APPLICATIONS OF EXTENSION THEOREMS OF LINEAR OPERATORS TO MAZUR - ORLICZ AND MOMENT PROBLEMS

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The aim of the present work is to give applications to concrete spaces of functions and operators of some theorems of extension of linear operators, with two constraints. Constrained interpolation problems are solved (particular cases of the general Markov moment problem), and applications of a variant of Mazur – Orlicz theorem are considered. The latter represents "the half" of the moment problem, in the sense that the interpolation equalities are replaced by inequalities. The general statements mentioned above are consequences of a more general theorem of extension of linear operators. The involved spaces have a natural topological linear order relation structure.

Key words: extesion of linear operators, moment problem, Mazur-Orlicz theorem
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1. Introduction

The aim of the present work is to apply general results on constrained extension of linear operators to the moment problem and Mazur-Orlicz theorem on concrete spaces. Using Hahn-Banach results in various applications (the moment problem, flows in infinite networks, transport problems, economic problems) is a useful technique (see [4]-[9], [12]-[15] and the references therein). In the present work, applications of a variant of Mazur-Orlicz theorem are considered as optimization problems with two opposite types of constraints (Section 2). The practical meaning is obvious and follows from the statement of Theorem 1. Being given the elements $x_j \in X, y_j \in Y, j \in J,$ an important problem is that of finding necessary and sufficient (or sufficient) conditions for the existence of a linear solution $F : X \to Y,$ of the interpolation problem $F(x_j) = y_j, j \in J,$ satisfying two constraints. Here X, Y are usually Banach lattices of functions or operators, Y being order-complete. For such spaces, the extension theorems of Hahn -

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Banach type do work. Due to these structures on $X, Y,$ we can apply the results from [2] and [17].

If x_j are the basic polynomials $t^j = t_1^{j_1} \cdots t_n^{j_n}, n \geq 2,$ then we have a multidimensional real classical moment problem. The upper constraint $F \leq P$ controls the continuity and the norm of the solution F. The lower constraint consists usually in the fact that F is positive on the positive cone of X. Hence the solution might have an integral representation by means of a positive scalar or vector measure. Two other important related questions appear namely the uniqueness and the construction of the solutions [1], [3], [14], [18]. As background of this work, we refer to [1], [2], [9], [16], [17]. The paper is organized as follows. Section 2 contains the statement of the general Mazur-Orlicz theorem and two related applications. Section 3 contains two general results on extension of linear operators. Each of these theorems is followed by an application to concrete spaces. All the general statements involved have the same root: Theorem 1 from [10], (see also [11]), recently recalled in [14], [15]. Section 4 concludes the paper.

2. Applications of Mazur-Orlicz theorem

We start this section by recalling the following generalization of the Mazur-Orlicz Theorem (see [12]). The practical meaning is obvious.

Theorem 1. (Theorem 5 [12]). *Let X be a preordered linear space, Y an order-complete vector lattice, $\{x_j; j \in J\} \subset X, \{y_j; j \in J\} \subset Y$ given finite or infinite families of elements. Let $P : X \to Y$ be a sublinear operator*

The following statements are equivalent:

(a) *there exists a linear operator $F \in L(X, Y)$ such that*

$$F(x_j) \geq y_j \ \forall j \in J, F(x) \geq 0 \ \forall x \in X_+, F(x) \leq P(x) \ \forall x \in X;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j; j \in J_0\} \subset R_+$, we have*:

$$\sum_{j \in J_0} \lambda_j x_j \leq x \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq P(x).$$

The next result of this Section uses the order relation given by the coefficients in spaces of analytic functions. On the other hand, let H be a complex Hilbert space, $U_0 \in \mathrm{A}(H)$ a selfadjoint operator from H into H. One defines:

$$Y_1 = \{U \in A(H); UU_0 = U_0U\}, Y = \{U \in Y_1; UV = VU \ \forall V \in Y_1\},$$
$$Y_+ = \{U \in Y; \langle U(h), h\rangle \geq 0 \ \forall h \in H\} \quad (1)$$

Obviously, Y defined by (1) is a commutative algebra of selfadjoint operators. Moreover, Y is a vector lattice, being complete with respect to the order relation [2], [7], and the operatorial norm on Y is solid:

$$|U| \leq |V| \Rightarrow \|U\| \leq \|V\|, U, V \in Y.$$

Recall that for an element U of the lattice Y, there exists (cf. [2])

$$|U| := \sup\{U, -U\} = U \vee (-U) = +\sqrt{U^2}.$$

The next result is an application of Theorem 1 to the space X of all absolutely convergent power series in the disc $|z| < r$, with real coefficients, continuous up to the boundary. The order relation is given by the coefficients: we write

$$\sum_{n \in N} \lambda_n z^n \prec \sum_{n \in N} \gamma_n z^n \Leftrightarrow (\lambda_n \leq \gamma_n, \forall n \in N).$$

Denote $\varphi_n(z) = z^n, n \in N, |z| \leq r.$ Let Y be the space defined by (1), $(B_n)_{n \in N}$ a sequence in Y, and $U \in Y_+$.

Theorem 2. *The following statements are equivalent:*

(a) *there is a linear positive operator* $F \in L_+(X, Y)$, *such that*

$$F(\varphi_n) \geq B_n, n \in N, |F(\varphi)| \leq \sum_{n \in N} |a_n| \cdot U^n, \forall \varphi = \sum_{n \in N} a_n \varphi_n \in X;$$

(b) *the following relations hold*

$$B_n \leq U^n, \quad n \in N;$$

Proof. $(b) \Rightarrow (a)$. One applies theorem 1 to $x_j = \varphi_j, j \in N.$ If

$$\sum_{j \in J_0} \lambda_j \varphi_j \leq \psi := \sum_{n \in N} \alpha_n \varphi_n \left(\Rightarrow \ \alpha_n \in R_+, n \in N, \lambda_j \leq \alpha_j, j \in J_0\right),$$

then the hypothesis and the above relations yield:

$$\lambda_j B_j \leq \lambda_j U^j \leq \alpha_j \cdot U^j, j \in N \Rightarrow$$
$$\sum_{j \in J_0} \lambda_j B_j \leq \sum_{j \in J_0} \alpha_j \cdot U^j \leq \sum_{n \in N} \alpha_n \cdot U^n = \sum_{n \in N} |\alpha_n| \cdot U^n =$$
$$|\psi|(U) = P(\psi), P(\varphi) := \sum_{n \in N} |a_n| \cdot U^n = P(-\varphi), \varphi = \sum_{n \in N} a_n \varphi_n \in X.$$

Notice that the definition of the order relation on the space X implies

$$\left|\sum_{n\in \mathrm{N}}\alpha_n\varphi_n\right| = |\psi| = \sum_{n\in \mathrm{N}}|\alpha_n|\varphi_n.$$

Hence, the assertions from (b), Theorem 1 are accomplished and the conclusion follows from a direct application of the latter theorem. On the other hand, $(a)\Rightarrow(b)$ is almost obvious, since $B_n \le F(\varphi_n)$ for all $n \in \mathrm{N}$ lead to

$$B_n \le F(\varphi_n) \le |F(\varphi_n)| \le P(\varphi_n) = U^n, \quad n \in \mathrm{N}.$$

This concluded the proof. □

Next we consider the space X of all absolutely convergent power series in the open unit polydisc $\prod_{k=1}^{n}\{|z_k|<1\}$, with real coefficients, and the order relation given by

$$\sum_{j\in \mathrm{N}^n}\alpha_j\varphi_j \prec \sum_{j\in \mathrm{N}^n}\beta_j\varphi_j \Leftrightarrow \left(\alpha_j \le \beta_j,\ j\in \mathrm{N}^n\right),$$

where

$$\varphi_j \in X,\ \varphi_j(z_1,\dots,z_n) = z_1^{j_1}\cdots z_n^{j_n},\ j=(j_1,\dots,j_n)\in N^n.$$

The relations (1) define the space Y. Let $A_k, B_k,\ k=1,\dots,n$ be positive elements of Y such that their norms are strictly smaller than one, and $(U_j)_{j\in N^n}$ a sequence in $Y, \alpha,\beta \in R_+$. Next we use the fact that X is a vector lattice. In particular, for positive elements $A_k, k=1,\dots,n$ of Y, we have

$$\psi = \sum_{j\in \mathrm{N}^n}\gamma_j\varphi_j \Rightarrow |\psi| = \sum_{j\in \mathrm{N}^n}|\gamma_j|\cdot\varphi_j,\quad |\psi|(A_1,\dots,A_n) = \sum_{j\in \mathrm{N}^n}|\gamma_j|\cdot A_1^{j_1}\cdots A_n^{j_n}.$$

The last definition has sense since the power series defining ψ converges absolutely in the unit polydisc, and Y is complete with respect to the norm – topology.

Theorem 3. *The following statements are equivalent:*

(a) *there exists a linear positive operator* $F \in L_+(X,Y)$ *such that*

$$U_j \le F(\varphi_j),\ j\in N^n,\ F(\psi) \le \alpha\cdot|\psi|(A_1,\dots,A_n) + \beta\cdot|\psi|(B_1,\dots,B_n),\ \psi\in X;$$

(b) $U_j \le \alpha\cdot A_1^{j_1}\cdots A_n^{j_n} + \beta\cdot B_1^{j_1}\cdots B_n^{j_n},\ \forall j=(j_1,\dots,j_n)\in N^n.$

Proof. The implication $(a)\Rightarrow(b)$ is obvious, since $\varphi_j = |\varphi_j|,\ j \in N^n$ and the hypothesis (a) lead to

$$U_j \le F(\varphi_j) \le \alpha \cdot |\varphi_j|(A_1,\ldots,A_n) + \beta \cdot |\varphi_j|(B_1,\ldots,B_n) =$$
$$\alpha \cdot A_1^{j_1} \cdots A_n^{j_n} + \beta \cdot B_1^{j_1} \cdots B_n^{j_n}.$$

In order to prove the converse implication, we apply Theorem 1, $(b)\Rightarrow(a)$, so that we have to verify the implication mentioned at point (b) of Theorem 1. Let $J_0 \subset N^n$ be a finite subset and $\{\lambda_j\}_{j\in J_0} \subset R_+$. Then using the hypothesis (b) of the present theorem, the following implications hold:

$$\sum_{j\in J_0} \lambda_j \varphi_j \le \psi = \sum_{j\in N^n} \gamma_j \varphi_j \Rightarrow 0 \le \lambda_j \le \gamma_j \ \forall j \in J_0, \gamma_j \ge 0, \forall j \in N^n,$$

$$U_j \le \alpha \cdot A_1^{j_1} \cdots A_n^{j_n} + \beta \cdot B_1^{j_1} \cdots B_n^{j_n},\ \forall j \in N^n \Rightarrow$$

$$\sum_{j\in J_0} \lambda_j U_j \le \sum_{j\in J_0} \lambda_j \left(\alpha \cdot A_1^{j_1} \cdots A_n^{j_n} + \beta \cdot B_1^{j_1} \cdots B_n^{j_n}\right) \le$$

$$\alpha \cdot \left(\sum_{j\in J_0} \gamma_j \cdot A_1^{j_1} \cdots A_n^{j_n}\right) + \beta \cdot \left(\sum_{j\in J_0} \gamma_j \cdot B_1^{j_1} \cdots B_n^{j_n}\right) \le$$

$$\alpha \cdot \left(\sum_{j\in N^n} \gamma_j \cdot A_1^{j_1} \cdots A_n^{j_n}\right) + \beta \cdot \left(\sum_{j\in N^n} \gamma_j \cdot B_1^{j_1} \cdots B_n^{j_n}\right) =$$

$$\alpha \cdot \psi(A_1,\ldots,A_n) + \beta \cdot \psi(B_1,\ldots,B_n) = \alpha \cdot |\psi|(A_1,\ldots,A_n) + \beta \cdot |\psi|(B_1,\ldots,B_n) = P(\psi).$$

Notice that $|\psi| = \psi$, since all the coefficients $\gamma_j,\ j \in N^n$ are nonnegative. Now a direct application of Theorem 1 leads to the existence of a linear positive operator F from X into Y, such that $F(\varphi_j) \ge U_j,\ j \in N^n,\ F(\psi) \le P(\psi), \psi \in X$.

This concludes the proof. □

Theorem 4. *Let*

$$X = L^1_\mu(M),\ \mu \ge 0,\ \{\varphi_j\}_{j\in J} \subset X,\ \{y_j\}_{j\in J} \subset R,\ \mu$$

being a $\sigma -$ finite measure. Assume that the intersection of the supports of two different functions $\varphi_{j_k} \ne \varphi_{j_l}$ has measure zero. The following statements are equivalent:

(a) *there exists* $h \in L_{\mu}^{\infty}(M)$ *such that*

$$0 \le h(x) \le 1 \ \mu - a.e., \int_M h\varphi_j d\mu \ge y_j, \ j \in J;$$

(b) *the following inequalities hold*

$$y_j \le \int_M \varphi_j^+ d\mu, \ j \in J.$$

Proof. The implication $(a) \Rightarrow (b)$ is almost obvious, because of the qualities of h. Namely we have:

$$y_j \le \int_M h\varphi_j d\mu \le \int_M h\varphi_j^+ d\mu \le \int_M \varphi_j^+ d\mu, \ j \in J.$$

For the converse, let $J_0 \subset J$ be a finite subset and $\{\lambda_j; j \in J_0\} \subset R_+$ be such that $\sum_{j \in J_0} \lambda_j \varphi_j \le \psi$. Using the hypothesis on the supports, we deduce

$$\sum_{j \in J_0} \lambda_j \varphi_j^+ = \left(\sum_{j \in J_0} \lambda_j \varphi_j \right)^+ \le \psi^+ \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le \sum_{j \in J_0} \lambda_j \int_M \varphi_j^+ d\mu =$$

$$\int_M \left(\sum_{j \in J_0} \lambda_j \varphi_j \right)^+ d\mu \le \int_M \psi^+ d\mu \le \int_M |\psi| d\mu = P(\psi).$$

We have used the fact that the scalars $\lambda_j, \ j \in J_0$ are nonnegative. Application of Theorem 1 to $P(\psi) = \|\psi\|_1$ leads to the existence of a linear positive form F of norm at most one, such that $F(\varphi_j) \ge y_j, \ j \in J$. This functional has a representation by means of a function h with the qualities stated at point (a). This concludes the proof. □

Remark 5. The set of the solutions h concerning theorem 4 is compact in the weak topology with respect to the dual pair (L^1, L^∞). For connections to extreme points and for results concerning the truncated moment problem on a bounded interval see [14, Section 4]. For a related moment problem on the positive semiaxes, see [15, Theorem 7 and Corollary 8].

3. On Markov moment problem

We recall the following abstract moment problem [12]. It is another way of writing a previous result [10], [11], in the moment problem setting (see [14], Theorem 2). This previous result is a generalization of H. Bauer's Theorem 5.4 [17].

Theorem 6. (see [12]). *Let* X, Y, *be as in Theorem 1,* $P : X \to Y$ *a convex operator,* $\{x_j\}_{j\in J} \subset X$, $\{y_j\}_{j\in J} \subset Y$ *given families. The following assertions are equivalent:*

(a) *there exists a linear positive operator* $F : X \to Y$ *such that*

$$F(x_j) = y_j \ \forall j \in J, \ F(x) \le P(x) \ \forall x \in X;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j\in J_0} \subset R$, *we have:*

$$\sum_{j\in J_0} \lambda_j x_j \le x \Rightarrow \sum_{j\in J_0} \lambda_j y_j \le P(x)$$

Theorem 7. *With the notations and under the hypothesis of Theorem 4, the following statements are equivalent;*

(a) *There exists* $h \in L^{\infty}_{\mu}(M)$, $0 \le h \le 1$, *such that* $\int_M \varphi_j h \, d\mu = y_j$, $j \in J$;

(b) *for any finite subset* $J_0 \subset J$ *and any family* $\{\lambda_j\}_{j\in J_0} \subset R$, *we have*

$$\sum_{j\in J_0} \lambda_j y_j \le \sum_{j\in J_0} \lambda_j \int_M \varphi_j^+ d\mu.$$

Proof. The implication $(a) \Rightarrow (b)$ is almost obvious, due to the qualities of h and using the assumption on the supports of φ_j, $j \in J$. Namely we have:

$$\sum_{j\in J_0} \lambda_j y_j = \int_M h \cdot \left(\sum_{j\in J_0} \lambda_j \varphi_j \right) d\mu \le$$

$$\int_M h \cdot \left(\sum_{j\in J_0} \lambda_j \varphi_j \right)^+ d\mu \le \int_M \left(\sum_{j\in J_0} \lambda_j \varphi_j \right)^+ d\mu = \int_M \left(\sum_{j\in J_0} \lambda_j \varphi_j^+ \right) d\mu.$$

For the converse implication, we apply $(b) \Rightarrow (a)$ of Theorem 6. We verify the implication mentioned at point (b), Theorem 6, also using the assumption (b) of the present theorem. We have:

$$\sum_{j\in J_0}\lambda_j\varphi_j \le \psi \Rightarrow \left(\sum_{j\in J_0}\lambda_j\varphi_j\right)^+ = \sum_{j\in J_0}\lambda_j\varphi_j^+ \le \psi^+ \Rightarrow$$

$$\sum_{j\in J_0}\lambda_j y_j \le \int_M\left(\sum_{j\in J_0}\lambda_j\varphi_j^+\right)d\mu = \int_M\left(\sum_{j\in J_0}\lambda_j\varphi_j\right)^+ d\mu$$

$$\le \int_M \psi^+ d\mu \le \int_M |\psi|\, d\mu = \|\psi\|_1,\ \psi \in X.$$

Applying Theorem 6, there is an $R-$ linear positive functional F on X, of norm at most one, which is the solution of the interpolation problem

$$F(\varphi_j) = y_j,\ j \in J.$$

By measure theory arguments [16], the conclusion follows. □

Next we recall an earlier result on the abstract Markov moment problem, in order to apply it to the multidimensional classical Markov moment problem. It is a general convexity and extension of linear operators result, with many applications [6]. This result represents a further consequence of Theorem 1 [10], recalled recently in [14], [15]. The proof of Theorem 8 from below can be found in [11]. If V is a convex neighborhood of the origin in a locally convex space X, we denote by p_V the gauge (Minkowski functional) [17] attached to V:

$$p_V(x) = \inf\{\lambda > 0;\ x \in \lambda V\},\ x \in X.$$

Theorem 8. (see [11], [6], [13]). *Let X be a locally convex space, Y an order complete vector lattice with strong order unit u_0 and $S \subset X$ a vector subspace. Let $A \subset X$ be a convex subset. Assume that the set A has the following properties*

(a) *there exists a (convex) neighborhood V of the origin such that $(S+V)\cap A = \Phi$, (Φ is the empty set: A and S are distanced);*

(b) *A is bounded.*

Then for any equicontinuous family of linear operators $\{f_j\}_{j\in J} \subset L(S,Y)$ and for any $\widetilde{y} \in Y_+ \setminus \{0\}$, there exists an equicontinuous family $\{F_j\}_{j\in J} \subset L(X,Y)$ such that

$$F_j|_S = f_j \text{ and } F_j|_A \ge \widetilde{y},\ \forall j \in J.$$

Moreover, if V is a neighborhood of the origin such that

$$f_j(V\cap S) \subset [-u_0, u_0],\quad (S+V)\cap A = \Phi,$$

$$0 < \alpha \in R \ s.t.\ p_V|_A \le \alpha, \quad \alpha_1 > 0 \ s.t.\ \widetilde{y} \le \alpha_1 u_0,$$

then the following relations hold

$$F_j(x) \le (1+\alpha+\alpha_1) p_V(x) \cdot u_0, \quad x \in X,\ j \in J.$$

In the next result X will be the space of all continuous functions φ in the closed polydisc $\prod_{k=1}^{n} \{|z_k| \le r_k\}$, that can be represented by an absolutely convergent series in the corresponding open polydisc. We denote

$$\varphi_j(z_1,\dots,z_n) = z_1^{j_1} \cdots z_n^{j_n},\ |z_k| < r_k,\ k=1,\dots,n,\ j=(j_1,\dots,j_n) \in \mathrm{N}^n.$$

Let H be a Hilbert space, and Y the space defined by (1). Let $A_1,\dots,A_n$ be positive elements of Y, such that $\|A_k\| < r_k,\ k=1,\dots,n.$ In the sequel $\{\psi_j\}_{j\in \mathrm{N}^n}$ will be a subset of X such that

$$\psi_j(0,\dots,0)=1,\ \|\psi_j\|_\infty \le M, \forall j \in \mathrm{N}^n.$$

Theorem 9. *With the notations and under the hypothesis from above, let*

$$(B_j)_{j\in N^n},\ j_k \ge m_0 \ge 1,\ k=1,\dots,n$$

be a sequence in Y *and* $\widetilde{B} \in Y_+ \setminus \{0\}$. *Assume that*

$$|B_j| \le A_1^{j_1} \cdots A_n^{j_n},\ j=(j_k)_{k=1}^n,\ j_k \ge m_0,\ k=1,\dots n.$$

Then there exists a linear operator $F \in L(X,Y)$ *such that*:

$$F(\varphi_j) = B_j,\ j \in \mathrm{N}^n,\ j_k \ge m_0,\ k=1,\dots,n,\ F(\psi_j) \ge \widetilde{B},\ j \in \mathrm{N}^n,$$

$$F(\varphi) \le \left(1+M+\|\widetilde{B}\| \cdot \prod_{k=1}^{n}\left(1-\frac{\|A_k\|}{r_k}\right)\cdot\left(\frac{r_k}{\|A_k\|}\right)^{m_0}\right)\cdot \|\varphi\|_\infty \cdot u_0,$$

$$u_0 := \prod_{k=1}^{n} \frac{r_k}{r_k-\|A_k\|}\cdot\left(\frac{\|A_k\|}{r_k}\right)^{m_0} \cdot I.$$

Proof. From the assumptions on the functions

$$\varphi_j,\ j_k \ge m_0 \ge 1,\ k=1,\dots,n,\ \psi_j,\ j \in \mathrm{N}^n,$$

we infer that

$$\|s-\psi\|_\infty \ge |s(0)-\psi(0)| = 1,\ \forall s \in S = Sp\{\varphi_j;\ j_k \ge m_0,\ k=1,\dots n\},$$
$$\forall \psi \in A := co(\{\psi_j;\ j \in \mathrm{N}^n\}),$$

the convex hull of the set $\{\psi_j;\ j \in N^n\}$. It follows that $(S + B(0,1)) \cap A = \Phi$, where Φ denotes the empty set, and $B(0,1)$ is the unit ball centered at the origin. Thus

$$V = B(0,1), \quad p_V = \|\ \|_\infty,\ p_{V|A} \le M =: \alpha,$$

where p_V stands for the gauge attached to $V = B(0,1)$. We also have $\widetilde{B} \le \|\widetilde{B}\| \cdot I$. Moreover if

$$s = \sum_{j \in J_0} \lambda_j \varphi_j \in S \cap B(0,1),\ j_k \ge m_0,\ k = 1,\dots,n,$$

then Cauchy inequalities, the fact that the $A_j,\ j = 1,\dots,n$ are commuting, as well as the relation $A \le \|A\| \cdot I$, (where A is a selfadjoint operator), yield

$$f(s) := \sum_{j \in J_0} \lambda_j B_j \le \sum_{j \in J_0} |\lambda_j| \cdot |B_j| \le \sum_{j \in J_0} \frac{1}{r_1^{j_1} \cdots r_n^{j_n}} A_1^{j_1} \cdots A_n^{j_n} \le$$

$$\left(\sum_{j_1 \ge m_0} \left(\frac{A_1}{r_1} \right)^{j_1} \right) \cdots \left(\sum_{j_n \ge m_0} \left(\frac{A_n}{r_n} \right)^{j_n} \right) \le \prod_{k=1}^{n} \left(\frac{\|A_k\|}{r_k} \right)^{m_0} \cdot \frac{r_k}{r_k - \|A_k\|} \cdot I := u_0.$$

It follows that

$$\widetilde{B} \le \|\widetilde{B}\| \cdot \prod_{k=1}^{n} \frac{r_k - \|A_k\|}{r_k} \cdot \left(\frac{r_k}{\|A_k\|} \right)^{m_0} u_0 := \alpha_1 \cdot u_0.$$

Now all conditions of Theorem 8 are accomplished. An application of the latter theorem leads to the existence of a linear operator

$$F \in L(X,Y),\ F(\varphi_j) = B_j,\ \forall j_k \ge m_0,\ F(\psi_j) \ge \widetilde{B},\ j \in \mathrm{N}^n,$$

$$F(\varphi) \le \left(1 + M + \|\widetilde{B}\| \cdot \prod_{k=1}^{n} \left(1 - \frac{\|A_k\|}{r_k} \right) \cdot \left(\frac{r_k}{\|A_k\|} \right)^{m_0} \right) \cdot \|\varphi\|_\infty \cdot u_0,$$

$$u_0 := \prod_{k=1}^{n} \frac{r_k}{r_k - \|A_k\|} \cdot \left(\frac{\|A_k\|}{r_k} \right)^{m_0} \cdot I.$$

This concludes the proof. □

4. Conclusions

Section 2 gives characterizations for the existence of the solutions of three concrete Mazur-Orlicz problems. In Section 3, we consider a similar problem to that of theorem 4, in the interpolation setting. On the other hand, sufficient condition for the existence of the solution of a Markov moment problem is stated and discussed. Both general results of Sections 2, respectively 3, are consequences of the same general theorem on extension of linear operators with two constraints [10], [11], recalled recently in [14, Theorem 1], [15]. The way of proving Theorems 4 and respectively 7 shows that Mazur-Orlicz-type results are quite different to the corresponding Markov moment problems, So, one can say that there are major differences in solving Mazur-Orlicz and Markov moment problems, even for similar statements. The functions ψ_j, $j \in \mathrm{N}^n$ of theorem 9 can be concrete elementary normalized entire functions (in each separate variable), related to the exponential function.

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