MAJORIZATION RESULTS FOR A GENERAL SUBCLASS OF MEROMORPHIC MULTIVALENT FUNCTIONS

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In the present paper we investigate a majorization problem for the new class $\mathcal{H}^{n,l}_{\lambda,p,\mu}(\alpha,\delta,\gamma,j;A,B)$ of meromorphic functions defined by using the multiplier transform $\mathcal{I}^{n,l}_{\lambda,p,\mu}$. Also, we point out some useful consequences of our main results.

Keywords: Meromorphic functions, multivalent functions, majorization property, multiplier transform, differential subordination

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1. Introduction

Let the functions f and g be analytic in an open subset G of the unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. We say that f is majorized by g in G (see [7]) and write

$$f(z) \ll g(z),\ z \in G,$$

if there exists a function φ analytic in $\mathbb{U}$, such that $|\varphi(z)| \leq 1$ for all $z \in \mathbb{U}$ and

$$f(z) = \varphi(z)g(z),\ z \in G. \tag{1}$$

If the functions f and g are analytic in $\mathbb{U}$, the function f is called to be *subordinate* to the function g, written $f(z) \prec g(z)$, if there exists a function w analytic in $\mathbb{U}$ with $|w(z)| < 1$, $z \in \mathbb{U}$, and $w(0) = 0$, such that $f = g \circ w$. In particular, if g is univalent in $\mathbb{U}$ then the following equivalence relationship holds true:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let Σ_p denote the class of all functions of the form

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^{k+1-p},\ (p \in \mathbb{N} := \{1, 2, \dots\}),$$

that are analytic in the punctured open unit disk $\mathbb{U}^* := \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$. In particular, we set $\Sigma_1 =: \Sigma$.

In a recent paper, Wang and Li [12] defined a new multiplier transform $\mathcal{I}^{n,l}_{\lambda,p,\mu}$ for the functions $f \in \Sigma_p$ by

$$\mathcal{I}^{n,l}_{\lambda,p,\mu} f(z) := z^{-p} + \sum_{k=0}^{\infty} \frac{(\mu)_{k+1}}{(k+1)!} \left(\frac{\lambda}{\lambda + l\,(k+1)}\right)^n a_k z^{k+1-p},\ z \in \mathbb{U}^*, \tag{2}$$

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where (and throughout this paper unless otherwise mentioned) the parameters n, l, p, λ and μ are constrained as follows:

$$n \geq 0,\ l \geq 0,\ p \in \mathbb{N},\ \lambda > 0,\ \text{and}\ \mu > 0,$$

and $(\mu)_k$ is the *Pochhammer symbol* defined by

$$(\mu)_k := \begin{cases} 1, & \text{if} \quad k = 0, \\ \mu(\mu+1)\cdots(\mu+k-1), & \text{if} \quad k \in \mathbb{N}. \end{cases}$$

It should be remembered that the operator $\mathcal{I}^{n,l}_{\lambda,p,\mu}$ generalize many other familiar operators considered by earlier authors (for details, see [12]).

It is readily verified from (2) (see [12]) that for all $f \in \Sigma_p$ we have

$$z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f\right)'(z) = \mu\mathcal{I}^{n,l}_{\lambda,p,\mu+1}f(z) - (\mu+p)\,\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z),\ z \in \mathbb{U}^*. \tag{3}$$

By making use of the operator $\mathcal{I}^{n,l}_{\lambda,p,\mu}$, we define a new subclass of meromorphic functions $f \in \Sigma_p$ as follows:

Definition 1.1. *Let $-1 \leq B < A \leq 1$, $p \in \mathbb{N}$, $0 \leq \alpha < p$, $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $\delta \geq 0$. A function $f \in \Sigma_p$ is said to be in the class $\mathcal{H}^{n,l}_{\lambda,p,\mu}(\alpha,\delta,\gamma,j;A,B)$ of meromorphic multivalent functions of complex order γ in $\mathbb{U}^*$ if and only if*

$$p - \frac{1}{\gamma}\left(\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)\right)^{(j+1)}}{\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)\right)^{(j)}} + p + j\right) - \delta\left|\frac{1}{\gamma}\left(\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)\right)^{(j+1)}}{\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)\right)^{(j)}} + p + j\right)\right| \prec \frac{p + [pB + (p-\alpha)(A-B)]\,z}{1+Bz}, \tag{4}$$

where $\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)\right)^{(j)}$ represents the "j" times derivative of $\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)$.

Remark 1.1. In the following some special cases of Definition 1.1 we show how the class of meromorphic multivalent functions $\mathcal{H}^{n,l}_{\lambda,p,\mu}(\alpha,\delta,\gamma,j;A,B)$, for suitable choices of the parameters, lead to certain new as well as known classes of meromorphic functions studied earlier in the literature.

(i) For $A = 1$ and $B = -1$, we get the class $\mathcal{H}^{n,l}_{\lambda,p,\mu}(\alpha,\delta,\gamma,j;1,-1) =: \mathcal{H}^{n,l}_{\lambda,p,\mu}(\alpha,\delta,\gamma,j)$ which consists of functions $f \in \Sigma_p$ satisfying the condition

$$\operatorname{Re}\left\{p - \frac{1}{\gamma}\left(\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)\right)^{(j+1)}}{\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)\right)^{(j)}} + p + j\right)\right\} - \delta\left|\frac{1}{\gamma}\left(\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)\right)^{(j+1)}}{\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)\right)^{(j)}} + p + j\right)\right| > \alpha,$$

with $0 \leq \alpha < p$.

(ii) If we set $\delta = 0$ in (i), then we get the class $\mathcal{H}^{n,l}_{\lambda,p,\mu}(\alpha,0,\gamma,j;1,-1) =: \mathcal{H}^{n,l}_{\lambda,p,\mu}(\alpha,\gamma,j)$ which consists of functions $f \in \Sigma_p$ satisfying the inequality

$$\operatorname{Re}\left\{p - \frac{1}{\gamma}\left(\frac{z\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)\right)^{(j+1)}}{\left(\mathcal{I}^{n,l}_{\lambda,p,\mu}f(z)\right)^{(j)}} + p + j\right)\right\} > \alpha,\ z \in \mathbb{U},\ 0 \leq \alpha < p.$$

(*iii*) Setting $n=0$ and $\mu=1$ in (*ii*), then we get $\mathcal{H}^{0,l}_{\lambda,p,1}\left(\alpha,0,\gamma,j;1,-1\right)=:\mathcal{H}_p\left(\alpha,\gamma,j\right)$ which represent the functions $f\in\Sigma_p$ such that

$$\operatorname{Re}\left\{p-\frac{1}{\gamma}\left(\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}+p+j\right)\right\}>\alpha,\ z\in\mathbb{U},\ 0\le\alpha<p.$$

(*iv*) For $p=1$ and $j=0$, the above classe mentioned at (*iii*) reduces to the class $\mathcal{H}^{0,l}_{\lambda,1,1}\left(\alpha,0,\gamma,0;1,-1\right)=:\mathcal{S}_\gamma\left(\alpha\right)$ consisting of functions $f\in\Sigma$ satisfying

$$\operatorname{Re}\left\{1-\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}+1\right)\right\}>\alpha,\ z\in\mathbb{U},\ 0\le\alpha<1.$$

The class $\mathcal{S}_\gamma\left(\alpha\right)$ is said to be the class of *meromorphic starlike functions of complex order* $\gamma\in\mathbb{C}^*$ and type α $(0\le\alpha<1)$ in $\mathbb{U}^*$.

(*v*) Taking $\gamma=1$ in (*iv*) we get the class $\mathcal{H}^{0,l}_{\lambda,1,1}\left(\alpha,0,1,0;1,-1\right)=:\mathcal{S}^*\left(\alpha\right)$ which consists of functions $f\in\Sigma$ with

$$\operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right)>\alpha,\ z\in\mathbb{U},\ 0\le\alpha<1.$$

The class $\mathcal{S}^*\left(\alpha\right)$ is said to be the class *meromorphic starlike functions of order* α $(0\le\alpha<1)$ in $\mathbb{U}^*$.

(*vi*) Considering $\alpha=0$ in (*v*) we get the class $\mathcal{H}^{0,l}_{\lambda,1,1}\left(0,1,0;1,-1\right)=:\mathcal{S}^*$ which consists of functions $f\in\Sigma$ such that

$$\operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right)>0,\ z\in\mathbb{U}.$$

(*vii*) Setting $p=1$, $j=1$ and $\gamma=1$ in (*iii*) we obtain the class $\mathcal{H}^{0,l}_{\lambda,1,1}\left(\alpha,0,1,1;1,-1\right)=:$ $\mathcal{K}\left(\alpha\right)$ which are the functions $f\in\Sigma$ satisfying the inequality

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha,\ z\in\mathbb{U},\ 0\le\alpha<1.$$

The class $\mathcal{K}\left(\alpha\right)$ is said to be the class of *meromorphic convex functions of order* α $(0\le\alpha<1)$ in $\mathbb{U}^*$.

(*viii*) For $\alpha=0$, the class of (*vii*) reduces to $\mathcal{H}^{0,l}_{\lambda,1,1}\left(0,0,1,1;1,-1\right)=:\mathcal{K}$ of functions $f\in\Sigma$ satisfying

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0,\ z\in\mathbb{U}.$$

The majorization problem for the normalized classes of starlike functions has recently been investigated by Altıntaş et al. [1] and MacGregor [7]. However, only a few articles deal with the majorization problem for the class of meromorphic functions (see [2, 3, 4, 5, 6, 9, 10, 11]). Motivated by these works, in the present paper we investigate a majorization problem for the new class $\mathcal{H}^{n,l}_{\lambda,p,\mu}\left(\alpha,\delta,\gamma,j;A,B\right)$ of meromorphic functions defined by using the multiplier transform $\mathcal{I}^{n,l}_{\lambda,p,\mu}$.

2. Majorization problem for the class $\mathcal{H}^{n,l}_{\lambda,p,\mu}\left(\alpha,\delta,\gamma,j;A,B\right)$

Unless otherwise mentioned we shall assume throughout the sequel that

$$-1\le B<A\le1,\ p\in\mathbb{N},\ 0\le\alpha<p,\ j\in\mathbb{N}_0,\ \gamma\in\mathbb{C}^*,\ \text{and}\ 0\le\delta\ne1.$$

Our main majorization result is given by the next theorem:

Theorem 2.1. *Let* $f \in \Sigma_p$ *and suppose that* $g \in \mathcal{H}_{\lambda,p,\mu}^{n,l}(\alpha,\delta,\gamma,j;A,B)$*. If*

$$\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} f(z)\right)^{(j)} \ll \left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j)},\ z \in \mathbb{U}^*,$$

then

$$\left|\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} f(z)\right)^{(j)}\right| \leq \left|\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} g(z)\right)^{(j)}\right|,$$

for $0 < |z| \leq r_0$*, where* $r_0 := r_0(\mu,\alpha,\delta,\gamma,p;A,B)$ *is the smallest positive root of the equation*

$$\kappa r^3 - (\mu + 2|B|)\, r^2 - (\kappa + 2)\, r + \mu = 0, \tag{5}$$

with

$$\kappa = \frac{(p-\alpha)(A-B)}{|1-\delta|}|\gamma| + \mu|B|.$$

Proof. Since $g \in \mathcal{H}_{\lambda,p,\mu}^{n,l}(\alpha,\delta,\gamma,j;A,B)$, from (4) and the definition of the subordination, there exists a function w that is analytic in $\mathbb{U}$ and satisfies the conditions $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{U}$, such that

$$p - \frac{1}{\gamma}\left(\frac{z\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j+1)}}{\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j)}} + p + j\right) - \delta\left|\frac{1}{\gamma}\left(\frac{z\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j+1)}}{\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j)}} + p + j\right)\right|$$
$$= \frac{p + [pB + (p-\alpha)(A-B)]\, w(z)}{1 + Bw(z)},\ z \in \mathbb{U}. \tag{6}$$

Setting

$$\varkappa := \varkappa(z) = p - \frac{1}{\gamma}\left(\frac{z\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j+1)}}{\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j)}} + p + j\right) \tag{7}$$

in (6) we get

$$\varkappa - \delta|\varkappa - p| = \frac{p + [pB + (p-\alpha)(A-B)]\, w(z)}{1 + Bw(z)},$$

which implies

$$\varkappa = \frac{p + \left(\frac{p\left(A - B\delta e^{-i\theta}\right) - \alpha(A-B)}{1-\delta e^{-i\theta}}\right) w(z)}{1 + Bw(z)} \tag{8}$$

for some $\theta \in \mathbb{R}$.

Replacing the value of $\varkappa$ from (8) in (7), we obtain

$$\frac{z\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j+1)}}{\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j)}} = -\frac{p + j + \left(\frac{(p-\alpha)(A-B)\gamma}{1-\delta e^{-i\theta}} + (p+j)\, B\right) w(z)}{1 + Bw(z)}. \tag{9}$$

From (3), from mathematical induction it follows that

$$z\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j+1)} = \mu\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} g(z)\right)^{(j)} - (\mu + p + j)\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j)},\ j \in \mathbb{N}. \tag{10}$$

Now, using the relation (10) and (9), and the fact that $|w(z)| \leq |z|$ for all $z \in \mathbb{U}$ (see [8]), we easily get for $z \in \mathbb{U}^*$

$$\left|\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j)}\right| \leq \frac{\mu\left(1+|B|\,|z|\right)}{\mu - \left|\frac{(p-\alpha)(A-B)\gamma}{1-\delta e^{-i\theta}} - \mu B\right|\,|z|}\left|\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} g(z)\right)^{(j)}\right|$$

$$\leq \frac{\mu\left(1+|B|\,|z|\right)}{\mu - \left[\frac{(p-\alpha)(A-B)}{|1-\delta|}\,|\gamma| + \mu\,|B|\right]\,|z|}\left|\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} g(z)\right)^{(j)}\right|. \qquad (11)$$

Next, since $\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} f(z)\right)^{(j)}$ is majorized by $\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j)}$ in $\mathbb{U}^*$, from (1) there exists a function φ analytic in $\mathbb{U}$, with $|\varphi(z)| \leq 1$ for all $z \in \mathbb{U}$, such that

$$\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} f(z)\right)^{(j)} = \varphi(z)\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j)},\ z \in \mathbb{U}^*.$$

Differentiating the last equality with respect to z and multiplying by z, we get

$$z\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} f(z)\right)^{(j+1)} = z\varphi'(z)\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j)} + z\varphi(z)\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j+1)},\ z \in \mathbb{U}^*,$$

and using (10) in the above relation it follows that

$$\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} f(z)\right)^{(j)} = \frac{z\varphi'(z)}{\mu}\left(\mathcal{J}_{\lambda,p,\mu}^{n,l} g(z)\right)^{(j)} + \varphi(z)\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} g(z)\right)^{(j)}\ z \in \mathbb{U}^*. \qquad (12)$$

Thus, noting that φ is an analytic function in $\mathbb{U}$ with $|\varphi(z)| \leq 1$ for $z \in \mathbb{U}$, satisfies the inequality (see, e.g. Nehari [8, page 168, relation (28)])

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2},\ z \in \mathbb{U}, \qquad (13)$$

and making use of (11) and (13) in (12), we get

$$\left|\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} f(z)\right)^{(j)}\right| \leq \left(|\varphi(z)| + \frac{|z|\left(1-|\varphi(z)|^2\right)}{1-|z|^2}\,\frac{1+|B|\,|z|}{\mu - \left[\frac{(p-\alpha)(A-B)}{|1-\delta|}\,|\gamma| + \mu\,|B|\right]\,|z|}\right)$$

$$\times\left|\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} g(z)\right)^{(j)}\right|,\ z \in \mathbb{U}^*. \qquad (14)$$

Upon setting

$$|z| = r,\ \text{and}\ |\varphi(z)| = \rho,\ 0 \leq \rho \leq 1,$$

the inequality (14) leads to

$$\left|\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} f(z)\right)^{(j)}\right| \leq \frac{\Theta(r,\rho)}{(1-r^2)\,(\mu-\kappa r)}\left|\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} g(z)\right)^{(j)}\right|,\ z \in \mathbb{U}^*, \qquad (15)$$

where

$$\Theta(r,\rho) = r\,(1+|B|\,r)\left(1-\rho^2\right) + \left(1-r^2\right)(\mu-\kappa r)\,\rho,$$

with

$$\kappa = \frac{(p-\alpha)\,(A-B)}{|1-\delta|}\,|\gamma| + \mu\,|B|\,.$$

If we denote

$$\Psi(r,\rho) := \frac{\Theta(r,\rho)}{(1-r^2)\,(\mu-\kappa r)},$$

then (15) becomes

$$\left|\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} f(z)\right)^{(j)}\right| \leq \Psi(r,\rho)\left|\left(\mathcal{J}_{\lambda,p,\mu+1}^{n,l} g(z)\right)^{(j)}\right|,\ z \in \mathbb{U}^*.$$

From the above relation, in order to prove our result, we need to determine

$$r_0 = \max\{r \in [0,1] : \Psi(r,\rho) \le 1, 0 \le \rho \le 1\}$$
$$= \max\{r \in [0,1] : \Lambda(r,\rho) \ge 0, 0 \le \rho \le 1\},$$

assuming that $\mu - \kappa r > 0$. Since $\kappa > 0$, this last inequality is equivalent to

$$r < \frac{\mu}{\kappa} =: r_*,$$

where

$$\Lambda(r,\rho) := -r\left(1+|B|\,r\right)\left(1-\rho^2\right) + \left(1-r^2\right)(\mu-\kappa r)(1-\rho) = (1-\rho)h(r,\rho),$$

with

$$h(r,\rho) = -r\left(1+|B|\,r\right)(1+\rho) + \left(1-r^2\right)(\mu-\kappa r).$$

It follows that the inequality $\Lambda(r,\rho) \ge 0$ is equivalent to $h(r,\rho) \ge 0$, while the function $h(r,\rho)$ takes its minimum value at $\rho = 1$, i.e.

$$\min\{h(r,\rho) : \rho \in [0,1]\} = h(r,1) = g(r),$$

where

$$g(r) := \kappa r^3 - (\mu + 2\,|B|)\,r^2 - (\kappa+2)\,r + \mu = 0.$$

Since $g(0) = \mu > 0$ and $g(1) = -2|B| \le 0$, it follows that $g(r) \ge 0$ for all $r \in [0, r_0]$, where r_0 is the smallest positive root of the equation (5). It is easy to check that

$$g(r_*) = g\left(\frac{\mu}{\kappa}\right) = -\frac{2\mu}{\kappa}\left(|B|\frac{\mu}{\kappa}+1\right) < 0,$$

and using the fact that r_0 is the smallest positive root of (5), it follows that $r_* > r_0$, and the proof of the theorem is complete. □

Setting $A = 1$ and $B = -1$ in Theorem 2.1 we get the following corollary:

Corollary 2.1. *Let the function $f \in \Sigma_p$ and suppose that $g \in \mathcal{H}^{n,l}_{\lambda,p,\mu}(\alpha,\delta,\gamma,j)$ with $0 \le \delta \ne 1$. If*

$$\left(\mathcal{J}^{n,l}_{\lambda,p,\mu} f(z)\right)^{(j)} \ll \left(\mathcal{J}^{n,l}_{\lambda,p,\mu} g(z)\right)^{(j)},\ z \in \mathbb{U}^*,$$

then

$$\left|\left(\mathcal{J}^{n,l}_{\lambda,p,\mu+1} f(z)\right)^{(j)}\right| \le \left|\left(\mathcal{J}^{n,l}_{\lambda,p,\mu+1} g(z)\right)^{(j)}\right|$$

for $0 < |z| \le r_1$, where $r_1 := r_1(\mu,\alpha,\delta,\gamma,p)$ is the smallest positive root of the equation

$$\kappa r^3 - (\mu+2)\,r^2 - (\kappa+2)\,r + \mu = 0,$$

with

$$\kappa = \frac{2\,(p-\alpha)}{|1-\delta|}\,|\gamma| + \mu.$$

For $\delta = 0$, Corollary 2.1 reduces to the next result:

Corollary 2.2. *Let the function $f \in \Sigma_p$ and suppose that $g \in \mathcal{H}^{n,l}_{\lambda,p,\mu}(\alpha,\gamma,j)$ with $0 \le \delta \ne 1$. If*

$$\left(\mathcal{J}^{n,l}_{\lambda,p,\mu} f(z)\right)^{(j)} \ll \left(\mathcal{J}^{n,l}_{\lambda,p,\mu} g(z)\right)^{(j)},\ z \in \mathbb{U}^*,$$

then

$$\left|\left(\mathcal{J}^{n,l}_{\lambda,p,\mu+1} f(z)\right)^{(j)}\right| \le \left|\left(\mathcal{J}^{n,l}_{\lambda,p,\mu+1} g(z)\right)^{(j)}\right|$$

for $0 < |z| \le r_2$, where $r_2 := r_2(\mu,\alpha,\gamma,p)$ is the smallest positive root of the equation

$$(2\,(p-\alpha)\,|\gamma| + \mu)\,r^3 - (\mu+2)\,r^2 - (2\,(p-\alpha)\,|\gamma| + \mu + 2)\,r + \mu = 0.$$

Further, putting $n = 0$ and $\mu = 1$ in Corollary 2.2 we get the following special case:

Example 2.1. *Let the function* $f \in \Sigma_p$ *and suppose that* $g \in \mathcal{H}_p(\alpha, \gamma, j)$. *If*

$$f^{(j)}(z) \ll g^{(j)}(z),\ z \in \mathbb{U}^*,$$

then

$$\left|zf^{(j+1)}(z) + (1+p+j)\, f^{(j)}(z)\right| \le \left|zg^{(j+1)}(z) + (1+p+j)\, g^{(j)}(z)\right|$$

for $0 < |z| \le r_3$, *where* $r_3 := r_3(\alpha, \gamma, p)$ *is the smallest positive root of the equation*

$$(2(p-\alpha)|\gamma| + 1)\, r^3 - 3r^2 - (2(p-\alpha)|\gamma| + 3)\, r + 1 = 0.$$

Also, putting $p = 1$ and $j = 0$ in Example 2.1 we obtain:

Example 2.2. *Let the function* $f \in \Sigma$ *and suppose that* $g \in \mathcal{S}_\gamma(\alpha)$. *If*

$$f(z) \ll g(z),\ z \in \mathbb{U}^*,$$

then

$$|zf'(z) + 2f(z)| \le |zg'(z) + 2g(z)|$$

for $0 < |z| \le r_4$, *where* $r_4 := r_4(\alpha, \gamma)$ *is the smallest positive root of the equation*

$$(2(1-\alpha)|\gamma| + 1)\, r^3 - 3r^2 - (2(1-\alpha)|\gamma| + 3)\, r + 1 = 0.$$

For $\gamma = 1$, Example 2.2 reduces to the next result:

Example 2.3. *Let the function* $f \in \Sigma$ *and suppose that* $g \in \mathcal{S}^*(\alpha)$. *If*

$$f(z) \ll g(z),\ z \in \mathbb{U}^*,$$

then

$$|zf'(z) + 2f(z)| \le |zg'(z) + 2g(z)|$$

for

$$0 < |z| \le \frac{3 - \alpha - \sqrt{\alpha^2 - 4\alpha + 6}}{3 - 2\alpha}.$$

Setting $\alpha = 0$ in Example 2.3 we have the following consequence:

Example 2.4. *Let the function* $f \in \Sigma$ *and suppose that* $g \in \mathcal{S}^*$. *If*

$$f(z) \ll g(z),\ z \in \mathbb{U}^*,$$

then

$$|zf'(z) + 2f(z)| \le |zg'(z) + 2g(z)|$$

for

$$0 < |z| \le \frac{3 - \sqrt{6}}{3}.$$

Setting $p = 1$, $j = 1$ and $\gamma = 1$ in Example 2.1 we get:

Example 2.5. *Let the function* $f \in \Sigma$ *and suppose that* $g \in \mathcal{K}(\alpha)$. *If*

$$f'(z) \ll g'(z),\ z \in \mathbb{U}^*,$$

then

$$|zf''(z) + 3f'(z)| \le |zg''(z) + 3g'(z)|$$

for

$$0 < |z| \le \frac{3 - \alpha - \sqrt{\alpha^2 - 4\alpha + 6}}{3 - 2\alpha}.$$

Putting $\alpha = 0$ in Example 2.5 we obtain the following special case:

Example 2.6. *Let the function $f \in \Sigma$ and suppose that $g \in \mathcal{K}$. If*

$$f'(z) \ll g'(z),\ z \in \mathbb{U}^*,$$

then

$$|zf''(z) + 3f'(z)| \leq |zg''(z) + 3g'(z)|$$

for

$$0 < |z| \leq \frac{3-\sqrt{6}}{3}.$$

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