A CHARACTERIZATION OF $(\sigma, \tau)-$ DERIVATIONS ON VON NEUMANN ALGEBRAS

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Let A be a von Neumann algebra and M be a Banach $A-$module. It is shown that for every homomorphisms σ, τ on A, every bounded linear map $f : A \to M$ with property that $f(p^2) = \sigma(p)f(p)+f(p)\tau(p)$ for every projection p in A is a $(\sigma, \tau)-$derivation. Also, it is shown that a bounded linear map $f : A \to M$ which satisfies $f(ab) = \sigma(a)f(b) + f(a)\tau(b)$ for all $a, b \in A$ with $ab = S$, is a $(\sigma, \tau)-$ derivation if $\tau(S)$ is left invertible for fixed S.

Keywords: $(\sigma, \tau)-$Jordan derivation; $(\sigma, \tau)-$derivation

MSC2000: Primary 46L10; Secondary 46L05, 46H25, 46L57

1. Introduction

Let A be a Banach algebra. An $A-$module M is a Banach $A-$module if M is a Banach space and the $A-$module maps $(a; x) \to ax; A \times M \to M$; and $(x; a) \to xa; M \times A \to M$; satisfy $max\{\|ax\|, \|xa\|\} \leq \|a\|\|x\|$ for all $a \in A$ and $x \in M$. Suppose that A is unital. We denote the identity of A by 1. A Banach $A-$module M is called unital provided that $1x = x = x1$ for each $x \in M$.

Recently, a number of authors [3, 9, 10] have studied various generalized notions of derivations in the context of Banach algebras. There are some applications in the other fields of research [6]. Such mappings have been extensively studied in pure algebra; cf. [1, 2, 5]. A generalized concept of derivation is as follows.

Let A be a Banach algebra and M be a Banach $A-$module. Let $\sigma, \tau \in BL(A)$ be bounded linear maps on A. A linear mapping $d : A \to M$ is called a

- $(\sigma, \tau)-$derivation if

$$d(ab) = \sigma(a)d(b) + d(a)\tau(b) \quad (a, b \in A). \tag{1}$$

- $(\sigma, \tau)-$Jordan derivation if

$$d(a^2) = \sigma(a)d(a) + d(a)\tau(a) \quad (a \in A). \tag{2}$$

For instance every ordinary derivation (Jordan derivation) of an algebra A into an A–module M is an $(id_A, id_A)-$derivation $((id_A, id_A)-$Jordan derivation), where id_A is the identity mapping on the algebra A. As another example, every homomorphism (Jordan homomorphism) $h : A \to A$ is a $(\frac{h}{2}, \frac{h}{2})-$derivation $((\frac{h}{2}, \frac{h}{2})-$Jordan derivation).

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Clearly, every $(\sigma, \tau)-$derivation is a $(\sigma, \tau)-$Jordan derivation. Using the fact that $ab+ba=(a+b)^2-a^2-b^2$, it is easy to prove that the $(\sigma, \tau)-$Jordan derivation identity is equivalent to

$$d(ab+ba)=\sigma(a)d(b)+d(a)\tau(b)+\sigma(b)d(a)+d(b)\tau(a) \quad (a,b\in A). \tag{3}$$

We refer to [11] for the general theory of these notions.

2. **Main result**

In 1996, Johnson [7] proved the following theorem (see also Theorem 2.4 of [4]).

Theorem 2.1. *Suppose A is a C^*- algebra and M is a Banach $A-$module. Then each Jordan derivation $d: A\to M$ is a derivation.*

As an application of this theorem, we give the following result for characterization of $(\sigma, \tau)-$derivations on von Neumann algebras.

Theorem 2.2. *Let A be a von Neumann algebra and let σ, τ be bounded homomorphisms on A. Let M be a Banach $A-$module and $d: A\to M$ be a bounded linear map with property that $d(p^2)=\sigma(p)d(p)+d(p)\tau(p)$ for every projection p in A. Then d is a $(\sigma, \tau)-$derivation.*

Proof. We prove the theorem in two steps as follows.
STEP I. Recall that σ, τ are bounded homomorphisms on A and $d: A\to M$ is a bounded linear map with property that $d(p^2)=\sigma(p)d(p)+d(p)\tau(p)$ for every projection p in A. We show that d is a $(\sigma, \tau)-$Jordan derivation. Let $p, q\in A$ be orthogonal projections in A. Then $p+q$ is a projection wherefore by assumption,

$$\begin{aligned}\sigma(p)d(p)+d(p)\tau(p)+\sigma(q)d(q)+d(q)\tau(q)&=d(p)+d(q)=d(p+q)\\ &=\sigma(p+q)d(p+q)+d(p+q)\tau(p+q)=\sigma(p)d(p)+d(p)\tau(p)\\ &+\sigma(q)d(q)+d(q)\tau(q)+\sigma(p)d(q)+d(p)\tau(q)+\sigma(q)d(p)+d(q)\tau(p).\end{aligned}$$

This means that

$$\sigma(p)d(q)+d(p)\tau(q)+\sigma(q)d(p)+d(q)\tau(p)=0. \tag{4}$$

Let $a=\sum_{j=1}^n \lambda_j p_j$ be a combination of mutually orthogonal projections $p_1, p_2, ..., p_n\in A$. Then we have

$$\sigma(p_i)d(p_j)+d(p_i)\tau(p_j)+\sigma(p_j)d(p_i)+d(p_j)\tau(p_i)=0 \tag{5}$$

for all $i,j\in\{1,2,...,n\}$ with $i\neq j$. So

$$d(a^2)=d(\sum_{j=1}^n \lambda_j^2 p_j)=\sum_{j=1}^n \lambda_j^2 d(p_j). \tag{6}$$

On the other hand by (5), we obtain that

$$\begin{aligned}\sigma(a)d(a)+d(a)\tau(a)&=\sigma(\sum_{j=1}^n \lambda_j p_j)\sum_{j=1}^n \lambda_j d(p_j)+\sum_{j=1}^n \lambda_j d(p_j)\tau(\sum_{j=1}^n \lambda_j p_j)\\ &=\sum_{j=1}^n \lambda_j^2 d(p_j). \qquad (7)\end{aligned}$$

Combining (6) by (7) to get $d(a^2)=\sigma(a)d(a)+d(a)\tau(a)$. By the spectral theorem (see Theorem 5.2.2 of [8]), every self adjoint element $a\in A_{sa}$ is the norm–limit of finite combinations of mutually orthogonal projections. Since d,σ,τ are bounded, then

$$d(a^2)=\sigma(a)d(a)+d(a)\tau(a) \tag{8}$$

for all $a\in A_{sa}$. Replacing a by $a+b$ in (8), we obtain

$$d(ab+ba)=\sigma(a)d(b)+d(a)\tau(b)+\sigma(b)d(a)+d(b)\tau(b) \tag{9}$$

for all $a,b\in A_{sa}$. Let $a\in A$. Then there are $a_1,a_2\in A_{sa}$ such that $a=a_1+ia_2$. Hence,

$$\begin{aligned} d(a^2) &= d(a_1^2-a_2^2+i(a_1a_2+a_2a_1)) = d(a_1^2)-d(a_2^2)+id(a_1a_2+a_2a_1)\\ &= \sigma(a_1)d(a_1)+d(a_1)\tau(a_1)-\sigma(a_2)d(a_2)-d(a_2)\tau(a_2)\\ &+ i(\sigma(a_1)d(a_2)+d(a_1)\tau(a_2)+\sigma(a_2)d(a_1)+d(a_2)\tau(a_1))\\ &= \sigma(a)d(a)+d(a)\tau(a). \end{aligned}$$

STEP II. We show that every $(\sigma,\tau)-$ Jordan derivation from A into M is a $(\sigma,\tau)-$derivation. Let $d:A\to M$ be a $(\sigma,\tau)-$ Jordan derivation. It is easy to see that M is a Banach $A-$module by the following module actions:

$$a\cdot m=\sigma(a)m, \qquad m\cdot a=m\tau(a) \qquad (a\in A, m\in M)$$

we denote $M_{(\sigma,\tau)}$ the above $A-$module. By definition of $(\sigma,\tau)-$derivation, we have

$$d(a^2)=\sigma(a)d(a)+d(a)\tau(a)$$

for all $a\in A$. This means that d is a Jordan derivation from A into $M_{(\sigma,\tau)}$. Form Theorem 2.1, d is a derivation from A into $M_{(\sigma,\tau)}$. Hence, d is a $(\sigma,\tau)-$ derivation from A into M. □

Suppose that A is a Banach algebra and M is an $A-$module. Let S be in A. We say that S is right separating point of M if the condition $mS=0$ for $m\in M$ implies $m=0$.

Theorem 2.3. *Let A be a unital Banach algebra and M be a Banach $A-$module. Let S be in A and $\sigma,\tau\in Hom(A)$ be bounded homomorphisms with the properties that $\tau(S)$ is a right separating point of M and $\sigma(1)=\tau(1)=1$. Let $f:A\to M$ be a bounded linear map. Then the following assertions are equivalent*

a) $f(ab)=\sigma(a)f(b)+f(a)\tau(b)$ for all $a,b\in A$ with $ab=S$.

b) f is a $(\sigma,\tau)-$Jordan derivation which satisfies $f(Sa)=\sigma(S)f(a)+f(S)\tau(a)$ and $f(aS)=\sigma(a)f(S)+f(a)\tau(S)$ for all $a\in A$.

Proof. First suppose that (a) holds. Then we have

$$f(S)=f(1S)=\sigma(1)f(S)+f(1)\tau(S)=f(S)+f(1)\tau(S)$$

hence, by hypothesis, we get that $f(1)=0$. Let $a \in A$. For scalars λ with $|\lambda| < \frac{1}{\|a\|}$, $1-\lambda a$ is invertible in A. Indeed, $(1-\lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. Then

$$\begin{aligned} f(S) &= f[(1-\lambda a)(1-\lambda a)^{-1}S] = \sigma((1-\lambda a))f((1-\lambda a)^{-1}S) \\ &+ f((1-\lambda a))\tau((1-\lambda a)^{-1}S) = \sigma((1-\lambda a))f(\sum_{n=0}^{\infty} \lambda^n a^n S) \\ &- \lambda f(a)\tau(\sum_{n=0}^{\infty} \lambda^n a^n S) = f(S) + \sum_{n=1}^{\infty} \lambda^n [f(a^n S) \\ &- f(a)\tau(a^{n-1}S) - \sigma(a)f(a^{n-1}S)]. \end{aligned}$$

So

$$\sum_{n=1}^{\infty} \lambda^n [f(a^n S) - f(a)\tau(a^{n-1}S) - \sigma(a)f(a^{n-1}S)] = 0$$

for all λ with $|\lambda| < \frac{1}{\|a\|}$. Consequently

$$f(a^n S) - f(a)\tau(a^{n-1}S) - \sigma(a)f(a^{n-1}S) = 0 \tag{10}$$

for all $n \in \mathbb{N}$. Put $n=1$ in (10) to get

$$f(aS) = \sigma(a)f(S) + f(a)\tau(S). \tag{11}$$

Similarly, using equation $f(S) = f[S(1-\lambda a)^{-1}(1-\lambda a)]$ we get

$$f(Sa) = \sigma(S)f(a) + f(S)\tau(a)$$

for all $a \in A$.

Now, put $n=2$ in (10) to get

$$f(a^2 S) = \sigma(a)f(aS) + f(a)\tau(aS). \tag{12}$$

Combining (11), (12) to obtain

$$f(a^2 S) = \sigma(a)(\sigma(a)f(S) + f(a)\tau(S)) + f(a)\tau(aS). \tag{13}$$

Replacing a by a^2 in (11), we get

$$f(a^2 S) = \sigma(a^2)f(S) + f(a^2)\tau(S). \tag{14}$$

It follows from (13), (14) that

$$(f(a^2) - \sigma(a)f(a) - f(a)\tau(a))\tau(S) = 0. \tag{15}$$

On the other hand $\tau(S)$ is right separating point of M. Then by (15) f is a $(\sigma,\tau)-$Jordan derivation.

Now suppose that the condition (b) holds. Let $a, b \in A$ which satisfy $ab = S$. The next relation follows from a straightforward computation using the $(\sigma,\tau)-$Jordan derivation identities (2) and (3).

$$
\begin{aligned}
f(Sa) = f(aba) &= \frac{1}{2}[f(a(ab+ba) + (ab+ba)a) - f(a^2b + ba^2)] \\
&= \frac{1}{2}[f(a)\tau(ab+ba) + \sigma(a)f(ab+ba) + f(ab+ba)\tau(a) + \sigma(ab+ba)f(a) \\
&\quad - f(a^2)\tau(b) - \sigma(a^2)f(b) - f(b)\tau(a^2) - \sigma(b)f(a^2)] \\
&= f(a)\tau(ba) + \sigma(a)f(b)\tau(a) + \sigma(ab)f(a) \\
&= f(a)\tau(ba) + \sigma(a)f(b)\tau(a) + f(Sa) - f(S)\tau(a).
\end{aligned}
$$

So

$$[f(S) - f(a)\tau(b) - \sigma(a)f(b)]\tau(a) = 0.$$

Hence,

$$[f(S) - f(a)\tau(b) - \sigma(a)f(b)]\tau(a)\tau(b) = [f(S) - f(a)\tau(b) - \sigma(a)f(b)]\tau(S) = 0.$$

Since $\tau(S)$ is a right separating point of M, then

$$f(S) = f(a)\tau(b) + \sigma(a)f(b).$$

□

By Theorems 2.2 and 2.3, we have the following corollaries.

Corollary 2.1. *Let A be a von Neumann algebra and let σ, τ be bounded homomorphisms on A satisfying $\sigma(1) = \tau(1) = 1$. Let M be a Banach $A-$module and $d : A \to M$ be a bounded linear map. Then the following assertions are equivalent*

a) $d(p^2) = \sigma(p)d(p) + d(p)\tau(p)$ for every projection p in A.
b) $\sigma(a)d(a^{-1}) + d(a)\tau(a^{-1}) = 0$ for all invertible $a \in A$.
c) d is a $(\sigma, \tau)-$derivation.

Corollary 2.2. *Let A be a von Neumann algebra and let M be a Banach $A-$module and $d : A \to M$ be a bounded linear map. Then the following assertions are equivalent*

a) $d(p^2) = pd(p) + d(p)p$ for every projection p in A.
b) $ad(a^{-1}) + d(a)a^{-1} = 0$ for all invertible $a \in A$.
c) d is a derivation.

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