RELATIVISTIC RESULTS OBTAINED BY CLASSICAL ARGUMENTS

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Problema reflecției luminii pe o suprafață în mişcare este tratată printr-un calcul nerelativist obținându-se aceleaşi rezultate ca în teoria relativității restrânse. Se studiază variațiile frecvenței undei reflectate şi ale unghiului de reflecție în funcție de unghiul de incidență şi de viteza suprafeței. Se discută efectul Doppler relativist în acelaşi context.

We present the problem of reflection at a moving surface and show that relativistic results may be deduced from simple non-relativistic computations. The variations of the frequency of the reflected wave and of the reflected angle are studied as functions of the incident angle and of the velocity of the surface. In the same framework we discuss relativistic Doppler effect.

Keywords: reflection at a moving boundary, Special relativity, non-relativistic computations

1. Introduction

The Special Relativity (SR) was introduced by Einstein in 1905 in order to explain the electromagnetism of moving bodies. It complies with all electromagnetic equations and elucidates all the experiments done with moving bodies since the middle of the XIX-th century till now, in particular those attempting to measure the influence of the source velocity with respect to the observer.

This paper does not have the purpose to contradict SR. It challenges only the way we understand some results of light interaction with moving media. More specifically we show that the same results given by relativistic arguments for reflection at a moving surface could be obtained also by non-relativistic computations. This is a rather unexpected result, the common belief being that non-relativistic formulae are at best only approximations of those of SR, with their validity restricted to small velocities. In the same context we discuss the connection between relativistic Doppler effect and the reflection at a moving surface.

The second chapter contains a sketch of the famous Michelson-Morley experiment which set forth a striking feature of reflection at a moving surface, namely that the Snell-Descartes laws are invalid. Not only the reflection angle is

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not equal to the incident one, but the frequencies of the two beams differ as well. We present the relativistic explanation of these relations, as well as a non-relativistic demonstration of the results. The fourth part contains graphical behaviour of the interesting relations. The final chapter includes some comments. We outline here the problem of refraction at a moving surface.

2. Reflection of light at a moving mirror

2.1. The Michelson-Morley experiment revisited

The Michelson-Morley experiment is so famous that we shall not describe it thouroughly. Fig. 1 presents only a part of this experiment, namely the transverse propagating beam. The light comes from the left and splits, half being reflected and half transmitted in the point *A*. One of the fractions goes along *AC*, in the direction in which Earth moves with velocity ***V***. The other part travels along *AB*, is reflected back in *B* and attains the point *C* in the same time in which the entire device moves with the Earth from *A* to *C*.

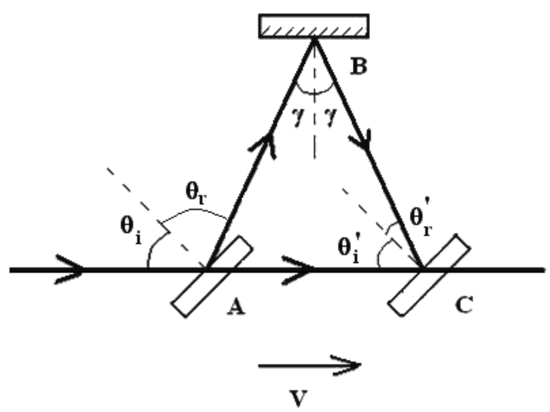

Fig. 1. The geometry of the transverse light ray in the Michelson-Morley experiment.

In Fig. 1 we have underlined an obvious feature of this reflection-refraction phenomenon: the reflection angles are not equal to the incident ones. Even if the angles are greatly exaggerated on the drawing it may be seen that the angles of reflection are not equal to the incident angles: $\theta_i \neq \theta_r$ and $\theta_i' \neq \theta_r'$. Is this an accurate result? Or we may think that the angles are equal and the rays just deviate a little from the centre of the beam-splitter?

In fact during the reflection at a moving surface the reflection angle is no more equal to the incident angle, as is the case for stationary media. The well known Snell law for refraction changes also. Besides, the frequencies of reflected and refracted waves are different from the frequency of the incident ray.

2.2. Relativistic proof for the reflection at a moving boundary

We follow here Sommerfeld [1]. It is obvious that the reflected and refracted quantities should depend on the incident angle, but also on the angle between the surface and the direction in which points the velocity of the surface movement. A detailed computation is rather tedious. Two important situations appear:

a) the mirror moves in a direction tangential to its plane surface; it may be shown that the laws of reflection and refraction do not change in this case

b) the mirror moves in the direction perpendicular to its surface; here the results are different from the usual Snell laws.

Therefore we shall present a simplified version of the situation, where the interface moves in a normal direction, as in Fig.2 below.

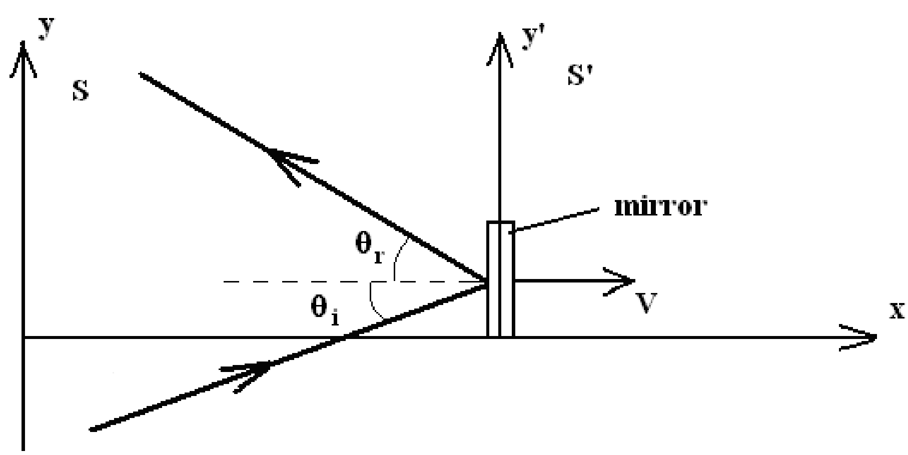

Fig 2. Simplified version of the reflection at a moving mirror.

The general case is obtained from the final relations (4) and (5) by replacing the velocity V by its normal component to the surface. Obviously we assume that the velocity is constant.

In SR the calculus begins by defining two reference systems: one associated with the light source considered at rest and designated by the letter S and the other moving with the mirror and denoted by S'. The idea of the explanation is to work in the reference system *S'* which moves with the mirror. In this system hold the usual laws of Snellius and Descartes: the reflection angle equals the incidence one and the beams have the same frequency. We shall shortly discuss the transmitted beam in the final chapter. In this system the wave 4-vector of the incident light is written as:

$$\ldots' = \left(k'_1,\, k'_2,\, k'_3,\, ik'_4\right) = \left(k'\cos\theta_i,\, k'\sin\theta_i,\, 0,\, i\frac{\omega'}{c}\right) \tag{1}$$

Here $k' = \omega'/c$ is the modulus of the wave-vector, ω' is the frequency of the wave in the S' system and c is the velocity of light in vacuum. After reflection on the mirror at rest in *S*', the only change is the sign of the k'_1 component and so the four-vector $\ldots'^{r}$ of the reflected light is

$$\ldots'^{r} = \left(-k'\cos\theta_i,\, k'\sin\theta_i,\, 0,\, i\frac{\omega'}{c}\right) \tag{2}$$

Now we go back to the laboratory reference system *S* by an inverse Lorentz transformation and get the components of $\ldots^{r}$, the reflected wave four-vector:

$$k_1^r = \gamma\left(k_1'^r - i\beta k_4'^r\right) \qquad k_2^r = k_2'^r \qquad k_3^r = k_3'^r \qquad k_4^r = \gamma\left(k_4'^r + i\beta k_1'^r\right) \tag{3}$$

Here $\beta = V/c$ and $\gamma = 1\big/\sqrt{1-\beta^2}$. We introduce in Eq. (3) the components from Eq. (2). After some algebra, we find eventually for the reflected frequency:

$$\omega_r = \omega_0 \frac{\left(1+\beta^2\right) - 2\beta\cos\theta_i}{1-\beta^2} \tag{4}$$

The result for the angle of reflection is:

$$\cos\theta_r = \frac{\left(1+\beta^2\right)\cos\theta_i - 2\beta}{\left(1+\beta^2\right) - 2\beta\cos\theta_i} \tag{5}$$

In the preceding relations ω_0 is the frequency of the incident beam.

2.3. Classical proof for the reflection at a moving boundary

We present here a non-relativistic proof of the above results. This demonstration is based on the continuity of the electromagnetic field across the surface between two media. To be specific, tangential components of the electric field are continuous on an interface without superficial current densities.

Assume a plane electromagnetic wave with frequency ω_0 travels in a medium with the refractive index n_1 and is incident at an angle θ_i to a plane surface of a transparent body with the refractive index n_2. The surface moves in the positive direction of the *x* axis, i.e. perpendicular to the surface. Hence its position is given by the equation $y = Vt$. The reflection and transmission angles are denoted by θ_r and θ_t, the frequencies of the two waves by ω_r and ω_t respectively. As we work in pre-relativistic physics the time is uniform in all systems. Let's postpone the transmitted wave for a while.

The phases of the plane waves in the incident and reflected rays are equal. This result is written as

$$\omega_0\left[t - \frac{n_1}{c}\left(x\sin\theta_i + Vt\cos\theta_i\right)\right] = \omega_r\left[t - \frac{n_1}{c}\left(x\sin\theta_r - Vt\cos\theta_r\right)\right] \quad (6)$$

The coefficients of *t*, as well as those of *x* must be equal. We get:

$$\omega_0\left(1 - \frac{n_1}{c}V\cos\theta_i\right) = \omega_r\left(1 + \frac{n_1}{c}V\cos\theta_r\right) \quad (7)$$

and

$$\omega_0 n_1 \sin\theta_i = \omega_r n_1 \sin\theta_r \quad (8)$$

Usual trigonometry gives the equation:

$$\left(1 - n_1^2\beta^2\right)\omega_r^{\ 2} - 2\omega_0\left(1 - n_1\beta\cos\theta_i\right)\omega_r + \omega_0^2\left(1 - 2n_1\beta\cos\theta_i + n_1^2\beta^2\right) = 0 \quad (9)$$

The two roots are real. The first one is the known result $\omega_r = \omega_0$ and using Eqs. (7) one gets the Snell law for reflection $\theta_r = \theta_i$; due to the Doppler effect this result must be discarded. The other root is exactly the relation (4), obtained above by relativistic arguments. Using this value in (7) one finds the reflection angle, which is given by the same relation (5) as before.

So, our non-relativistic approach gives precisely the results obtained from Special Relativity. It is strange enough that such an intricate relativistic result may be obtained from classical considerations. Usually non-relativist relations are just approximations of the accurate relativist results, correct only at low velocities. In our situation the paradox is enhanced because we study electromagnetic fields. We shall return later to this situation.

3. Graphical variation of important physical quantities

In this chapter we present graphically the variation of the reflection angle θ_r and of the frequency of the reflected beam ω_r for a mirror moving perpendicular to its plane surface. The two independent variables are the incident angle θ_i and the ratio $\beta = V/c$.

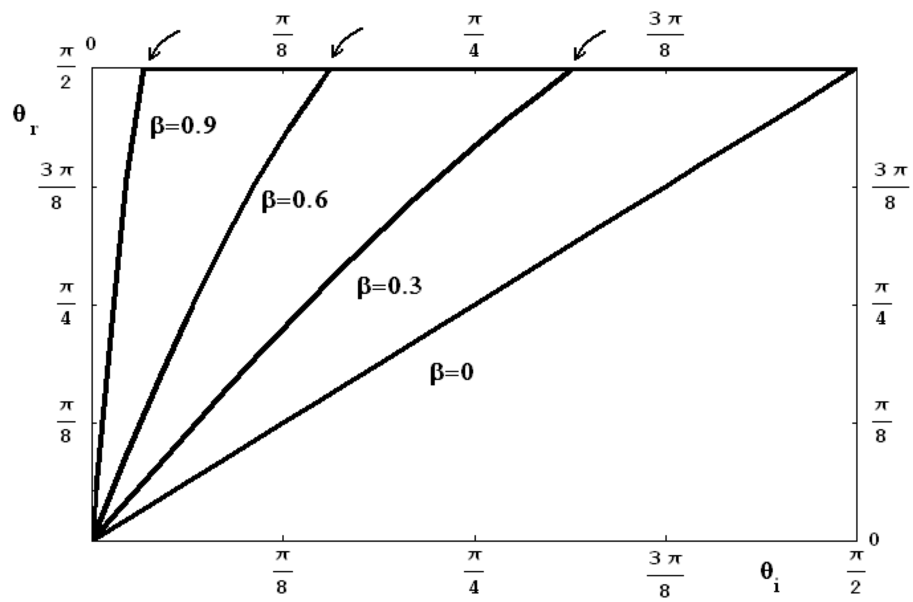

Fig. 3. Variation of the reflection angle θ_r as a function of the incident angle θ_i, with $\beta = V/c$ as a parameter. The arrows designate points where $\theta_r = \pi/2$, or $\cos\theta_r = 0$

The bisector $\beta = 0$ represents the usual Snellius-Descartes law $\theta_r = \theta_i$. But for moving media the dependence $\theta_r = \theta_r(\theta_i, \beta = const)$ is nonlinear. The reflection angle becomes $\pi/2$ at certain values of the incident angle smaller than $\pi/2$, values indicated on Fig. 3 by small arrows. These values are obviously given by the equation $(1+\beta^2)\cos\theta_i - 2\beta = 0$, obtained by cancelling the numerator of (5). For $\beta = 0.3$ reflection parallel to the surface appears at an incident angle of roughly 57º; for this angle drops to about 6 º for *b*=0.9. Another numerical example is $\theta_r = \theta_r\left(\pi/4, \sqrt{2}-1\right) = \pi/2$. As an angle of reflection greater than $\pi/2$ has no physical meaning, we have continued the curves with a straight horizontal line. This occurrence appears only at positive values of the velocity, i.e. when the surface moves away from the light source. All happens as if the surface would "refuse" to reflect light; the reflected beam is similar to the whispering modes in wave-guides.

Fig. 4 shows another view of the variation of the reflection angle, this time the continuous function $\theta_r = \theta_r(\theta_i, \beta)$. The ridge on this graph continued by the horizontal region in the upper right corner represents points where $\theta_r = \pi/2$.

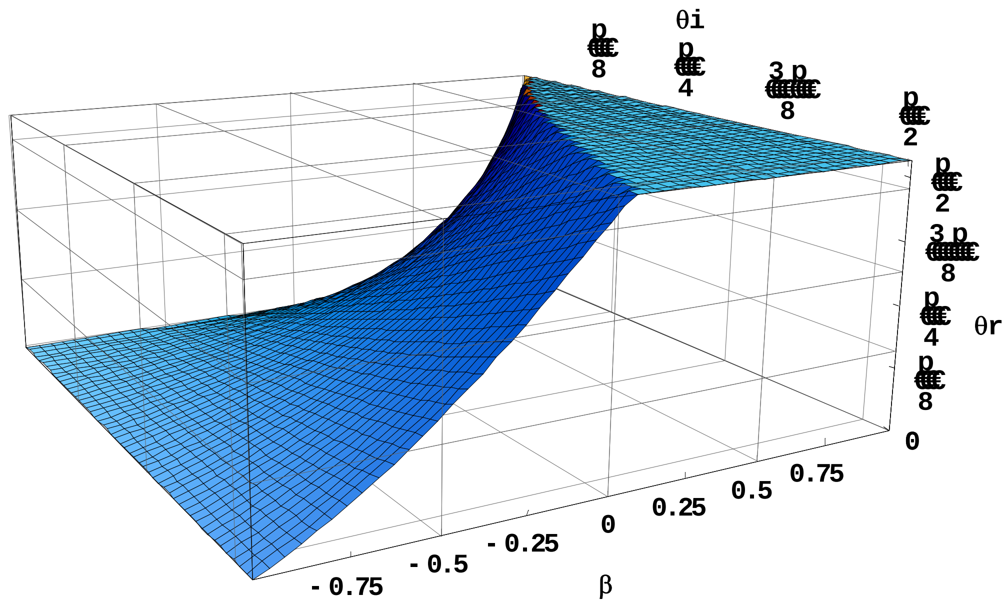

Fig. 4. Variation of the reflection angle θ_r as a function of the incident angle θ_i and of the velocity of the interface β. We let the interface move in both directions.

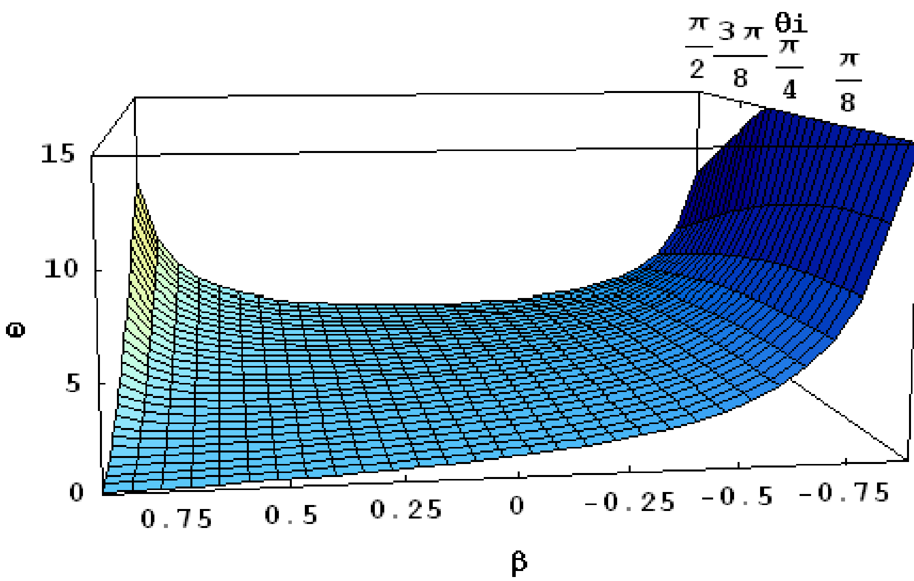

Fig. 5. The ratio between the frequency of the reflected light and that of the incident light, denoted by ω, versus the incident angle θ_i and the velocity β of the interface.

Fig. 5 presents the variation of the frequency of the reflected beam as a function of the incident angle and of the surface velocity.

For usual velocities the frequency does not differ too much from that of the incident light. Significant changes appear for $|\beta| > 0.25$. Take for example $\beta = 0.98$; for incidence at an angle of 45º the reflected beam has a frequency 14.5 times larger than the incident light. If $\beta = -0.98$, that is if the surface moves towards the incident light, the frequency is enhanced by reflection 84 times. In the last chapter we shall present a situation when such high velocities where actually attained in experiments.

4. Discussion

Our results are contained in relations (4) and (5). These are not new results, they may be found in any classic textbook on special relativity. However, the important point is not the formulae in themselves, but the way we have deduced them. On one hand they are extremely relativistic relations, accurate to very high velocities of the bodies in movement. On the other hand they are concerned with light. It would have been normal to treat them only in the realm of Special Relativity. However, we have shown that exactly the same relations are consequences of a completely non-relativistic treatment.

The situation seems to be a paradox. A classical approach should give only non-relativistic results. These ought to be only approximations to relativistic relations. The former are just approximations of the latter for slow motion. It seems not to be the situation here.

Let's try to explain this fact. The starting point of the demonstration given in paragraph 2.3 is the continuity of the electromagnetic field across the surface between two media and the relation (6). They are pre-relativistic achievements of electromagnetism. Special Relativity itself is based on classical electrodynamics. Therefore it is quite normal than the two theories agree.

There is another point we have to discuss. Why did we not treat the problem of the transmitted light? Isn't it the refracted beam a plane wave itself? Apparently we could have written instead of Eq. (6) the following more complex equality:

$$\omega_0\left[t-\frac{n_1}{c}\left(x\sin\theta_i+Vt\cos\theta_i\right)\right]=\omega_r\left[t-\frac{n_1}{c}\left(x\sin\theta_r-Vt\cos\theta_r\right)\right]=$$

$$\omega_t\left[t-\frac{n_2}{c}\left(x\sin\theta_t+Vt\cos\theta_t\right)\right] \qquad (6')$$

Hence the transmitted wave would be handled as the reflected one. But in fact this is not an acceptable approach. Electrodynamics of moving media must be treated using relativist four-vectors, as e.g. in [2]. It appears that Eq, (6') is not correct. Therefore we have limited ourselves to reflected radiation.

As can be readily seen from figures 3 to 5 the influence of the movement is significant only at very high velocities; one has to deal with mirrors traveling at speeds greater than say *c*/4. At the end of chapter 3 we have declared that such movements were in fact experimentally detected. Indeed, during experiments made with very intense and very short laser pulses incident upon solid thin foils electrons are pushed away from the target [3, 4]. Almost all the electrons are driven out in the direction of the light pulse, being expelled from the sample. Some of them are attracted back by the remaining ions from the target and travel backwards. Therefore one deals with two "flying mirrors" made by electrons in vacuum. The velocities of such flying mirrors are big enough to measure light with frequency 12 times larger than that of the laser: if the incident light is in the visible range, in the experiments was detected radiation in the *X*-domain. As can be seen from Eq. (4) and Fig. (5) backward electrons are more efficient in increasing the frequency of the reflected light. These flying mirrors must be imagined as thin sheets of matter. They have two interfaces with the vacuum around. These surfaces are by no means plane and parallel. One may think that the preceding analysis applies to the first surface between vacuum and electrons. However, after a study of the transmission light through the electron leaf, it would be interesting to consider also the reflection on the second interface, between electrons and vacuum.

The dependence between the reflected and the incident frequencies given in Eq. (5) could be compared with the relativistic Doppler effect. The frequency measured by an observer traveling with velocity $V_o = \beta_o c$ is given by (see [2]):

$$\omega_o' = \omega_o \frac{1-\beta\cos\theta_o}{\sqrt{1-\beta^2}} \tag{10}$$

where θ_0 is the angle between the direction of the light and the direction in which moves the observer (or the light source). As a particular occurrence, take the case when the source and the observer move at right angle, $\theta_o = \pi/2$. One finds the famous transverse Doppler effect which does not exist in pre-relativistic physics:

$$\omega_o'(\pi/2) = \omega_o \sqrt{\frac{1-\beta}{1+\beta}} \tag{11}$$

The question is if the relations (10) and (11) may be deduced from relation (5). In this case our demonstration of Eq. (5) could be used to induce – in a pre-relativistic way – a formula for the transverse Doppler effect. Actually relations (5) and (10) are not the same. Even so, one could infer from Eq. (5) a relation which, for low velocities, would give numerical results similar to those of Eq. (10). It seems that the two problems deal with different physics. But we must remember that in any actual experiments, in particular in that of Ives and Stilwell [5], light is reflected by mirrors in relative movement with respect to the light source. Light is also transmitted through different windows until it reaches detectors, so in order to understand correctly this instance refracted light has to be thoroughly studied.

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