MODIFIED ITERATIVE ALGORITHMS WITH ACCELERATION TERMS FOR FIXED POINT AND EQUILIBRIUM PROBLEMS

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This paper proposes a double inertial subgradient extragradient algorithm for finding common solutions to pseudomonotone equilibrium problems and fixed points of a family of demicontractive mappings. The proposed method introduces a non-monotone step size selection strategy and sets up two steps inertial extrapolation process to accelerate the convergence speed. We establish strong convergence result for the algorithm without requiring prior knowledge of the Lipschitz-type constants of the bifunction.

Keywords: equilibrium problem, fixed point problem, pseudomonotone bifunction, demicontractive mapping, inertial item.

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1. Introduction

Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $f : H\times H \to \mathbb{R}$ be a bifunction with $f(x, x) = 0, \forall x \in C$. The equilibrium problem is to find $x^* \in C$ satisfying

$$f(x^*, y) \geq 0,\ \forall y \in C. \tag{1}$$

The solution set of problem (1) is denoted by $EP(f, C)$. It is well known that equilibrium problems can be applied to the study of a series of mathematical problems, such as variational inequality problems, minimax problems, optimization problems, Nash equilibrium problems and saddle point problems, see ([1, 9, 11, 19, 24, 25, 28, 33, 35, 38, 41, 42]). Due to the significance of equilibrium problem, many authors have extensively investigated it in recent years, see [8, 18, 34, 36]. One of the most popular methods is the proximal point method, see [12, 17, 21]. However, if the bifunction f is pseudomonotone, the convergence of the proximal method cannot be guaranteed. To overcome this issue, Tran et al. [26] employed the idea of Korpelevichs extragradient method and proposed the following algorithm where f is pseudomonotone and Lipschitz-type continuous:

$$\begin{cases} v_n = \arg\min\left\{\lambda f(u_n, u) + \frac{1}{2}\|u - u_n\|^2 : u \in C\right\}, \\ u_{n+1} = \arg\min\left\{\lambda f(v_n, u) + \frac{1}{2}\|u - u_n\|^2 : u \in C\right\}. \end{cases} \tag{2}$$

This algorithm needs to calculate two strongly convex programming problems in each iteration step. However, in cases where the two-valued function or the feasible set has a complex structure, the evaluation of the subprograms contained in the algorithm can be expensive. Lyashko et al. [14] adopted the slack projection technique, replaced the feasible set

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in the second step projection with a half space, and proposed the subgradient extragradient method:

$$\begin{cases} v_n = \arg\min\left\{\lambda f(w_n, u) + \frac{1}{2}\|u - w_n\|^2 : u \in C\right\}, \\ w_{n+1} = \arg\min\left\{\lambda f(v_n, u) + \frac{1}{2}\|u - w_n\|^2 : u \in C_n\right\}, \end{cases} \tag{3}$$

where C_n is a half space. Several variant forms of (3) have been proposed and studied, see [32, 34, 37, 43].

Use $CB(H)$ to denote the family of all nonempty closed bounded subsets of H. Let $T : H \to CB(H)$ be a multivalued mapping. Recall that the fixed point problem is to find $u \in H$ such that

$$u \in Tu. \tag{4}$$

The solution set of problem (4) is denoted by $F(T)$.

Iterative methods for fixed point problems are widely applied in optimization, image processing, signal processing and related fields, see [4, 5, 20, 24, 27, 31, 39, 40]. A fundamental approach to solve fixed point problems is the Mann iteration algorithm [16], which has the following manner: (i) $\{\alpha_n\} \subset (0, 1)$ and $x_0 \in H$ is any initial point; (ii) for known x_n, define

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)w_n, w_n \in Tx_n. \tag{5}$$

Another interesting method for solving (4) is Halpern's method [7] which generates a sequence $\{x_n\}$ as follows: (i) x_0 is an initial point and $u \in H$ is a fixed point; (ii) for given x_n, the next step x_{n+1} is defined by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)w_n, w_n \in Tx_n. \tag{6}$$

Due to the fact that some mathematical models' constraints are expressed through fixed points and equilibrium problems, such as in signal processing and image recovery, see [6, 13, 15, 22, 23], many researchers have recently focused on the common solution of equilibrium problems and fixed point problems, see [30, 37, 43]. Especially, Hieu [8] proposed a subgradient extragradient method to solve fixed point problems and equilibrium problems. Yang and Liu [32] proposed a subgradient extragradient algorithm that does not require prior knowledge of the Lipschitz constants of f. In 2020, Jolaoso et al. [10] proposed an inertial subgradient extragradient algorithm to solve the common solutions of equilibrium problems and fixed point problems and provided strongly convergent results (Algorithm 1.1). This method adopts an inertial extrapolation step to enhance the convergence speed of the algorithm.

Algorithm 1.1. Initialization: *Choose parameters* $\lambda_1 > 0$, $0 < \mu < 1$ *and* $\alpha \geq 3$. *Pick* $t_0, t_1 \in C$, *and put* $n = 1$.

Step 1. *Let* t_{n-1} *and* t_n *be given and choose* α_n *such that* $0 \leq \alpha_n \leq \bar{\alpha}_n$, *where*

$$\bar{\alpha}_n = \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\epsilon_n}{\|t_n - t_{n-1}\|}\right\}, & \text{if } t_n \neq t_{n-1}, \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases}$$

Compute

$$u_n = t_n + \alpha_n(t_n - t_{n-1}).$$

Step 2. *Calculate*

$$v_n = \arg\min\left\{\lambda_n f(u_n, y) + \frac{1}{2}\|u_n - y\|^2, y \in C\right\}.$$

If $u_n = v_n$, *then set* $u_n = v_n$ *and go to Step 4. Otherwise, go to Step 3.*

Step 3. *Choose* $g_n \in \partial_2 f(u_n, \cdot)(v_n)$ *such that* $u_n - \lambda_n g_n - v_n \in N_c(v_n)$. *Compute*

$$w_n = \arg\min \left\{ \xi \lambda_n f(v_n, y) + \frac{1}{2}\|u_n - y\|^2, y \in T_n \right\},$$

where

$$T_n := \{x \in H : \langle u_n - \lambda_n g_n - v_n, x - v_n \rangle \leq 0\}.$$

Step 4. *Compute*

$$t_{n+1} = \delta_n \phi(t_n) + (1 - \delta_n)(\gamma_{n,0} w_n + \sum_{i=1}^{n} \gamma_{n,i} y_{n,i}),$$

where $y_{n,i} \in T_i(w_n)$. *Update*

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\mu(\|u_n - v_n\|^2 + \|w_n - v_n\|^2)}{2[f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n)]}, \lambda_n \right\}, & \text{if } f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n) \neq 0, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Step 5. *Put* $n := n + 1$ *and go to* ***Step 1***.

Motivated by the work on algorithms for solving a common solution of pseudomonotone equilibrium problems and fixed point problems of demicontractive mappings, in this paper, we propose a double inertial subgradient extragradient algorithm. The proposed algorithm incorporates a non-monotone stepsize strategy and integrates an inertial extrapolation step inspired by Algorithm 1.1, thereby accelerating the convergence rate of the algorithm. Our method combines the viscosity method and the Mann-type algorithm to obtain the strong convergence result.

2. **Preliminaries**

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. $'\rightarrow'$ and $'\rightharpoonup'$ denote the strong convergence and the weak convergence, respectively. A mapping S: $H \to H$ is said to be contractive if there exists $\mu \in [0, 1)$ such that $\|Sx - Sy\| \leq \mu\|x - y\|, \forall x, y \in H$.

A subset $A \subset H$ is called the proximal set if for any $x \in H$, there exists $y \in A$ such that $\|x - y\| = d(x, A)$. The Hausdorff metric on $CB(H)$ is defined as:

$$\mathcal{H}(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}, \forall A, B \in CB(H).$$

Recall that a multivalued mapping $T : H \to CB(H)$ is called

(i) nonexpansive if $\mathcal{H}(Tx, Ty) \leq \|x - y\|, \forall x, y \in H$.

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $\mathcal{H}(Tx, Tp) \leq \|x - p\|, \forall x \in H, p \in F(T)$.

(iii) κ-demicontractive if $F(T) \neq \emptyset$ and there exists $\kappa \in [0, 1)$ such that

$$\mathcal{H}(Tx, Tp)^2 \leq \|x - p\|^2 + \kappa d(x, Tx)^2, \forall x \in H, p \in F(T).$$

A mapping $I - T$ is said to be demiclosed at zero if for any sequence $x_n \subset H$ with $x_n \rightharpoonup x^*$ and $x_n \to y_n$ for $y_n \in Tx_n$, then $x^* \in F(T)$.

Let C be a nonempty closed convex subset of H. The metric projection $P_C : H \to C$ is defined as $P_C(x) = \arg\min \{\|x - y\| : y \in C\}, \forall x \in H$. For any $x \in H$, $z = P_C(x)$ if and only if $\langle z - x, y - z \rangle \leq 0, \forall y \in C$.

Let C be a nonempty closed convex subset of H. A bifunction $f : H \times H \to \mathbb{R}$ is called

(i) strongly monotone on C if there is $\mu > 0$ satisfying

$$f(u, v) + f(v, u) \leq -\mu\|u - v\|^2, \forall u, v \in C;$$

(ii) monotone on C if $f(u, v) + f(v, u) \leq 0, \forall u, v \in C$;

(iii) pseudomonotone on C if $f(u,v) \geq 0 \Longrightarrow f(v,u) \leq 0, \forall u,v \in C$;
(iv) strongly pseudomonotone on C if there is $\mu > 0$ satisfying

$$f(u,v) \geq 0 \Rightarrow f(v,u) \leq -\mu\|u-v\|^2, \forall u,v \in C;$$

(v) Lipschtiz-type continuous on H if there are $c_1 > 0$ and $c_2 > 0$ such that

$$f(u,v) + f(v,u) \geq f(u,w) - c_1\|u-v\|^2 - c_2\|v-w\|^2, \forall u,v,w \in H.$$

A function $g : H \to \mathbb{R}$ is lower semicontinuous at $x \in H$ if and only if for all $x_n \to x$, we have $g(x) \leq \liminf_{n\to\infty} g(x_n)$. g is upper semicontinuous at x if $-g$ is lower semicontinuous at x.

For each $x, y \in H$, the subdifferential of a convex function $f(x,.)$ at y is denoted by $\partial_2 f(x,y)$, i.e., $\partial_2 f(x,y) := \{u \in H : f(x,z) \geq f(x,y) + \langle u, z-y\rangle, \forall z \in H\}$. In particular, $\partial_2 f(x,x) = \{u \in H : f(x,x) \geq \langle u, z-x\rangle, \forall z \in H\}$.

Lemma 2.1 ([29]). *Let $\{a_n\} \subset [0,+\infty)$, $\{\theta_n\} \subset (0,1)$ and $\{b_n\} \subset \mathbb{R}$ be three real sequences such that $a_{n+1} \leq (1-\theta_n)a_n + \theta_n b_n, \forall n \geq 1$. Then,*
(i) *If $\{b_n\}$ is bounded, then $\{a_n\}$ is also bounded.*
(ii) *If $\sum_{n=0}^{\infty} \theta_n = +\infty$ and $\limsup_{n\to\infty} b_n \leq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.2 ([3]). *Let C be a convex subset of H and $f : C \to \mathbb{R}$ be subdifferentiable on C. Then, x^* is a solution to the following convex problem:*

$$\min\{f(x) : x \in C\}$$

if and only if $0 \in \partial f(x^*) + N_C(x^*)$, where $N_C(x^*) := \{y \in H : \langle y, z - x^*\rangle \leq 0, \forall z \in C\}$ is the normal cone of C at x^*.

Lemma 2.3 ([2]). *Let H be a real Hilbert space, $t_i \in H (1 \leq i \leq n)$ and $\{\beta_i\} \subset (0,1)$ with $\sum_{i=1}^{n} \beta_i = 1$. Then,*

$$\|\sum_{i=1}^{n} \beta_i t_i\|^2 = \sum_{i=1}^{n} \beta_i\|t_i\|^2 - \sum_{i,j=1, i\neq j}^{n} \beta_i\beta_j\|t_i - t_j\|^2.$$

Lemma 2.4 ([15]). *Let $\{a_k\}$ be a sequence of real numbers such that there exists a nondecreasing subsequence $\{a_{k_i}\}$of $\{a_k\}$ such that $a_{k_i} < a_{k_i+1}$, for all $i \in N$. Then there exists a nondecreasing $\{m_n\} \subset N$ such that $\lim\limits_{n\to\infty} m_n = \infty$ and the following properties are satisfied for all (sufficiently large) numbers $n \in N$:*

$$a_{m_n} \leq a_{m_n+1} \quad \text{and} \quad a_n \leq a_{m_n+1}.$$

3. **Main results**

In this section, we introduce an algorithm for solving the equilibrium problem and fixed point problem and prove a strong convergence result.

Assumption 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. For $i \in N$, $T_i : H \to CB(H)$ is κ_i-demicontractive mapping, $I - T_i$ is demiclosed at zero and $T_i(x) = \{x\}$ for all $x \in F(T_i)$. Let ϕ be a τ-contractive on H. Suppose that $EP(f) \cap \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $f : H \times H \to \mathbb{R}$ be a bifunction satisfying the following conditions (A1)-(A4):
(A1) for every $x \in H$, $f(x,.)$ is convex, subdifferentiable and lower semicontinous on H;
(A2) f is psuedomonotone on C and $f(x,x) = 0, \forall x \in C$;
(A3) f is Lipshitz-type continous on H;
(A4) for every $y \in C$, $f(\cdot, y)$ is sequentially weakly upper semicontinuous on C.
Assume that the involved parameters satisfy the following conditions:
(B1) $\{\delta_n\} \subset (0,1)$, $\lim\limits_{n\to\infty} \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$;

(B2) $\{\epsilon_n\} \subset [0,\infty)$, $\{\nu_n\} \subset [0,\infty)$, $\lim\limits_{n\to\infty} \dfrac{\epsilon_n}{\delta_n} = 0$ and $\lim\limits_{n\to\infty} \dfrac{\nu_n}{\delta_n} = 0$;
(B3) $\{\gamma_{n,i}\}_{i=0}^{\infty} \subset (0,1)$, $\sum_{i=0}^{n} \gamma_{n,i} = 1$;
(B4) $\liminf_{n\to\infty}(\gamma_{n,0} - \kappa)\gamma_{n,i} > 0$ and $\kappa = \max\{\kappa_i\}$ for all $i \in \mathbb{N}\setminus\{0\}$.

Algorithm 3.1. ***Initialization.*** *Choose parameters: $\lambda_1 > 0$, $\mu, \eta \in (0,1)$, $\theta, \sigma \in [0,1)$, $\xi \in (0,1]$, $\{\rho_n\} \subset [0,\infty)$ and $\sum_{n=0}^{\infty} \rho_n < \infty$. Take $t_0, t_1 \in C$ and set $n = 1$.*

Step 1. *Given the current iterates t_{n-1} and t_n, choose α_n, β_n satisfying $0 \le \alpha_n \le \bar{\alpha}_n$ and $0 \le \beta_n \le \bar{\beta}_n$, where*

$$\bar{\alpha}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|t_n - t_{n-1}\|}\right\}, & \text{if } t_n \neq t_{n-1}, \\ \theta, & \text{otherwise}, \end{cases} \tag{7}$$

and

$$\bar{\beta}_n = \begin{cases} \min\left\{\sigma, \frac{\nu_n}{\|t_n - t_{n-1}\|}\right\}, & \text{if } t_n \neq t_{n-1}, \\ \sigma, & \text{otherwise}. \end{cases} \tag{8}$$

Compute

$$\begin{cases} s_n = t_n + \alpha_n(t_n - t_{n-1}), \\ u_n = t_n + \beta_n(t_n - t_{n-1}). \end{cases}$$

Step 2. *Calculate*

$$v_n = \arg\min\left\{\lambda_n f(u_n, y) + \frac{1}{2}\|u_n - y\|^2, y \in C\right\}.$$

If $u_n = v_n$, then put $u_n = v_n$ and go to Step 4. Otherwise, go to Step 3.

Step 3. *Choose $g_n \in \partial_2 f(u_n, \cdot)(v_n)$ such that $u_n - \lambda_n g_n - v_n \in N_c(v_n)$. Compute*

$$w_n = \arg\min\left\{\xi\lambda_n f(v_n, y) + \frac{1}{2}\|u_n - y\|^2, y \in T_n\right\},$$

where

$$T_n := \{x \in H : \langle u_n - \lambda_n g_n - v_n, x - v_n\rangle \le 0\}.$$

Step 4. *Calculate*

$$\begin{cases} x_n = \gamma_{n,0} w_n + \sum_{i=1}^{n} \gamma_{n,i} y_{n,i}, \\ z_n = (1-\eta)s_n + \eta x_n, \\ t_{n+1} = \delta_n \phi(z_n) + (1-\delta_n)x_n, \end{cases}$$

where $y_{n,i} \in T_i(w_n)$ and

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu(\|u_n - v_n\|^2 + \|w_n - v_n\|^2)}{2\left[f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n)\right]}, \lambda_n + \rho_n\right\}, \\ \qquad\qquad \text{if } f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n) > 0, \\ \lambda_n + \rho_n, \qquad \text{otherwise}. \end{cases}$$

Step 5. *Put $n := n + 1$ and go to* ***Step 1****.*

Lemma 3.1. *The sequence $\{\lambda_n\}$ generated by Algorithm 3.1 satisfies: $\lim\limits_{n\to\infty} \lambda_n = \lambda$ and $\min\left\{\dfrac{\mu}{2\max\{c_1, c_2\}}, \lambda_1\right\} \le \lambda \le \lambda_1 + P$, where $P = \sum_{n=1}^{\infty} \rho_n$.*

Proof. $\{\lambda_n\}$ is bounded. In fact, since f satisfies the Lipschtiz-type condition with constants c_1 and c_2, when $f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n) > 0$, we have

$$\frac{\mu(\|u_n - v_n\|^2 + \|w_n - v_n\|^2)}{2\left[f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n)\right]} \geq \frac{\mu(\|u_n - v_n\|^2 + \|w_n - v_n\|^2)}{2(c_1\|u_n - v_n\|^2 + c_2\|v_n - w_n\|^2)}$$
$$\geq \frac{\mu(\|u_n - v_n\|^2 + \|w_n - v_n\|^2)}{2\max\{c_1, c_2\}(\|u_n - v_n\|^2 + \|v_n - w_n\|^2)} \geq \frac{\mu}{2\max\{c_1, c_2\}}.$$

Using the definition of λ_{n+1}, we have $\min\left\{\frac{\mu}{2\max\{c_1, c_2\}}, \lambda_1\right\} \leq \lambda_n \leq \lambda_1 + P$. Next, we prove the convergence of the sequence $\{\lambda_n\}$. From the definition of $\{\lambda_n\}$, it follows that $\sum_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^+ \leq \sum_{n=0}^{\infty}\rho_n < \infty$ where $(\lambda_{n+1} - \lambda_n)^+ = \max\{0, \lambda_{n+1} - \lambda_n\}, (\lambda_{n+1} - \lambda_n)^- = \max\{0, -(\lambda_{n+1} - \lambda_n)\}$. Thus, $\sum_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^+$ is convergent. On the other hand, suppose for contradiction that $\sum_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^-$ is not convergent. Since

$$\lambda_{n+1} - \lambda_n = (\lambda_{n+1} - \lambda_n)^+ - (\lambda_{n+1} - \lambda_n)^-,$$
$$\lambda_{m+1} - \lambda_0 = \sum_{n=0}^{m}(\lambda_{n+1} - \lambda_n)^+ - \sum_{n=0}^{m}(\lambda_{n+1} - \lambda_n)^-. \tag{9}$$

From (9), if take $m \to \infty$, we obtain $\lambda_m \to +\infty$, which leads to a contradiction. Hence $\sum_{n=0}^{\infty}(\lambda_{n+1} - \lambda_0)^-$ must be convergent. Applying $m \to \infty$ in (9), we can conclude $\lim_{n\to\infty}\lambda_n = \lambda$. It is evident that $\min\left\{\frac{\mu}{2\max\{c_1, c_2\}}, \lambda_1\right\} \leq \lambda \leq \lambda_1 + P$. □

Lemma 3.2. *For all $p \in EP(f) \cap \bigcap_{i=1}^{\infty} F(T_i)$ and $n \geq 1$, we have*

$$\|w_n - p\|^2 \leq \|u_n - p\|^2 - (1 - \xi)\|u_n - w_n\|^2 - \xi(1 - \mu\frac{\lambda_n}{\lambda_{n+1}})(\|u_n - v_n\|^2 + \|w_n - v_n\|^2).$$

Proof. By the definition of w_n and Lemma 2.2, we have

$$0 \in \partial_2\left\{\xi\lambda_n f(v_n, w_n) + \frac{1}{2}\|w_n - u_n\|^2\right\} + N_{T_n}(w_n).$$

Then, there exist $u \in \partial_2 f(v_n, w_n)$ and $q \in N_{T_n}(w_n)$ such that

$$\xi\lambda_n u + w_n - u_n + q = 0. \tag{10}$$

By the subdifferentiability of f, we have

$$f(v_n, y) - f(v_n, w_n) \geq \langle u, y - w_n\rangle, \forall y \in C. \tag{11}$$

Besides, from $q \in N_{T_n}(w_n)$, we have

$$\langle q, w_n - y\rangle \geq 0, \forall y \in T_n. \tag{12}$$

By combining (10) and (12), we derive

$$\langle u_n - w_n, y - w_n\rangle \leq \xi\lambda_n\langle u, y - w_n\rangle, \forall y \in T_n. \tag{13}$$

From (11) and (13) to get $\langle u_n - w_n, y - w_n\rangle \leq \xi\lambda_n[f(v_n, y) - f(v_n, w_n)], \forall y \in T_n$. Let $y = p \in EP(f, C) \subset C \subset T_n$, we have $\langle u_n - w_n, p - w_n\rangle \leq \xi\lambda_n[f(v_n, p) - f(v_n, w_n)]$. Since f is pseudomonotone and $v_n \in C$, we obtain $f(v_n, p) \leq 0$. Then,

$$\xi\lambda_n f(v_n, w_n) \leq \langle u_n - w_n, w_n - p\rangle. \tag{14}$$

Similarly, since $g_n \in \partial_2 f(u_n, \cdot)(v_n)$, we obtain

$$f(u_n, z) - f(u_n, v_n) \geq \langle g_n, z - v_n\rangle, \forall z \in H.$$

Choose $z = w_n$ to obtain

$$f(u_n, w_n) - f(u_n, v_n) \geq \langle g_n, w_n - v_n\rangle. \tag{15}$$

Since $w_n \in T_n$, we have $\langle u_n - \lambda_n g_n - v_n, w_n - v_n\rangle \le 0$. It follows that

$$\lambda_n \langle g_n, w_n - v_n\rangle \ge \langle u_n - v_n, w_n - v_n\rangle. \tag{16}$$

Combining (15) and (16), we have

$$\lambda_n(f(u_n, w_n) - f(u_n, v_n)) \ge \langle u_n - v_n, w_n - v_n\rangle. \tag{17}$$

Note that $\lambda_{n+1}(f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n)) \le \frac{\mu}{2}(\|u_n - v_n\|^2 + \|w_n - v_n\|^2)$. It is equivalent to

$$\lambda_n(f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n)) \le \frac{\lambda_n}{\lambda n + 1}\frac{\mu}{2}(\|u_n - v_n\|^2 + \|w_n - v_n\|^2). \tag{18}$$

From (14), (17) and (18), we have

$$\mu\frac{\lambda_n}{\lambda_{n+1}}(\|u_n - v_n\|^2 + \|w_n - v_n\|^2) \ge 2\langle u_n - v_n, w_n - v_n\rangle + \frac{2}{\xi}\langle u_n - w_n, p - w_n\rangle.$$

On the other hand, since $2\langle u_n - v_n, w_n - v_n\rangle = \|u_n - v_n\|^2 + \|w_n - v_n\|^2 - \|u_n - w_n\|^2$, and

$$2\langle u_n - w_n, p - w_n\rangle = \|u_n - w_n\|^2 + \|p - w_n\|^2 - \|u_n - p\|^2,$$

we deduce

$$\|w_n - p\|^2 \le \|u_n - p\|^2 - (1-\xi)\|u_n - w_n\|^2 - \xi(1 - \mu\frac{\lambda_n}{\lambda_{n+1}})(\|u_n - v_n\|^2 + \|w_n - v_n\|^2). \tag{19}$$

□

Theorem 3.1. *Suppose that conditions (A1)-(A4)and (B1)-(B5) hold. Then, the sequence* $\{t_n\}$ *generated by Algorithm 3.1 converges strongly to* p*, where* $p = P_{EP(f)\cap\bigcap_{i=1}^{\infty} F(T_i)}(p)$.

Proof. By Lemma 2.3, we have

$$\begin{aligned}
\|x_n - p\|^2 &= \|\gamma_{n,0} w_n + \sum_{i=1}^{n}\gamma_{n,i} y_{n,i} - p\|^2 \\
&\le \gamma_{n,0}\|w_n - p\|^2 + \sum_{i=1}^{n}\gamma_{n,i}\|y_{n,i} - p\|^2 - \sum_{i=1}^{n}\gamma_{n,0}\gamma_{n,i}\|w_n - y_{n,i}\|^2 \\
&\le \gamma_{n,0}\|w_n - p\|^2 + \sum_{i=1}^{n}\gamma_{n,i} H^2(T_i w_n, T_i p) - \sum_{i=1}^{n}\gamma_{n,0}\gamma_{n,i}\|w_n - y_{n,i}\|^2 \\
&\le \gamma_{n,0}\|w_n - p\|^2 + \sum_{i=1}^{n}\gamma_{n,i}(\|w_n - p\|^2 + \kappa_i\|w_n - y_{n,i}\|^2) - \sum_{i=1}^{n}\gamma_{n,0}\gamma_{n,i}\|w_n - y_{n,i}\|^2 \\
&= \|w_n - p\|^2 - \sum_{i=1}^{n}(\gamma_{n,0} - \kappa)\gamma_{n,i}\|w_n - y_{n,i}\|^2 \qquad (20) \\
&\le \|w_n - p\|^2.
\end{aligned}$$

Since $\xi \in (0, 1]$, $\mu \in (0, 1)$ and $\lim_{n\to\infty} \lambda_n = \lambda$, we have that there exists $N_1 \ge 0$ such that for all $n \ge N_1$, $\xi(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}) \ge 0$. By (19), we have $\|w_n - p\|^2 \le \|u_n - p\|^2$. Therefore, from the definition of z_n, we have

$$\begin{aligned}\|z_n - p\| &= \|(1-\eta)s_n + \eta x_n - p\| \\ &\le (1-\eta)\|s_n - p\| + \eta\|x_n - p\| \\ &\le (1-\eta)\|s_n - p\| + \eta\|u_n - p\| \\ &\le (1-\eta)\|t_n - p\| + (1-\eta)\alpha_n\|t_n - t_{n-1}\| + \eta\|t_n - p\| + \eta\beta_n\|t_n - t_{n-1}\| \\ &= \|t_n - p\| + [(1-\eta)\alpha_n + \eta\beta_n]\|t_n - t_{n-1}\|.\end{aligned}$$

Combining the above inequality with the definition of t_{n+1}, we get

$$\begin{aligned}\|t_{n+1} - p\| &= \|\delta_n\phi(z_n) + (1-\delta_n)x_n - p\| \\ &= \|\delta_n(\phi(z_n) - p) + (1-\delta_n)(x_n - p)\| \\ &\le \delta_n\|\phi(z_n) - p\| + (1-\delta_n)\|x_n - p\| \\ &= \delta_n\|\phi(z_n) - \phi(p) + \phi(p) - p\| + (1-\delta_n)\|w_n - p\| \\ &\le \delta_n\tau\|z_n - p\| + \delta_n\|\phi(p) - p\| + (1-\delta_n)\|u_n - p\| \\ &= \delta_n\tau\|z_n - p\| + \delta_n\|\phi(p) - p\| + (1-\delta_n)\|t_n + \beta_n(t_n - t_{n-1}) - p\| \\ &\le \delta_n\tau\|z_n - p\| + \delta_n\|\phi(p) - p\| + (1-\delta_n)(\|t_n - p\| + \beta_n\|t_n - t_{n-1}\|) \\ &\le \delta_n\tau\|t_n - p\| + [(1-\eta)\alpha_n + \eta\beta_n + (1-\delta_n)\beta_n]\|t_n - t_{n-1}\| \\ &\quad + (1-\delta_n)\|t_n - p\| + \delta_n\|\phi(p) - p\| \\ &= (1-(1-\tau)\delta_n)\|t_n - p\| + (1-\tau)\delta_n[\frac{\phi(p) - p}{1-\tau} \\ &\quad + \frac{(1-\eta)\alpha_n + \eta\beta_n + (1-\delta_n)\beta_n}{\delta_n(1-\tau)}\|t_n - t_{n-1}\|].\end{aligned}$$

Let $M = 2\max\left\{\frac{\|\phi(p) - p\|}{1-\tau}, \sup_{n\ge N_1} \frac{(1-\eta)\alpha_n + \eta\beta_n + (1-\delta_n)\beta_n}{\delta_n(1-\tau)}\|t_n - t_{n-1}\|\right\}$. Then, we have

$$\|t_{n+1} - p\| \le (1-(1-\tau)\delta_n)\|t_n - p\| + 1 - \tau\delta_n M.$$

Applying Lemma 2.1(i), we conclude that $\{\|t_{n+1} - p\|\}$ is bounded. This implies that $\{t_n\}$ is bounded. Therefore, $\{s_n\}$, $\{u_n\}$, $\{w_n\}$, $\{v_n\}$ and $\{T_i w_n\}$ are bounded.

By the definition of u_n, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|t_n + \beta_n(t_n - t_{n-1}) - p\|^2 \\ &\le \|t_n - p\|^2 + \beta_n^2\|t_n - t_{n-1}\|^2 + 2\beta_n\|t_n - p\|\|t_n - t_{n-1}\| \\ &= \|t_n - p\|^2 + \beta_n\|t_n - t_{n-1}\|(\beta_n\|t_n - t_{n-1}\| + 2\|t_n - p\|) \\ &\le \|t_n - p\|^2 + \beta_n M_1\|t_n - t_{n-1}\|.\end{aligned}$$

where $M_1 = \sup_{n\ge N_1}(\beta_n\|t_n - t_{n-1}\| + 2\|t_n - p\|)$. Hence,

$$\begin{aligned}\|t_{n+1} - p\|^2 &= \|\delta_n\phi(z_n) + (1-\delta_n)x_n - p\|^2 \\ &\le (1-\delta_n)^2\|w_n - p\|^2 + 2\delta_n\langle\phi(z_n) - p, t_{n+1} - p\rangle \\ &\le (1-\delta_n)^2\|u_n - p\|^2 + 2\delta_n\|\phi(z_n) - \phi(p)\|\|t_{n+1} - p\| \\ &\quad + 2\delta_n\langle\phi(p) - p, t_{n+1} - p\rangle \\ &\le (1-\delta_n)^2(\|t_n - p\|^2 + \beta_n M_1\|t_n - t_{n-1}\|) + 2\delta_n\tau\|z_n - p\|\|t_{n+1} - p\| \\ &\quad + 2\delta_n\langle\phi(p) - p, t_{n+1} - p\rangle \\ &\le (1-\delta_n)^2(\|t_n - p\|^2 + \beta_n M_1\|t_n - t_{n-1}\|) + \delta_n\tau(\|z_n - p\|^2 + \\ &\quad \|t_{n+1} - p\|^2) + 2\delta_n\langle\phi(p) - p, t_{n+1} - p\rangle. \qquad (21)\end{aligned}$$

Note that

$$\begin{aligned}\|z_n - p\|^2 &\le (\|t_n - p\| + ((1-\eta)\alpha_n + \eta\beta_n)\|t_n - t_{n-1}\|)^2\\ &= \|t_n - p\|^2 + [(1-\eta)\alpha_n + \eta\beta_n]^2\|t_n - t_{n-1}\|^2\\ &\quad + 2[(1-\eta)\alpha_n + \eta\beta_n]\|t_n - p\|\|t_n - t_{n-1}\|. \qquad (22)\end{aligned}$$

Combining (21) and (22), we have

$$\begin{aligned}\|t_{n+1} - p\|^2 &\le (\frac{1 - 2\delta_n + \delta_n\tau}{1-\delta_n\tau})\|t_n - p\|^2 + \frac{\delta_n^2}{1-\delta_n\tau}\|t_n - p\|^2\\ &\quad + \frac{\beta_n M_1(1-\delta_n)^2}{1-\delta_n\tau}\|t_n - t_{n-1}\| + \frac{\delta_n\tau[(1-\eta)\alpha_n + \eta\beta_n]^2}{1-\delta_n\tau}\|t_n - t_{n-1}\|^2\\ &\quad + \frac{2\delta_n\tau[(1-\eta)\alpha_n + \eta\beta_n]}{1-\delta_n\tau}\|t_n - p\|\|t_n - t_{n-1}\|\\ &\quad + \frac{2\delta_n}{1-\delta_n\tau}\langle\phi(p) - p, t_{n+1} - p\rangle\\ &= (1 - \frac{2\delta_n(1-\tau)}{1-\delta_n\tau})\|t_n - p\|^2 + \frac{\delta_n^2}{1-\delta_n\tau}\|t_n - p\|^2\\ &\quad + \frac{\beta_n M_1(1-\delta_n)^2}{1-\delta_n\tau}\|t_n - t_{n-1}\| + \frac{\delta_n\tau[(1-\eta)\alpha_n + \eta\beta_n]^2}{1-\delta_n\tau}\|t_n - t_{n-1}\|^2\\ &\quad + \frac{2\delta_n\tau[(1-\eta)\alpha_n + \eta\beta_n]}{1-\delta_n\tau}\|t_n - p\|\|t_n - t_{n-1}\|\\ &\quad + \frac{2\delta_n}{1-\delta_n\tau}\langle\phi(p) - p, t_{n+1} - p\rangle. \qquad (23)\end{aligned}$$

We investigate two possible cases: Case 1 and Case 2.

Case 1. There is $N \ge N_1$ such that $\{\|t_n - p\|^2\}$ is a non-increasing sequence for all $n \ge N$. This implies the existence of $\lim_{n\to\infty}\|t_n - p\|^2$. Since $\{t_n\}$ is bounded, we have

$$\lim_{n\to\infty}(\|t_n - p\|^2 - \|t_{n+1} - p\|^2) = 0.$$

From (20), we have

$$\begin{aligned}\|t_{n+1} - p\|^2 &\le (1-\delta_n)\|x_n - p\|^2 + 2\delta_n\langle\phi(z_n) - p, t_{n+1} - p\rangle\\ &\le (1-\delta_n)[\|w_n - p\|^2 - \sum_{i=1}^{n}(\gamma_{n,0} - \kappa)\gamma_{n,i}\|w_n - y_{n,i}\|^2]\\ &\quad + 2\delta_n\langle\phi(z_n) - p, t_{n+1} - p\rangle\\ &\le (1-\delta_n)[\|u_n - p\|^2 - \sum_{i=1}^{n}(\gamma_{n,0} - \kappa)\gamma_{n,i}\|w_n - y_{n,i}\|^2]\\ &\quad + 2\delta_n\langle\phi(z_n) - p, t_{n+1} - p\rangle\\ &\le (1-\delta_n)[\|t_n - p\|^2 + \beta_n M_1\|t_n - t_{n-1}\|\\ &\quad - \sum_{i=1}^{n}(\gamma_{n,0} - \kappa)\gamma_{n,i}\|w_n - y_{n,i}\|^2] + 2\delta_n\langle\phi(z_n) - p, t_{n+1} - p\rangle.\end{aligned}$$

Then

$$\begin{aligned}(1-\delta_n)\sum_{i=1}^{n}(\gamma_{n,0} - \kappa)\gamma_{n,i}\|w_n - y_{n,i}\|^2 &\le (1-\delta_n)\|t_n - p\|^2 - \|t_{n+1} - p\|^2\\ &\quad + 2\delta_n\langle\phi(z_n) - p, t_{n+1} - p\rangle + (1-\delta_n)\beta_n M_1\|t_n - t_{n-1}\|.\end{aligned}$$

Since $\lim_{n\to\infty} \delta_n = 0$ and $\lim_{n\to\infty} \beta_n\|t_n - t_{n-1}\| = 0$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} (\gamma_{n,0} - \kappa)\gamma_{n,i}\|w_n - y_{n,i}\|^2 = 0.$$

According to (B4), we obtain

$$\lim_{n\to\infty} \|w_n - y_{n,i}\| = 0. \tag{24}$$

Based on (19), we obtain

$$\begin{aligned}
\|t_{n+1} - p\|^2 &\le (1 - \delta_n)\|x_n - p\|^2 + 2\delta_n\langle \phi(z_n) - p, t_{n+1} - p\rangle \\
&\le (1 - \delta_n)\|w_n - p\|^2 + 2\delta_n\langle \phi(z_n) - p, t_{n+1} - p\rangle \\
&\le (1 - \delta_n)[\|u_n - p\|^2 - (1 - \xi)\|u_n - w_n\|^2 - \xi(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}) \\
&\quad (\|u_n - v_n\|^2 + \|w_n - v_n\|^2)] + 2\delta_n\langle \phi(z_n) - p, t_{n+1} - p\rangle \\
&\le (1 - \delta_n)[\|t_n - p\|^2 + \beta_n M_1\|t_n - t_{n-1}\| - (1 - \xi)\|u_n - w_n\|^2 \\
&\quad - \xi(1 - \mu\frac{\lambda_n}{\lambda_{n+1}})(\|u_n - v_n\|^2 + \|w_n - v_n\|^2)] + 2\delta_n\langle \phi(z_n) - p, t_{n+1} - p\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
&(1 - \delta_n)[(1 - \xi)\|u_n - w_n\|^2 + \xi(1 - \mu\frac{\lambda_n}{\lambda_{n+1}})(\|u_n - v_n\|^2 + \|w_n - v_n\|^2)] \\
&\le (1 - \delta_n)\|t_n - p\|^2 - \|t_{n+1} - p\|^2 + (1 - \delta_n)\beta_n M_1\|t_n - t_{n-1}\| \\
&\quad + 2\delta_n\langle \phi(z_n) - p, t_{n+1} - p\rangle.
\end{aligned}$$

Similarly, we have

$$\lim_{n\to\infty} (1 - \xi)\|u_n - w_n\|^2 + \xi(1 - \mu\frac{\lambda_n}{\lambda_{n+1}})(\|u_n - v_n\|^2 + \|w_n - v_n\|^2) = 0.$$

Since $\forall n \ge N$, $\xi(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}) \ge 0$, $\xi \in (0, 1]$, we have

$$\lim_{n\to\infty} \|u_n - v_n\| = 0, \tag{25}$$

and

$$\lim_{n\to\infty} \|w_n - v_n\| = 0. \tag{26}$$

Combining (25) and (26), we have

$$\lim_{n\to\infty} \|u_n - w_n\| = 0. \tag{27}$$

From the definition of β_n, (24) and $\delta_n \to 0$, we have

$$\|t_n - u_n\| = \beta_n\|t_n - t_{n-1}\| = \frac{\beta_n}{\delta_n}\|t_n - t_{n-1}\|\delta_n \to 0, \tag{28}$$

$$\|x_n - w_n\| = \gamma_{n,0}\|w_n - w_n\| + \sum_{i=1}^{n} \|y_{n,i} - w_n\| \to 0, \tag{29}$$

and

$$\|t_{n+1} - x_n\| = \delta_n\|\phi(z_n) - x_n\| + (1 - \delta_n)\|x_n - x_n\| \to 0. \tag{30}$$

Take into account of (27)-(30), we obtain

$$\lim_{n\to\infty} \|t_{n+1} - t_n\| = 0. \tag{31}$$

Since $\{t_n\}$ is bounded, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $t_{n_k} \rightharpoonup t \in H$ and

$$\limsup_{n\to\infty} \langle \phi(p) - p, t_n - p\rangle = \lim_{k\to\infty} \langle \phi(p) - p, t_{n_k} - p\rangle = \langle \phi(p) - p, t - p\rangle.$$

So,

$$\limsup_{n\to\infty} \langle \phi(p) - p, t_n - p\rangle = \langle \phi(p) - p, t - p\rangle \leq 0. \tag{32}$$

Combining (31) and (32), we get

$$\limsup_{n\to\infty} \langle \phi(p) - p, t_{n+1} - p\rangle \leq 0. \tag{33}$$

Using (25)-(28), we have $v_{n_k} \rightharpoonup t$, $w_{n_k} \rightharpoonup t$ and $u_{n_k} \rightharpoonup t$, as $k \to \infty$. Since C is closed and convex set, so C is weakly closed, we can deduce that $t \in C$. From the definition of λ_{n+1} and (17), we have

$$\begin{aligned} \xi\lambda_{n_k} f(v_{n_k}, y) &\geq \xi\lambda_{n_k} f(v_{n_k}, w_{n_k}) + \langle u_{n_k} - w_{n_k}, y - w_{n_k}\rangle \\ &\geq \xi\lambda_{n_k}(f(u_{n_k}, w_{n_k}) - f(u_{n_k}, v_{n_k}) - \frac{\mu}{2\lambda_{n_k+1}}(\|u_{n_k} - v_{n_k}\|^2 \\ &\quad + \|w_{n_k} - v_{n_k}\|^2)) + \langle u_{n_k} - w_{n_k}, y - w_{n_k}\rangle \\ &\geq \xi\langle u_{n_k} - v_{n_k}, w_{n_k} - v_{n_k}\rangle - \frac{\xi\mu\lambda_{n_k}}{2\lambda_{n_k+1}}\|u_{n_k} - v_{n_k}\|^2 \\ &\quad - \frac{\xi\mu\lambda_{n_k}}{2\lambda_{n_k+1}}\|w_{n_k} - v_{n_k}\|^2 + \langle u_{n_k} - w_{n_k}, y - w_{n_k}\rangle, \forall y \in C. \end{aligned}$$

Taking the limit as $k \to \infty$ on the right-hand side of the above inequality and using the sequentially weakly upper semicontinutity of f, we have

$$0 \leq \limsup_{n\to\infty} f(v_{n_k}, y) \leq f(t, y), \forall y \in C.$$

Hence, we get $t \in EP(f, C)$. Since T_i are demiclosed at zero and (24), we obtain $t \in F(T_i)$. Therefore, $t \in EP(f) \cap \bigcap_{i=1}^{\infty} F(T_i)$.
It follows that

$$\begin{aligned} \lim_{n\to\infty}(&\frac{\beta_n M_1(1-\delta_n)^2}{2\delta_n}\|t_n - t_{n-1}\| + \frac{\delta_n\tau[(1-\eta)\alpha_n + \eta\beta_n]^2}{2\delta_n}\|t_n - t_{n-1}\|^2 \\ &+ \frac{2\delta_n\tau[(1-\eta)\alpha_n + \eta\beta_n]}{2\delta_n}\|t_n - p\|\|t_n - t_{n-1}\| + \frac{\delta_n^2}{2\delta_n}\|t_n - p\|^2) = 0. \end{aligned} \tag{34}$$

Using (24), (34) and Lemma 2.1(ii), we have

$$\lim_{n\to\infty} \|t_n - p\| = 0.$$

Case 2. Suppose that $\{\|t_n - p\|^2\}$ is not non-increasing.There exists a subsequence $\{\|t_{n_i} - p\|^2\}$ of $\{\|t_n - p\|^2\}$ such that $\{\|t_{n_i} - p\|^2\} \leq \{\|t_{n_i+1} - p\|^2\}$ holds for all $i \in N$. Then, according to Lemma 2.4, there exists a non-decreasing sequence $\{m_n\}$ such that $\lim_{n\to\infty} m_n = \infty$ and

$$\|t_{m_n} - p\|^2 \leq \|t_{m_n+1} - p\|^2 \quad \text{and} \quad \|t_n - p\|^2 \leq \|t_{m_n+1} - p\|^2, \forall n \in N.$$

Following a similar argument as in Case 1, we get

$$\lim_{n\to\infty} \|u_{m_n} - w_{m_n}\| = 0, \quad \lim_{n\to\infty} \|w_{m_n} - v_{m_n}\| = 0,$$

$$\lim_{n\to\infty} \|u_{m_n} - v_{m_n}\| = 0, \quad \lim_{n\to\infty} \|t_{m_n} - t_{m_n-1}\| = 0,$$

$$\lim_{n\to\infty} \|w_{m_n} - y_{m_n,i}\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \langle \phi(p) - p, t_{m_n+1} - p\rangle \leq 0.$$

By (23), we have

$$
\begin{aligned}
0 &\le \|t_{m_n+1}-p\|^2 - \|t_{m_n}-p\|^2 \\
&\le (1-\frac{2\delta_{m_n}(1-\tau)}{1-\delta_{m_n}\tau})\|t_{m_n}-p\|^2 + \frac{\delta_{m_n}^2}{1-\delta_{m_n}\tau}\|t_{m_n}-p\|^2 \\
&\quad + \frac{\beta_{m_n}M_1(1-\delta_{m_n})^2}{1-\delta_{m_n}\tau}\|t_{m_n}-t_{m_n-1}\| + \frac{\delta_{m_n}\tau[(1-\eta)\alpha_{m_n}+\eta\beta_{m_n}]^2}{1-\delta_{m_n}\tau}\|t_{m_n}-t_{m_n-1}\|^2 \\
&\quad + \frac{2\delta_{m_n}\tau[(1-\eta)\alpha_{m_n}+\eta\beta_{m_n}]}{1-\delta_{m_n}\tau}\|t_{m_n}-p\|\|t_{m_n}-t_{m_n-1}\| \\
&\quad + \frac{2\delta_{m_n}}{1-\delta_{m_n}\tau}\langle \phi(p)-p, t_{m_n+1}-p\rangle - \|t_{m_n}-p\|^2.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\frac{2(1-\tau)}{1-\delta_{m_n}\tau}\|t_{m_n}-p\|^2 \le{}& \frac{\delta_{m_n}}{1-\delta_{m_n}\tau}\|t_{m_n}-p\|^2 + \frac{\beta_{m_n}M_1(1-\delta_{m_n})^2}{\delta_{m_n}(1-\delta_{m_n}\tau)}\|t_{m_n}-t_{m_n-1}\| \\
&+ \frac{\tau[(1-\eta)\alpha_{m_n}+\eta\beta_{m_n}]^2}{1-\delta_{m_n}\tau}\|t_{m_n}-t_{m_n-1}\|^2 \\
&+ \frac{2\tau[(1-\eta)\alpha_{m_n}+\eta\beta_{m_n}]}{1-\delta_{m_n}\tau}\|t_{m_n}-p\|\|t_{m_n}-t_{m_n-1}\| \\
&+ \frac{2}{1-\delta_{m_n}\tau}\langle \phi(p)-p, t_{m_n+1}-p\rangle.
\end{aligned}
$$

As $n \to \infty$, the right-hand side of the above inequality tends to 0. This implies that $\lim_{n\to\infty} \|t_{m_n}-p\|^2 = 0$. Therefore, we have

$$0 \le \|t_n - p\|^2 \le \|t_{m_n+1}-p\|^2 \to 0, n \to \infty.$$

Hence, we conclude that the sequence $\{t_n\}$ converges strongly to p. □

4. **Conclusion**

In this paper, we introduce a strongly convergent iterative algorithm that approximates the common solution of the pseudomonotone equilibrium problem and the fixed point problem of demicontractive mappings. The algorithm we proposed sets up two inertial extrapolation steps, combines the Mann-type method and the viscosity approximation method, and involves the non-monotone stepsize rule. The strong convergence theorem is given under mild conditions.

REFERENCES

[1] *E. Blum and W. Oettli*, From optimization and variational inequalities to equilibrium problems, Math. Student., **63**(1994), 127-149.

[2] *C. E. Chidume and J. N. Ezeora*, Krasnoselskii-type algorithm for family of multi-valued strictly pseudo-contractive mappings, Fixed Point Theory Appl., **2014**(2014), 1-7.

[3] *P. Daniele, F. Giannessi and A. Maugeri*, Equilibrium problems and variational models, Kluwer, 2003.

[4] *Q. L. Dong, L. Liu and Y. Yao*, Self-adaptive projection and contraction methods with alternated inertial terms for solving the split feasibility problem, J. Nonlinear Convex Anal., **23**(2022), 591-605.

[5] *Q. L. Dong, Y. Peng and Y. Yao*, Alternated inertial projection methods for the split equality problem, J. Nonlinear Convex Anal., **22**(2021), 53-67.

[6] *Z. Fu, Y. Lin, D. Yang and S. Yang*, Fractional fourier transforms meet Riesz potentials and image processing, SIAM J. Imaging Sci., **17**(2024), 476-500.

[7] *B. Halpern*, Fixed points of nonexpanding maps, B. Amer. Math. Soc., **73**(1967), 957-961.

[8] *D. V. Hieu*, Halpern subgradient extragradient method extended to equilibrium problems, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., **111**(2017), 823-840.

[9] *Z. Jia*, Global boundedness of weak solutions for an attraction-repulsion chemotaxis system with p-Laplacian diffusion and nonlinear production, Discrete Cont. Dyn. Syst.-Ser. B, **28**(2023), 4847-4863.

[10] *L. O. Jolaoso and M. Aphane*, A self-adaptive inertial subgradient extragradient method method for pseudomonotone equilibrium and common fixed point problems, Fixed Point Theory Appl., **2020**(2020), 1-22.

[11] *I. V. Konnov*, Equilibrium models and variational inequalities, Elsevier, Amsterdam, 2007.

[12] *I. V. Konnov*, Combined relaxation methods for variational inequalities, Lecture Notes in Economics and Mathematical Systems, **1**(2001), 495.

[13] *G. Liu, X. Chen, Z. Shen, Y. Liu and X. Jia*, Reachable set estimation for continuous delayed singularly perturbed systems with bounded disturbances, Appl. Math. Comput., **416**(2022), Article number 126751.

[14] *S. I. Lyashko, V. V. Semenov and T. A. Voitova*, Low-cost modification of Korpelevich's methods for monotone equilibrium problems, Cybernet. Systems Anal., **47**(2011), 631-639.

[15] *P. E. Mainge*, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal., **16**(2008), 899-912.

[16] *W. R. Mann*, Mean value methods in iteration, Proc. Amer. Math. Soc., **4**(1953), 506-510.

[17] *B. Martinet*, Regularisation d'inequations variationnells par approximations successive, Rev. Fran. Inform. Rech. Oper., **4**(1970), 154-158.

[18] *G. Mastroeni*, Gap function for equilibrium problems, J. Global Optim., **27**(2003), 411-426.

[19] *G. Mastroeni*, On auxiliary principle for equilibrium problems, Equilibrium Problems and Variational Models., **68**(2003), 289-298.

[20] *A. Moudafi*, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl., **241**(2000), 46-55.

[21] *R. T. Rockafellar*, Monotone operators and the proximal point algorithm, SIAM J. Control Optim., **14**(1976), 877-898.

[22] *S. Shi, Z. Fu and Q. Wu*, On the average operators, oscillatory integrals, singulars, singular integrals and their applications, J. Appl. Anal. Comput., **14**(2024), 334-378.

[23] *S. Shi, Z. Zhai and L. Zhang*, Characterizations of the viscosity solution of a non-local and nonlinear equation induced by the fractional p-Laplace and the fractional p-convexity, Adv. Calc. Var., **17**(2024), 195-207.

[24] *S. Shi and L. Zhang*, Dual characterization of fractional capacity via solution of fractional p-Laplace equation, Math. Nach., **293**(2020), 2233-2247.

[25] *B. S. Thakur and M. Postolache*, Existence and approximation of solutions for generalized extended nonlinear variational inequalities, J. Inequal. Appl., **2013**(2013), Article number 590.

[26] *D. Q. Tran, M. L. Dung and V. H. Mguyen*, Extragradient algorithms extended to equilibrium problems, Optim., **57**(2008), 749-776.

[27] *Q. Sun, J. Ren and F. Zhao*, Sliding mode control of discrete-time interval type-2 fuzzy Markov jump systems with the preview target signal, Appl. Math. Comput., **435**(2022), Article number 127479.

[28] *X. S. Wang, Z. Q. Wang and Z. Jia*, Global weak solutions for an attraction-repulsion chemotaxis system with p-Laplacian diffusion and logistic source, Acta Math. Sci., **44**(2024), 909-924.

[29] *H. K. Xu*, Iterative algorithms for nonlinear operators, J. London. Math. Soc., **66**(2002), 240-256.

[30] *M. R. Xu, S. Liu and Y. Lou*, Persistence and extinction in the anti-symmetric Lotka-Volterra systems, J. Differ. Equations, **387**(2024), 299-323.

[31] *Y. M. Xue, J. K. Han, Z. Q. Tu and X. Y. Chen*, Stability analysis and design of cooperative control for linear delta operator system, AIMS Math., **8**(2023), 12671-12693.

[32] *J. Yang and H. Liu*, The subgradient extragradient method extended to pseudomonotone equilibrium problems and fixed point problems in Hilbert space, Optim. Lett., **14**(2020), 1-14.

[33] *Y. Yao, A. Adamu and Y. Shehu*, Strongly convergent inertial forward-backward-forward algorithm without on-line rule for variational inequalities, Acta Math. Sci., **44**(2024), 551-566.

[34] *Y. Yao, O. S. Iyiola and Y. Shehu*, Three inertial acceleration algorithms for solving non-monotone equilibrium problems in Hilbert spaces, Acta Math. Sci., **45**(2025), 1674-1700.

[35] *Y. Yao, L.O. Jolaoso and Y. Shehu*, C-FISTA type projection algorithm for quasi-variational inequalities, Numer. Algorithms, 98(2025), 1781-1798.

[36] *Y. Yao, L. O. Jolaoso, Y. Shehu and J. C. Yao*, New Douglas-Rashford Splitting Algorithms for Generalized DC Programming with Applications in Machine Learning, J. Sci. Comput., **103**(2025), article numbe 88.

[37] *Y. Yao, H. Li and M. Postolache*, Iterative algorithms for split equilibrium problems of monotone operators and fixed point problems of pseudo-contractions, Optim., **71(9)**(2022), 2451-2469.

[38] *Y. Yao and M. Postolache*, Iterative methods for pseudomonotone variational inequalities and fixed point problems, J. Optim. Theory Appl., **155**(2012), 273-287.

[39] *Y. Yao, N. Shahzad, M. Postolache and J. C. Yao*, Convergence of self-adaptive Tseng-type algorithms for split variational inequalities and fixed point problems, Carpathian J. Math., **40**(2024), 755-770.

[40] *L. J. Zhu, J. C. Yao and Y. Yao*, Approximating solutions of a split fixed point problem of demicontractive operators, Carpathian J. Math., **40**(2024), 195-206.

[41] *L. J. Zhu and Y. Yao*, Algorithms for approximating solutions of split variational inclusion and fixed point problems, Mathematics, **11** (2023), Article number 641.

[42] *L. J. Zhu and Y. Yao*, Modified splitting algorithms for approximating solutions of split variational inclusions in Hilbert spaces, Fixed Point Theory, **25**(2024), 773-784.

[43] *L. J. Zhu, Y. Yao and M. Postolache*, Projection methods with linesearch technique for pseudomonotone equilibrium problems and fixed point problems, U.P.B. Sci. Bull., Series A, **83**(2021), 3-14.