_{k>0}\|\mathbf{U}_k-\mathbf{V}_k\|^p\right] = 0.$$

At $k = 0$, it is evident that

$$\lim_{\|\mathbf{U}_0-\mathbf{V}_0\|\to 0} E\left[\|\mathbf{U}_0 - \mathbf{V}_0\|^p\right] = 0.$$

By combining the two aforementioned equations, we obtain

$$\lim_{\|\mathbf{U}_0-\mathbf{V}_0\|\to 0} E\left[\sup_{k\geq 0} \|\mathbf{U}_k - \mathbf{V}_k\|^p\right] = 0.$$

Therefore, we have established stability in p-th moment of system (1). This concludes the theorem.

Theorem 3.2. *If uncertain singular system* (1) *is stable in p-th moment, then it is stable in measure.*

Proof *From Definition 3.1, for two solutions* $\mathbf{U}_k$ *and* $\mathbf{V}_k$ *with different initial values* $\mathbf{U}_0$ *and* $\mathbf{V}_0$*, respectively. Then it follows from the definition of stability in p-th moment that*

$$\lim_{\|\mathbf{U}_0-\mathbf{V}_0\|\to 0} E\left[\sup_{k\geq 0} \|\mathbf{U}_k - \mathbf{V}_k\|^p\right] = 0, \forall p > 0. \tag{6}$$

By Markov inequality, for any given real number $\epsilon > 0$*, we have*

$$\begin{aligned}
&\lim_{\|\mathbf{U}_0-\mathbf{V}_0\|\to 0} \mathcal{M}\{\|\mathbf{U}_k - \mathbf{V}_k\| > \epsilon\} \leq \\
&\lim_{\|\mathbf{U}_0-\mathbf{V}_0\|\to 0} \frac{E[\|\mathbf{U}_k - \mathbf{V}_k\|^p]}{\epsilon^p} \leq \\
&\lim_{\|\mathbf{U}_0-\mathbf{V}_0\|\to 0} \frac{E\left[\sup_{k\geq 0} \|\mathbf{U}_k - \mathbf{V}_k\|^p\right]}{\epsilon^p} \to 0, \forall k \geq 0.
\end{aligned}$$

Therefore, from Definition 2.6, p-th moment stability implies the stability in measure. This concludes the theorem.

Theorem 3.3. *For any two real numbers* p_1 *and* p_2 $(0 < p_1 < p_2 < +\infty)$*, if uncertain singular systems* (1) *is stable in* p_2*-th moment, then it is stable in* p_1*-th moment.*

Proof *From Definition 3.1, for two solutions* $\mathbf{U}_k$ *and* $\mathbf{V}_k$ *with different initial values* $\mathbf{U}_0$ *and* $\mathbf{V}_0$*, respectively. Then it follows from the definition of stability in* p_2*-th moment that*

$$\lim_{\|\mathbf{U}_0-\mathbf{V}_0\|\to 0} E\left[\sup_{k\geq 0} \|\mathbf{U}_k - \mathbf{V}_k\|^p\right] = 0, \forall p > 0.$$

According to Holder's inequality, we have

$$\begin{aligned}
&E[\|\mathbf{U}_k - \mathbf{V}_k\|^{p_1}] \\
=&E[\|\mathbf{U}_k - \mathbf{V}_k\|^{p_1} \cdot \mathbf{1}] \\
\leq& \sqrt[p_2/p_1]{E[\|\mathbf{U}_k - \mathbf{V}_k\|^{p_1 \cdot p_2/p_1}]} \cdot \sqrt[p_2/(p_2-p_1)]{E[\mathbf{1}^{p_2/(p_2-p_1)}]} \\
=& \sqrt[p_2/p_1]{E[\|\mathbf{U}_k - \mathbf{V}_k\|^{p_2}]}, \forall k > 0.
\end{aligned}$$

Thus, stability in p_2*-th moment implies stability in* p_1*-th moment when* $p_1 < p_2$*. This concludes the theorem.*

4. Numerical example

To demonstrate the effectiveness of Theorem 3.1, a numerical example will be provided.

Example 4.1. *We will now examine the uncertain singular system given by:*

$$\begin{cases} Fd\mathbf{U}_k = g(k)\mathfrak{A}\mathbf{U}_k dk + h(k)\mathfrak{B}\mathbf{U}_k dC_k, \\ \mathbf{U}|_{k=0} = \mathbf{U}_0, \quad k \geq 0, \end{cases} \tag{7}$$

where

$$g(k)=2e^{-\frac{k}{3}}, \quad h(k) = \frac{1}{10}e^{-\frac{k}{2}},$$

and

$$\mathfrak{F}=\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathfrak{A}=\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix},$$
$$\mathfrak{B} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Applying a similar methodology as demonstrated in Example 1 of Ref [36]*, it can be determined that the system* (1) *possesses a unique and sample-continuous solution. Furthermore, we observe that* $2e^{-\frac{k}{3}}$ *is integrable over the interval* $[0, +\infty)$*, and* $\frac{1}{10}e^{-\frac{k}{2}}$ *satisfies*

$$\int_0^{+\infty} \frac{1}{10}e^{-\frac{s}{2}} ds = \frac{1}{5} < \frac{\pi}{\sqrt{3}p\|\mathfrak{B}_1\|} = \frac{4\sqrt{3}\pi}{27p},$$

where

$$\mathfrak{B}_1 = \begin{bmatrix} 1 & -\frac{1}{4} \\ 0 & -1 \end{bmatrix}, 0 < p < \frac{20\sqrt{3}\pi}{27}.$$

Consequently, utilizing Theorem 3.1, we can conclude that system (1) *is stable in p-th moment. Upon matrix calculations, we obtain*

$$\mathfrak{P}\mathfrak{F}\mathfrak{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathfrak{P}\mathfrak{A}\mathfrak{Q} = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathfrak{P}\mathfrak{B}\mathfrak{Q} \ = \begin{bmatrix} 1 & -\frac{1}{4} & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

where

$$\mathfrak{P} = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathfrak{Q} = \begin{bmatrix} 0 & 1 & 0 \\ 4 & 1 & -1 \\ 4 & 0 & 0 \end{bmatrix}.$$

Letting

$$\mathbf{U}_k = \mathfrak{Q}\begin{bmatrix} \mathbf{U}_{1,k} \\ \mathbf{U}_{2,k} \end{bmatrix}$$

for all $k > 0$*, where* $\mathbf{U}_{1,k} \in R^2$ *and* $\mathbf{U}_{2,k} \in R$*, system* (1) *can be expressed as*

$$\begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} d\mathbf{U}_{1,k} = 2e^{-\frac{k}{3}} \begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix} \mathbf{U}_{1,k} dk \\ \qquad\qquad + \frac{1}{10}e^{-\frac{k}{2}} \left\{ \begin{bmatrix} 1 & -\frac{1}{4} \\ 0 & -1 \end{bmatrix} \mathbf{U}_{1,k} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{U}_{2,k} \right\} dC_k, \\ 0 = e^{-k}\mathbf{U}_{2,k} dk, \end{cases}$$

or

$$\begin{cases} d\mathbf{U}_{1,k} = 2e^{-\frac{k}{3}} \begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix} \mathbf{U}_{1,k} dk \\ \qquad\qquad + \frac{1}{10}e^{-\frac{k}{2}} \begin{bmatrix} 1 & -\frac{1}{4} \\ 0 & -1 \end{bmatrix} \mathbf{U}_{1,k} dC_k, \\ 0 = \mathbf{U}_{2,k}, \end{cases}$$

for all $k \geq 0$. *The aforementioned equations can be expressed in the form of the following system of equations:*

$$\begin{cases} du_1(k) = 2e^{-\frac{k}{3}}[u_1(k) + \dfrac{1}{4}u_2(k)]dk \\ \qquad\qquad +\frac{1}{10}e^{-\frac{k}{2}}[u_1(k) - \frac{1}{4}u_2(k)]dC_k \\ du_2(k) = 2e^{-\frac{k}{3}}u_2(k)dk - \dfrac{1}{10}e^{-\frac{k}{2}}u_2(k)dC_k, \end{cases} \tag{8}$$

where $\mathbf{U}_k = [u_1(k), u_2(k)]^T$ *for all* $k \geq 0$. *According to Definition 2.4, the corresponding ODEs of Eq.*(8) *are*

$$\begin{cases} du_1^\alpha(k) = 2e^{-\frac{k}{3}}[u_1^\alpha(k) + \dfrac{1}{4}u_2^\alpha(k)]dk \\ \qquad\qquad +\frac{1}{10}e^{-\frac{k}{2}} \mid u_1^\alpha(k) - \frac{1}{4}u_2^\alpha(k) \mid \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}dk \\ du_2^\alpha(k) = 2e^{-\frac{k}{3}}u_2^\alpha(k)dk \\ \qquad\qquad +\frac{1}{10}e^{-\frac{t}{2}} \mid u_2^\alpha(k) \mid \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}dk, \quad k \geq 0. \end{cases} \tag{9}$$

*In FIGURE.*1, *when* $(u_1^{0.2}(0), u_2^{0.2}(0)) = (1, 1)$, *the trajectories of* $u_1^{0.2}(k)$ *and* $u_2^{0.2}(k)$ *are represented by* u_1 *and* u_2, *respectively. When* $(u_1^{0.2}(0), u_2^{0.2}(0)) = (1.01, 1.01)$, *the trajectories of* $u_1^{0.2}(k)$ *and* $u_2^{0.2}(k)$ *are denoted by* u_1^* *and* u_2^*, *respectively.*

*In FIGURE.*2, *when* $(u_1^{0.4}(0), u_2^{0.4}(0)) = (1, 1)$, *the trajectories of* $u_1^{0.4}(k)$ *and* $u_2^{0.4}(k)$ *are represented by* u_1 *and* u_2, *respectively. When* $(u_1^{0.4}(0), u_2^{0.4}(0)) = (1.01, 1.01)$, *the trajectories of* $u_1^{0.4}(k)$ *and* $u_2^{0.4}(k)$ *are denoted by* u_1^* *and* u_2^*, *respectively.*

As depicted in the aforementioned figures, if the variation in initial values is sufficiently small, the solutions of Eq.(9) *under different belief degrees that correspond to these initial values will be closer and closer as time increases. The aforementioned phenomenon indicates the stability of Eq.*(9). *Therefore, it can be inferred that Eq.*(8) *and system* (1) *are stable in p-th moment.*

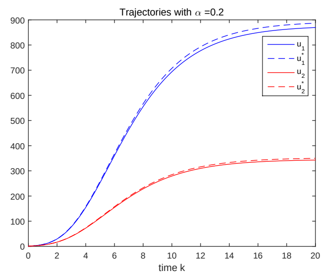

FIGURE 1. Trajectories of $u_1^\alpha(k), u_2^\alpha(k)$ when $\alpha = 0.2$

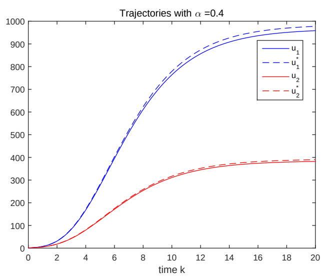

FIGURE 2. Trajectories of $u_1^\alpha(k), u_2^\alpha(k)$ when $\alpha = 0.4$

5. Conclusions

In this paper, we investigated the stability in p-th moment of an uncertain singular system, building upon the concept of stability in p-th moment and exploring the system's behavior across different p values. By making suitable assumptions, we derived sufficient conditions for determining the stability in p-th moment. In addition, the relationships among stability almost surely, stability in measure, and stability in p-th moment for the uncertain singular systems are also discussed. To illustrate our findings, we provided an example that verifies the stability of the system. This analysis reveals the intrinsic nature of the system's stability across different moments and provides valuable insights for further research.

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