FIXED POINT THEOREMS ON MODULAR FUZZY METRIC SPACES.

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In this paper, we present a new space which is a melange between a fuzzy metric space and a modular metric space. We state some properties and examples of our new space. Then, we formulate and prove the existence and uniqueness results of a fixed point for continuous mappings under this new space. To support our results, we introduce some examples and an application.

Keywords: Fuzzy metric space, modular metric space, modular fuzzy metric space, fixed point.

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1. Introduction

In 1922, Banach [4] put the main stone of the fixed point theorem on metric spaces. Then after, many authors came to treat this topic in their researches. Some scientists implemented different spaces to enhance the Banach contraction theorem to larger spaces, see as an example [2, 5, 11, 13, 20, 21, 25, 28, 32, 33].
In 1965, Zadeh [38] presented the concept of a fuzzy set. Ten years later, Kramosil and Michalek [26] stated the definition of fuzzy metric spaces. In 1988, Grabiec [18] implemented the notion of fuzzy metric space to extend the Banach contraction theorem over this space. Posteriorly, George and Veeramani [15] employed the definition of t-norm to formulate and introduce some results on the notion of a fuzzy metric space. Then, several researchers presented different contraction conditions over fuzzy metric spaces, see [12]-[16].
In 2010, Chistyakov [8] introduced the notion of modular metric spaces. Then, numerous mathematicians discussed different results in their works over modular metric spaces, for example, look at the references [1]-[29].
In this paper, we introduce a new space named modular fuzzy metric space. We launch some fixed point results over a modular fuzzy metric spaces. To analyse our work, we state some examples, corollaries, and an application.

2. Preliminaries

In this section, we will recall some definitions which are crucial in this paper.

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Definition 2.1. [38] *Let Ξ be any set. A fuzzy set E in Ξ is a function with domain Ξ and values in* $[0, 1]$.

Definition 2.2. [36] *Given a binary operation* $* : [0, 1] \times [0, 1] \to [0, 1]$. *An operator $*$ is a continuous t-norm if $\forall \alpha, \beta, \gamma, \delta \in [0, 1]$ satisfy:*

(1) $\alpha * \beta = \beta * \alpha$.
(2) $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$.
(3) $\alpha * 1 = \alpha$.
(4) *If $\alpha \leq \gamma$ and $\beta \leq \delta$, then $\alpha * \beta \leq \gamma * \delta$.*

George and Veeramani [15] in 1994 introduced the concept of a fuzzy metric space using the defintion of t-norm as follows:

Definition 2.3. [15] *The triplet $(\Xi, \Lambda, *)$ is called a fuzzy metric space if Ξ is an arbitrary set, $*$ is a continuous t-norm and Λ is a fuzzy set on $\Xi \times \Xi \times (0, \infty) \to [0, 1]$; for all ι, η, ϑ in Ξ, and for all $s, t > 0$ satisfying the following conditions:*

(1) $\Lambda(\iota, \eta, 0) = 0$, $\Lambda(\iota, \eta, t) > 0$, $\forall t > 0$.
(2) $\Lambda(\iota, \eta, t) = 1$ *if and only if* $\iota = \eta$, *for all* $t > 0$.
(3) $\Lambda(\iota, \eta, t) = \Lambda(\eta, \iota, t)$.
(4) $\Lambda(\iota, \eta, t) * \Lambda(\eta, \vartheta, s) \leq \Lambda(\iota, \vartheta, t + s)$.
(5) $\Lambda(\iota, \eta, .) : (0, \infty) \to [0, 1]$ *is left continuous.*

Here, Λ called a fuzzy metric on Ξ.

Example 2.1. *Let (Ξ, d) be a metric space. Define $\alpha * \beta = \alpha\beta$ for all $\alpha, \beta \in [0, 1]$, and $\Lambda : \Xi \times \Xi \times (0, \infty) \to [0, 1]$ as*

$$\Lambda(\iota, \eta, t) = \frac{t}{t + d(\iota, \eta)}$$

$\forall \iota, \eta \in \Xi$ and $t > 0$. Then $(\Xi, \Lambda, *)$ is a fuzzy metric space; called fuzzy metric induced by the metric d.

The notions of convergence, completeness and compactness on fuzzy metric spaces were presented in [15] as follows:

Definition 2.4. [15] *Let $(\Xi, \Lambda, *)$ be a fuzzy metric space.*

(1) *A sequence $\{\iota_\kappa\}_{\kappa \in \mathbb{N}}$ in Ξ is convergent to an element $\iota \in \Xi$ if $\lim_{\kappa \to \infty} \Lambda(\iota_\kappa, \iota, t) = 1$, for all $t > 0$.*
(2) *A sequence $\{\iota_\kappa\}_{\kappa \in \mathbb{N}}$ in Ξ is Cauchy if for all $0 < \epsilon < 1$ and for $t > 0$, there exists a number $\kappa_0 \in \mathbb{N}$ such that $\Lambda(\iota_\kappa, \iota_\xi, t) > 1 - \epsilon$ for each $\kappa, \xi \geq \kappa_0$.*
(3) *A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*
(4) *A fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.*

In 2010, Chistyakov [8] defined the notion of modular metric spaces as follows:

Definition 2.5. [8] *A modular metric on a nonempty set Ξ is a function $\Theta : (0, \infty) \times \Xi \times \Xi \to [0, \infty)$ that will be written as $\Theta_\varrho(\iota, \eta) = \Theta(\varrho, \iota, \eta)$; for all $\iota, \eta, \vartheta \in \Xi$ and for all $\varrho, \sigma > 0$, satisfy the following three conditions:*

(1) $\Theta_\varrho(\iota, \eta) = 0$ *if and only if* $\iota = \eta$, $\forall \varrho > 0$ *and* $\iota, \eta \in \Xi$.
(2) $\Theta_\varrho(\iota, \eta) = \Theta_\varrho(\eta, \iota)$, $\forall \varrho > 0$ *and* $\iota, \eta \in \Xi$.
(3) $\Theta_{\varrho+\sigma}(\iota, \eta) \leq \Theta_\varrho(\iota, \vartheta) + \Theta_\sigma(\vartheta, \eta)$; *for all* $\varrho, \sigma > 0$ *and* $\iota, \eta, \vartheta \in \Xi$.

Remark 2.1. *Let Θ be modular on a set Ξ. Then for given $\iota, \eta \in \Xi$, the function $0 < \varrho \to \Theta_\varrho(\iota, \eta) \in (0, \infty)$ is non increasing on $(0, \infty)$. In fact if $0 < \varrho < \sigma$, then by above definition $\Theta_\sigma(\iota, \eta) \leq \Theta_{\sigma-\varrho}(\iota, \iota) + \Theta_\varrho(\iota, \eta) = \Theta_\varrho(\iota, \eta)$ for all $\iota, \eta \in \Xi$.*

Definition 2.6. [10] *Given a modular Θ on Ξ, a sequence $\{\iota_\kappa\}_{\kappa\in\mathbb{N}}$ in Ξ_Θ is said to be modular convergent to an element $\iota \in \Xi_\Theta$ if there exists a number $\varrho > 0$, possibly depending on $\{\iota_\kappa\}$ and ι, such that $\lim_{\kappa\to\infty} \Theta_\varrho(\iota_\kappa, \iota) = 0$. i.e $\iota_\kappa \to \iota$ as $\kappa \to \infty$.*

Definition 2.7. [10] *Given a modular Θ on Ξ, a sequence $\{\iota_\kappa\}_{\kappa\in\mathbb{N}}$ in Ξ_Θ is said to be modular Cauchy if there exists a number $\varrho = \varrho(\{\iota_\kappa\}) > 0$ such that $\lim_{\kappa,\xi\to\infty} \Theta_\varrho(\iota_\kappa, \iota_\xi) = 0$.*

Definition 2.8. [10] *A modular space Ξ_Θ is said to be modular complete if each Cauchy sequence in Ξ_Θ is modular convergent. In fact, if $\{\iota_\kappa\} \subset \Xi_\Theta$ and there exists $\varrho = \varrho(\{\iota_\kappa\}) > 0$ such that $\lim_{\kappa,\xi\to\infty} \Theta_\varrho(\iota_\kappa, \iota_\xi) = 0$, then there exists $\iota \in \Xi_\Theta$, such that $\lim_{\kappa\to\infty} \Theta_\varrho(\iota_\kappa, \iota) = 0$.*

Definition 2.9. [1] *A modular Θ on Ξ is said to be satisfied the Δ_2-condition if $\lim_{n\to\infty} \Theta_\varrho(\iota_\kappa, \iota) = 0$, for some $\varrho > 0$ implies that $\lim_{n\to\infty} \Theta_\varrho(\iota_\kappa, \iota) = 0$, for all $\varrho > 0$.*

3. Main results

In this section, we construct a new space called a modular fuzzy metric space. We present some examples of this space. Also, we formulate and prove some new fixed point results under this space. We start by presenting the following definitions.

Definition 3.1. *A modular fuzzy metric space is the triplet $(\Xi, \zeta_\varrho, *)$ such that Ξ is an arbitrary set, $(*)$ is a continuous t-norm and ζ_ϱ is a fuzzy set on $(0,\infty) \times \Xi \times \Xi \times (0,\infty) \to [0,1]$; for all ι, η, ϑ in Ξ, and $s, t > 0$ satisfying the following conditions:*

(1) *$\zeta_\varrho(\iota, \eta, 0) = 0$, $\zeta_\varrho(\iota, \eta, t) > 0$, for all $t, \varrho > 0$.*
(2) *$\zeta_\varrho(\iota, \eta, t) = 1$ if and only if $\iota = \eta$, for all $t, \varrho > 0$.*
(3) *$\zeta_\varrho(\iota, \eta, t) = \zeta_\varrho(\eta, \iota, t)$, for all $t, \varrho > 0$.*
(4) *$\zeta_\sigma(\iota, \eta, t) * \zeta_\varrho(\eta, \vartheta, s) \leq \zeta_{\sigma+\varrho}(\iota, \vartheta, t+s)$, for all $t, s, \sigma, \varrho > 0$.*
(5) *$\zeta_\varrho(\iota, \eta, .) : (0,\infty) \to [0,1]$ is left continuous.*

Here, ζ_ϱ is called a modular fuzzy metric.

Definition 3.2. *Let $(\Xi, \zeta_\varrho, *)$ be a modular fuzzy metric space.*

(1) *A sequence $\{\iota_\kappa\}_{\kappa\in\mathbb{N}}$ in Ξ is convergent to an element $\iota \in \Xi$ if $\lim_{\kappa\to\infty} \zeta_\varrho(\iota_\kappa, \iota, t) = 1$ for all $t > 0$ and some $\varrho > 0$.*
(2) *A sequence $\{\iota_\kappa\}_{\kappa\in\mathbb{N}}$ in Ξ is Cauchy if for all $0 < \epsilon < 1$, there exists a number $\kappa_0 \in \mathbb{N}$ such that $\zeta_\varrho(\iota_\kappa, \iota_\xi, t) > 1 - \epsilon$, for each $\kappa, \xi \geq \kappa_0$ and some $\varrho > 0$.*
(3) *A modular fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*
(4) *A modular fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.*

Definition 3.3. *A fuzzy modular metric ζ_ϱ on Ξ is said to be satisfied the Δ_{2_t}-condition if $\lim_{\kappa\to\infty} \zeta_\varrho(\iota_\kappa, \iota, t) = 1$, for some $\varrho > 0$ and for some $t > 0$ imply that $\lim_{\kappa\to\infty} \zeta_\varrho(\iota_\kappa, \iota, t) = 1$, for all $\varrho > 0$, and for all $t > 0$.*

Example 3.1. *Let (Ξ, Θ_ϱ) be a modular metric space. Define $\alpha * \beta = \alpha\beta$ for all $\alpha, \beta \in [0,1]$, and $\zeta_\varrho : (0,\infty) \times \Xi \times \Xi \times (0,\infty) \to [0,1]$ by*

$$\zeta_\varrho(\iota, \eta, t) = \frac{t}{t + \Theta_\varrho(\iota, \eta)}.$$

Then $(\Xi, \zeta_\varrho, *)$ is a modular fuzzy metric space.

Proof. Let $\iota, \eta \in \Xi$ and $t, \varrho > 0$. Then, ζ_ϱ satisfies (1), (2) and (3) of the definition of the modular fuzzy metric space. Now, we will prove that $\zeta_\sigma(\iota, \eta, t) * \zeta_\varrho(\eta, \vartheta, s) \leq \zeta_{\sigma+\varrho}(\iota, \vartheta, t+s)$. By the definition of the modular metric space, we have

$$\Theta_{\sigma+\varrho}(\iota, \vartheta) \leq \Theta_\sigma(\iota, \eta) + \Theta_\varrho(\eta, \vartheta).$$

Hence

$$\Theta_{\sigma+\varrho}(\iota, \vartheta) \leq \frac{t+s}{t}\Theta_\sigma(\iota, \eta) + \frac{t+s}{s}\Theta_\varrho(\eta, \vartheta).$$

So

$$\frac{1}{t+s}\Theta_{\sigma+\varrho}(\iota, \vartheta) \leq \frac{1}{t}\Theta_\sigma(\iota, \eta) + \frac{1}{s}\Theta_\varrho(\eta, \vartheta) = \frac{s\Theta_\sigma(\iota, \eta) + t\Theta_\varrho(\eta, \vartheta)}{ts}.$$

Then

$$1 + \frac{1}{t+s}\Theta_{\sigma+\varrho}(\iota, \vartheta) \leq 1 + \frac{s\Theta_\sigma(\iota, \eta) + t\Theta_\varrho(\eta, \vartheta)}{ts}.$$

Since $\Theta_\sigma(\iota, \eta)\Theta_\varrho(\eta, \vartheta) \geq 0$, we have

$$\begin{aligned}\frac{t+s+\Theta_{\sigma+\varrho}(\iota, \vartheta)}{t+s} &\leq \frac{ts + s\Theta_\sigma(\iota, \eta) + t\Theta_\varrho(\eta, \vartheta)}{ts} \\ &\leq \frac{ts + s\Theta_\sigma(\iota, \eta) + t\Theta_\varrho(\eta, \vartheta) + \Theta_\sigma(\iota, \eta)\Theta_\varrho(\eta, \vartheta)}{ts},\end{aligned}$$

which gives:

$$\frac{ts}{ts + s\Theta_\sigma(\iota, \eta) + t\Theta_\varrho(\eta, \vartheta) + \Theta_\sigma(\iota, \eta)\Theta_\varrho(\eta, \vartheta)} \leq \frac{t+s}{t+s+\Theta_{\sigma+\varrho}(\iota, \vartheta)}.$$

Hence

$$\frac{t}{t+\Theta_\sigma(\iota, \eta)} \times \frac{s}{s+\Theta_\varrho(\eta, \vartheta)} \leq \frac{t+s}{t+s+\Theta_{\sigma+\varrho}(\iota, \vartheta)}.$$

Thus

$$\zeta_\sigma(\iota, \eta, t) * \zeta_\varrho(\eta, \vartheta, s) \leq \zeta_{\sigma+\varrho}(\iota, \vartheta, t+s).$$

□

Remark 3.1. (1) *For* $\Theta_\varrho(\iota, \eta) = \frac{|\iota-\eta|}{\varrho}$, *we have* $\zeta_\varrho(\iota, \eta, t) = \frac{t}{t+\frac{|\iota-\eta|}{\varrho}}$ *is a modular fuzzy metric.*

(2) *For* $\Theta_\varrho(\iota, \eta) = \frac{|\iota-\eta|}{\varrho+1}$, *we have* $\zeta_\varrho(\iota, \eta, t) = \frac{t}{t+\frac{|\iota-\eta|}{\varrho+1}}$ *is a modular fuzzy metric.*

Example 3.2. *Let* (Ξ, Θ_ϱ) *be a modular metric space. Define* $\alpha * \beta = \alpha\beta$ *for all* $\alpha, \beta \in [0, 1]$, *and* $\zeta_\varrho : (0, \infty) \times \Xi \times \Xi \times (0, \infty) \to [0, 1]$ *by*

$$\zeta_\varrho(\iota, \eta, t) = \exp\Big\{-\frac{\Theta_\varrho(\iota, \eta)}{t}\Big\}.$$

Then $(\Xi, \zeta_\varrho, *)$ *is a modular fuzzy metric space.*

Proof. Let $\iota, \eta \in \Xi$ and $t, \varrho > 0$. Then ζ_ϱ satisfies (1), (2), (3) and (5) of the definition of the modular fuzzy metric space. Now, we will prove that $\zeta_\sigma(\iota, \eta, t) * \zeta_\varrho(\eta, \vartheta, s) \leq \zeta_{\sigma+\varrho}(\iota, \vartheta, t+s)$. By the definition of the modular metric space, we have:

$$\Theta_{\sigma+\varrho}(\iota, \vartheta) \leq \Theta_\sigma(\iota, \eta) + \Theta_\varrho(\eta, \vartheta),$$

which gives:

$$\Theta_{\sigma+\varrho}(\iota, \vartheta) \leq \frac{t+s}{t}\Theta_\sigma(\iota, \eta) + \frac{t+s}{s}\Theta_\varrho(\eta, \vartheta).$$

So

$$\frac{1}{t+s}\Theta_{\sigma+\varrho}(\iota, \vartheta) \leq \frac{1}{t}\Theta_\sigma(\iota, \eta) + \frac{1}{s}\Theta_\varrho(\eta, \vartheta),$$

which imply that

$$-\frac{1}{t+s}\Theta_{\sigma+\varrho}(\iota,\vartheta) \geq -\frac{1}{t}\Theta_{\sigma}(\iota,\eta) - \frac{1}{s}\Theta_{\varrho}(\eta,\vartheta).$$

Hence

$$\begin{aligned}\exp\Big\{-\frac{1}{t+s}\Theta_{\sigma+\varrho}(\iota,\vartheta)\Big\} &\geq \exp\Big\{-\frac{1}{t}\Theta_{\sigma}(\iota,\eta) - \frac{1}{s}\Theta_{\varrho}(\iota,\vartheta)\Big\}\\ &= \exp\Big\{-\frac{1}{t}\Theta_{\sigma}(\iota,\eta)\Big\} \times \exp\Big\{-\frac{1}{s}\Theta_{\varrho}(\iota,\vartheta)\Big\}.\end{aligned}$$

Thus

$$\zeta_{\sigma+\varrho}(\iota,\vartheta,t+s) \geq \zeta_{\sigma}(\iota,\eta,t) * \zeta_{\varrho}(\iota,\vartheta,s).$$

□

Remark 3.2. (1) *For* $\Theta_\varrho(\iota,\eta) = \frac{|\iota-\eta|}{\varrho}$, *we have* $\zeta_\varrho(\iota,\eta,t) = \exp\Big\{-\frac{|\iota-\eta|}{t\varrho}\Big\}$ *is a modular fuzzy metric.*

(2) *For* $\Theta_\varrho(\iota,\eta) = \frac{|\iota-\eta|}{\varrho+1}$, *we have* $\zeta_\varrho(\iota,\eta,t) = \exp\Big\{-\frac{|\iota-\eta|}{t(\varrho+1)}\Big\}$ *is a modular fuzzy metric.*

Theorem 3.1. *On a complete modular fuzzy metric space* $(\Xi,\zeta_\varrho,*)$, *consider a continuous mapping* $\Gamma:\Xi\to\Xi$. *Suppose there exist a strictly decreasing, continuous function* $\Upsilon:(0,1]\to[0,\infty)$ *with* $\Upsilon(1)=0$ *and a real number* H *with* $0<H<1$ *such that*

$$\Upsilon(\zeta_\varrho(\Gamma\iota,\Gamma\eta,t)) \leq H\Upsilon(\zeta_\varrho(\iota,\eta,t)) \qquad (\mathbf{3}.1)$$

for all $\iota,\eta\in\Xi$, $\iota\neq\eta$. *Then* Γ *has a unique fixed point in* Ξ.

Proof. Let ι_0 be an arbitrary point in Ξ. Choose $\iota_1\in\Xi$ such that $\iota_1=\Gamma\iota_0$. Continuing this process, we construct a sequence (ι_κ) such that $\iota_{\kappa+1}=\Gamma\iota_\kappa$, for $\kappa=0,1,2..$

Let $\iota=\iota_{\kappa-1}$ and $\eta=\iota_\kappa$. Replacing this in (**3**.1), we get

$$\begin{aligned}\Upsilon(\zeta_\varrho(\Gamma\iota,\Gamma\eta,t)) &= \Upsilon(\zeta_\varrho(\Gamma\iota_{\kappa-1},\Gamma\iota_\kappa,t))\\ &= \Upsilon(\zeta_\varrho(\iota_\kappa,\iota_{\kappa+1},t))\\ &\leq H\Upsilon(\zeta_\varrho(\iota_{\kappa-1},\iota_\kappa,t)) \qquad (\mathbf{3}.2)\\ &< \Upsilon(\zeta_\varrho(\iota_{\kappa-1},\iota_\kappa,t)).\end{aligned}$$

Since Υ is a strictly decreasing function, we obtain

$$\zeta_\varrho(\iota_\kappa,\iota_{\kappa+1},t) > \zeta_\varrho(\iota_{\kappa-1},\iota_\kappa,t). \qquad (\mathbf{3}.3)$$

We use the same method for $\iota=\iota_{\kappa-2}$ and $\eta=\iota_{\kappa-1}$, we get

$$\zeta_\varrho(\iota_\kappa,\iota_{\kappa-1},t) > \zeta_\varrho(\iota_{\kappa-1},\iota_{\kappa-2},t). \qquad (\mathbf{3}.4)$$

Therefore, (**3**.3) and (**3**.4) imply that $\{\zeta_\varrho(\iota_\kappa,\iota_{\kappa+1},t)\}$ is a strictly increasing sequence of positive real numbers in $[0,1]$.

Put $\Sigma_\kappa(\varrho,t)=\zeta_\varrho(\iota_\kappa,\iota_{\kappa+1},t)$. Then $\{\Sigma_\kappa(\varrho,t)\}$ is a strictly increasing sequence.

So $\exists$ $\Sigma(\varrho,t)$ such that $\lim_{\kappa\to\infty}\Sigma_\kappa(\varrho,t)=\Sigma(\varrho,t)$. Assume that $0<\Sigma(\varrho,t)<1$.

By (**3**.2), we have

$$\Upsilon(\Sigma_\kappa(\varrho,t)) \leq H\Upsilon(\Sigma_{\kappa-1}(\varrho,t)).$$

So,

$$\lim_{\kappa\to\infty}\Upsilon(\Sigma_\kappa(\varrho,t)) \leq \lim_{\kappa\to\infty}H\Upsilon(\Sigma_{\kappa-1}(\varrho,t)).$$

The continuity of Υ implies that

$$\Upsilon(\Sigma(\varrho,t)) \leq H\Upsilon(\Sigma(\varrho,t)),$$

a contradiction. Then $\Sigma(\varrho,t)=1$.

Now, we will prove that $\{\iota_\kappa\}$ is a Cauchy sequence. Assume not, then for $0<\epsilon<1$, there exist two sub-sequences $\{\iota_{\xi(i)}\}$ and $\{\iota_{\kappa(i)}\}$ such that for each $i \in \mathbb{N}$, let $\kappa(i), \xi(i) \in \mathbb{N}$ satisfying $\kappa(i), \xi(i) \geq \kappa$ and $\kappa(i) > \xi(i) > i$, such that

$$\zeta_\varrho(\iota_{\kappa(i)}, \iota_{\xi(i)}, t) \leq 1-\epsilon,\ \zeta_\varrho(\iota_{\kappa(i)-1}, \iota_{\xi(i)-1}, t) > 1-\epsilon,\ \zeta_\varrho(\iota_{\kappa(i)-1}, \iota_{\xi(i)}, t) > 1-\epsilon. \qquad (\mathbf{3}.5)$$

Consider

$$1-\epsilon \geq \zeta_\varrho(\iota_{\kappa(i)}, \iota_{\xi(i)}, t) \geq \zeta_{\frac{\varrho}{2}}(\iota_{\kappa(i)}, \iota_{\kappa(i)-1}, \frac{t}{2}) * \zeta_{\frac{\varrho}{2}}(\iota_{\kappa(i)-1}, \iota_{\xi(i)}, \frac{t}{2}).$$

By definition of Δ_{2_t}-condition on Ξ and ($\mathbf{3}$.5), we have

$$\zeta_{\frac{\varrho}{2}}(\iota_{\kappa(i)-1}, \iota_{\xi(i)}, \frac{t}{2}) > 1-\epsilon.$$

Hence

$$1-\epsilon \geq \zeta_\varrho(\iota_{\kappa(i)}, \iota_{\xi(i)}, t) > \zeta_{\frac{\varrho}{2}}(\iota_{\kappa(i)}, \iota_{\kappa(i)-1}, \frac{t}{2}) * 1-\epsilon.$$

If $i \to \infty$, we have $\Sigma_{\kappa(i)}(\frac{\varrho}{2}, \frac{t}{2}) = \zeta_{\frac{\varrho}{2}}(\iota_{\kappa(i)}, \iota_{\kappa(i)-1}, \frac{t}{2}) \to 1$.
So

$$\zeta_\varrho(\iota_{\kappa(i)}, \iota_{\xi(i)}, t) \to 1-\epsilon.$$

Then by ($\mathbf{3}$.1), we have

$$\Upsilon(\zeta_\varrho(\iota_{\kappa(i)}, \iota_{\xi(i)}, t)) \leq H\Upsilon(\zeta_\varrho(\iota_{\kappa(i)-1}, \iota_{\xi(i)-1}, t)) < \Upsilon(\zeta_\varrho(\iota_{\kappa(i)-1}, \iota_{\xi(i)-1}, t)).$$

Thus

$$1-\epsilon \geq \zeta_\varrho(\iota_{\kappa(i)}, \iota_{\xi(i)}, t) > \zeta_\varrho(\iota_{\kappa(i)-1}, \iota_{\xi(i)-1}, t) > 1-\epsilon,$$

which is impossible. Hence $\{\iota_\kappa\}$ is a Cauchy sequence in a complete modular fuzzy metric space. So $\exists\ \varpi \in \Xi$ such that $\lim_{\kappa\to\infty} \iota_\kappa = \varpi$, that means $\lim_{\kappa\to\infty} \zeta_\varrho(\iota_\kappa, \varpi, t) = 1$.
To show ϖ is a fixed point of Γ, we have :
Γ is continuous: $\iota_\kappa \to \varpi \Rightarrow \Gamma\iota_\kappa \to \Gamma\varpi$.
By ($\mathbf{3}$.1), we have

$$\Upsilon(\zeta_\varrho(\iota_\kappa, \Gamma\iota_\kappa, t)) \leq H\Upsilon(\zeta_\varrho(\iota_{\kappa-1}, \iota_\kappa, t)).$$

Since $\Upsilon(1) = 0$ and for $\kappa \to \infty$, we get

$$\Upsilon(\zeta_\varrho(\varpi, \Gamma\varpi, t)) \leq H\Upsilon(\zeta_\varrho(\varpi, \varpi, t)) = H\Upsilon(1) = 0.$$

So

$$\zeta_\varrho(\varpi, \Gamma\varpi, t) = 1.$$

Hence, $\zeta_\varrho(\varpi, \Gamma\varpi, t) = 1 \Rightarrow \Gamma\varpi = \varpi$. Thus ϖ is a fixed point of Γ.
Now, we will prove that ϖ is unique. Assume not, $\exists w \in \Xi$, such that $\Gamma w = w$ where $w \neq \varpi$ and $\lim_{\kappa\to\infty} \iota_\kappa = w$. Then

$$\begin{aligned}\Upsilon(\zeta_\varrho(w, \varpi, t)) &= \Upsilon(\zeta_\varrho(\Gamma w, \Gamma\varpi, t)) \\ &\leq H\Upsilon(\zeta_\varrho(w, \varpi, t)) \\ &\leq H\Upsilon\Big(\zeta_{\frac{\varrho}{2}}(w, \iota_\kappa, \frac{t}{2}) * \zeta_{\frac{\varrho}{2}}(\iota_\kappa, \varpi, \frac{t}{2})\Big).\end{aligned}$$

Since $\Upsilon(1) = 0$ and for $\kappa \to \infty$ on both sides, we have

$$\Upsilon(\zeta_\varrho(w, \varpi, t)) \leq H\Upsilon(\zeta_{\frac{\varrho}{2}}(w, w, \frac{t}{2}) * \zeta_{\frac{\varrho}{2}}(\varpi, \varpi, \frac{t}{2})) = H\Upsilon(1) = 0.$$

So

$$\Upsilon(\zeta_\varrho(w, \varpi, t)) = 0.$$

Hence, $\zeta_\varrho(w, \varpi, t) = 1 \Rightarrow w = \varpi$. Thus Γ has a unique fixed point ϖ. $\square$

Theorem 3.2. *On a complete modular fuzzy metric space* $(\Xi, \zeta_\varrho, *)$*, consider a continuous mapping* $\Gamma : \Xi \to \Xi$*. Suppose there exist a strictly decreasing, continuous function* $\Upsilon : (0,1] \to [0,\infty)$ *with* $\Upsilon(1) = 0$ *and a real number* H *with* $0 < H < 1$ *such that*

$$\Upsilon(\zeta_\varrho(\Gamma\iota, \Gamma\eta, , t)) \leq H\Upsilon\Big(\frac{\zeta_\varrho(\iota, \eta, t) + \zeta_\varrho(\Gamma\iota, \iota, t)}{4} + \frac{\zeta_\varrho(\eta, \Gamma\eta, t)}{2}\Big) \tag{3.6}$$

for all $\iota, \eta \in \Xi$*,* $\iota \neq \eta$*. Then* Γ *has a unique fixed point in* Ξ*.*

Proof. Let ι_0 be an arbitrary point in Ξ. Choose $\iota_1 \in \Xi$ such that $\iota_1 = \Gamma\iota_0$. Continuing this process, we construct a sequence (ι_κ) such that $\iota_{\kappa+1} = \Gamma\iota_\kappa$, for $\kappa = 0, 1, 2..$
Let $\iota = \iota_{\kappa-1}$ and $\eta = \iota_\kappa$. Replacing this in (**3**.6), we get

$$\begin{aligned}\Upsilon(\zeta_\varrho(\Gamma\iota, \Gamma\eta, t)) &= \Upsilon(\zeta_\varrho(\Gamma\iota_{\kappa-1}, \Gamma\iota_\kappa, t)) \\ &= \Upsilon(\zeta_\varrho(\iota_\kappa, \iota_{\kappa+1}, t)) \\ &\leq H\Upsilon\Big(\frac{\zeta_\varrho(\iota_{\kappa-1}, \iota_\kappa, t) + \zeta_\varrho(\Gamma\iota_{\kappa-1}, \iota_{\kappa-1}, t)}{4} + \frac{\zeta_\varrho(\iota_\kappa, \Gamma\iota_\kappa, t)}{2}\Big) \\ &< \Upsilon\Big(\frac{\zeta_\varrho(\iota_{\kappa-1}, \iota_\kappa, t) + \zeta_\varrho(\Gamma\iota_{\kappa-1}, \iota_{\kappa-1}, t)}{4} + \frac{\zeta_\varrho(\iota_\kappa, \Gamma\iota_\kappa, t)}{2}\Big) \\ &= \Upsilon\Big(\frac{\zeta_\varrho(\iota_{\kappa-1}, \iota_\kappa, t) + \zeta_\varrho(\iota_\kappa, \iota_{\kappa-1}, t)}{4} + \frac{\zeta_\varrho(\iota_\kappa, \iota_{\kappa+1}, t)}{2}\Big).\end{aligned} \tag{3.7}$$

Since Υ is a strictly decreasing function, we obtain

$$\zeta_\varrho(\iota_\kappa, \iota_{\kappa+1}, t) > \frac{\zeta_\varrho(\iota_{\kappa-1}, \iota_\kappa, t) + \zeta_\varrho(\iota_\kappa, \iota_{\kappa-1}, t)}{4} + \frac{\zeta_\varrho(\iota_\kappa, \iota_{\kappa+1}, t)}{2}.$$

Hence

$$\zeta_\varrho(\iota_\kappa, \iota_{\kappa+1}, t) > \zeta_\varrho(\iota_\kappa, \iota_{\kappa-1}, t). \tag{3.8}$$

We use the same method for $\iota = \iota_{\kappa-2}$ and $\eta = \iota_{\kappa-1}$, we get

$$\zeta_\varrho(\iota_\kappa, \iota_{\kappa-1}, t) > \zeta_\varrho(\iota_{\kappa-1}, \iota_{\kappa-2}, t). \tag{3.9}$$

Therefore, (**3**.8) and (**3**.9) imply that $\{\zeta_\varrho(\iota_\kappa, \iota_{\kappa+1}, t)\}$ is a strictly increasing sequence of positive real numbers in $[0, 1]$.
Put $\Sigma_\kappa(\varrho, t) = \zeta_\varrho(\iota_\kappa, \iota_{\kappa+1}, t)$. Then $\{\Sigma_\kappa(\varrho, t)\}$ is a strictly increasing sequence.
So $\exists\ \Sigma(t, \varrho)$ such that $\lim_{\kappa\to\infty} \Sigma_\kappa(\varrho, t) = \Sigma(\varrho, t)$. Assume that $0 < \Sigma(\varrho, t) < 1$.
By (**3**.7), we have

$$\Upsilon(\Sigma_\kappa(\varrho, t)) \leq H\Upsilon\Big(\frac{\Sigma_{\kappa-1}(\varrho, t)}{2} + \frac{\Sigma_\kappa(\varrho, t)}{2}\Big).$$

So,

$$\lim_{\kappa\to\infty} \Upsilon(\Sigma_\kappa(\varrho, t)) \leq \lim_{\kappa\to\infty} H\Upsilon\Big(\frac{\Sigma_{\kappa-1}(\varrho, t)}{2} + \frac{\Sigma_\kappa(\varrho, t)}{2}\Big).$$

By the continuity of Υ, we have

$$\Upsilon(\Sigma(\varrho, t)) \leq H\Upsilon\Big(\frac{\Sigma(\varrho, t)}{2} + \frac{\Sigma(\varrho, t)}{2}\Big),$$

a contradiction. Then $\Sigma(\varrho, t) = 1$.
Now, we will prove that $\{\iota_\kappa\}$ is a Cauchy sequence. Assume not, then for $0 < \epsilon < 1$, there exists two sub-sequences $\{\iota_{\xi(i)}\}$ and $\{\iota_{\kappa(i)}\}$ such that for each $i \in \mathbb{N}$, let $\kappa(i), \xi(i) \in \mathbb{N}$ satisfying $\kappa(i), \xi(i) \geq \kappa$ and $\kappa(i) > \xi(i) > i$, such that

$$\zeta_\varrho(\iota_{\kappa(i)}, \iota_{\xi(i)}, t) \leq 1 - \epsilon,\ \zeta_\varrho(\iota_{\kappa(i)-1}, \iota_{\xi(i)-1}, t) > 1 - \epsilon,\ \zeta_\varrho(\iota_{\kappa(i)-1}, \iota_{\xi(i)}, t) > 1 - \epsilon. \tag{3.10}$$

Consider

$$1 - \epsilon \geq \zeta_\varrho(\iota_{\kappa(i)}, \iota_{\xi(i)}, t) \geq \zeta_{\frac{\varrho}{2}}(\iota_{\kappa(i)}, \iota_{\kappa(i)-1}, \frac{t}{2}) * \zeta_{\frac{\varrho}{2}}(\iota_{\kappa(i)-1}, \iota_{\xi(i)}, \frac{t}{2}).$$

By definition of Δ_{2_t}-condition on Ξ and (**3**.10), we have

$$\zeta_{\frac{\varrho}{2}}(\iota_{\kappa(i)-1}, \iota_{\xi(i)}, \frac{t}{2}) > 1 - \epsilon.$$

Thus

$$1 - \epsilon \geq \zeta_\varrho(\iota_{\kappa(i)}, \iota_{\xi(i)}, t) > \zeta_{\frac{\varrho}{2}}(\iota_{\kappa(i)}, \iota_{\kappa(i)-1}, \frac{t}{2}) * 1 - \epsilon.$$

If $i \to \infty$, we have $\Sigma_{\kappa(i)}(\frac{\varrho}{2}, \frac{t}{2}) = \zeta_{\frac{\varrho}{2}}(\iota_{\kappa(i)}, \iota_{\kappa(i)-1}, \frac{t}{2}) \to 1$.
So

$$\zeta_\varrho(\iota_{\kappa(i)}, \iota_{\xi(i)}, t) \to 1 - \epsilon.$$

Then by (**3**.6), we have

$$\begin{aligned}\Upsilon(\zeta_\varrho(\iota_{\kappa(i)}, \iota_{\xi(i)}, t)) &\leq H\Upsilon\Big(\frac{\zeta_\varrho(\iota_{\kappa(i)-1}, \iota_{\xi(i)-1}, t) + \zeta_\varrho(\Gamma\iota_{\kappa(i)-1}, \iota_{\kappa(i)-1}, t)}{4} + \frac{\zeta_\varrho(\iota_{\xi(i)-1}, \Gamma\iota_{\xi(i)-1}, t)}{2}\Big) \\ &< \Upsilon\Big(\frac{\zeta_\varrho(\iota_{\kappa(i)-1}, \iota_{\xi(i)-1}, t) + \zeta_\varrho(\Gamma\iota_{\kappa(i)-1}, \iota_{\kappa(i)-1}, t)}{4} + \frac{\zeta_\varrho(\iota_{\xi(i)-1}, \Gamma\iota_{\xi(i)-1}, t)}{2}\Big) \\ &= \Upsilon\Big(\frac{\zeta_\varrho(\iota_{\kappa(i)-1}, \iota_{\xi(i)-1}, t) + \zeta_\varrho(\iota_{\kappa(i)}, \iota_{\kappa(i)-1}, t)}{4} + \frac{\zeta_\varrho(\iota_{\xi(i)-1}, \iota_{\xi(i)}, t)}{2}\Big).\end{aligned}$$

Since Υ is a strictly decreasing function, (**3**.10) implies that

$$1 - \epsilon > \frac{1 - \epsilon + 1}{4} + \frac{1}{2} = 1 - \frac{\epsilon}{4} > 1 - \epsilon,$$

which is impossible. Hence $\{\iota_\kappa\}$ is a Cauchy sequence in a complete modular fuzzy metric space. So $\exists\ \varpi \in \Xi$ such that $\lim_{\kappa\to\infty} \iota_\kappa = \varpi$, that means $\lim_{\kappa\to\infty} \zeta_\varrho(\iota_\kappa, \varpi, t) = 1$.
To show ϖ is a fixed point of Γ, we have :
Γ is continuous: $\iota_\kappa \to \varpi \Rightarrow \Gamma\iota_\kappa \to \Gamma\varpi$.
By (**3**.6), we have

$$\begin{aligned}\Upsilon(\zeta_\varrho(\iota_\kappa, \Gamma\iota_\kappa, t)) &\leq H\Upsilon\Big(\frac{\zeta_\varrho(\iota_{\kappa-1}, \iota_\kappa, t) + \zeta_\varrho(\Gamma\iota_{\kappa-1}, \iota_{\kappa-1}, t)}{4} + \frac{\zeta_\varrho(\iota_\kappa, \Gamma\iota_\kappa, t)}{2}\Big) \\ &\leq H\Upsilon\Big(\frac{\zeta_\varrho(\iota_{\kappa-1}, \iota_\kappa, t) + \zeta_\varrho(\iota_\kappa, \iota_{\kappa-1}, t)}{4} + \frac{\zeta_\varrho(\iota_\kappa, \iota_{\kappa+1}, t)}{2}\Big).\end{aligned}$$

Since $\Upsilon(1) = 0$ and for $\kappa \to \infty$, we get

$$\Upsilon(\zeta_\varrho(\varpi, \Gamma\varpi, t)) \leq H\Upsilon\Big(\frac{\zeta_\varrho(\varpi, \varpi, t) + \zeta_\varrho(\varpi, \varpi, t)}{4} + \frac{\zeta_\varrho(\varpi, \varpi, t)}{2}\Big) = H\Upsilon(\zeta_\varrho(\varpi, \varpi, t)) = H\Upsilon(1) = 0.$$

Hence, $\zeta_\varrho(\varpi, \Gamma\varpi, t) = 1 \Rightarrow \Gamma\varpi = \varpi$. Thus ϖ is a fixed point of Γ.
Now, we will prove that ϖ is unique. Assume not, $\exists w \in \Xi$, such that $\Gamma w = w$ where $w \neq \varpi$ and $\lim_{\kappa\to\infty} \iota_\kappa = w$. By (**3**.6), we have

$$\begin{aligned}\Upsilon(\zeta_\varrho(w, \varpi, t)) &= \Upsilon(\zeta_\varrho(\Gamma w, \Gamma\varpi, t)) \\ &\leq H\Upsilon\Big(\frac{\zeta_\varrho(w, \varpi, t) + \zeta_\varrho(\Gamma w, w, t)}{4} + \frac{\zeta_\varrho(\varpi, \Gamma\varpi, t)}{2}\Big) \\ &= H\Upsilon\Big(\frac{\zeta_\varrho(w, \varpi, t) + 1}{4} + \frac{1}{2}\Big).\end{aligned}$$

Hence

$$\zeta_\varrho(w, \varpi, t) \geq 1.$$

Thus, $\zeta_\varrho(w, \varpi, t) = 1 \Rightarrow w = \varpi$. So Γ has a unique fixed point ϖ. □

The two following examples satisfy Theorem (**3**.1).

Example 3.3. *Let* $\Xi = [0,1]$ *and* $\zeta_\varrho(\iota,\eta,t) = \frac{t}{t+\frac{|\iota-\eta|}{\varrho}}$. *Define* $\Gamma : [0,1] \to [0,1]$ *via* $\Gamma(\iota) = \frac{\iota}{3}$. *Also, define* $\Upsilon : (0,1] \to [0,\infty)$ *via* $\Upsilon(\iota) = \frac{1}{\iota} - 1$. *Note that* Υ *is a strictly decreasing, continuous function and* $\Upsilon(1) = 0$. *Now, we have:*

$$\zeta_\varrho(\Gamma\iota,\Gamma\eta,t) = \frac{3\varrho t}{3\varrho t + |\iota-\eta|}, \quad \zeta_\varrho(\iota,\eta,t) = \frac{\varrho t}{\varrho t + |\iota-\eta|}.$$

So, $\Upsilon(\zeta_\varrho(\Gamma\iota,\Gamma\eta,t)) = \dfrac{1}{\zeta_\varrho(\Gamma\iota,\Gamma\eta,t)} - 1 = \dfrac{|\iota-\eta|}{3\varrho t}$; $\Upsilon(\zeta_\varrho(\iota,\eta,t)) = \dfrac{1}{\zeta_\varrho(\iota,\eta,t)} - 1 = \dfrac{|\iota-\eta|}{\varrho t}$.

Hence, for $H = \frac{1}{3}$, *we get* $\Upsilon(\zeta_\varrho(\Gamma\iota,\Gamma\eta,t)) = H\Upsilon(\zeta_\varrho(\iota,\eta,t))$. *Thus, Theorem (***3***.1) implies that* Γ *has a unique fixed point* $0 \in \Xi$.

Example 3.4. *Let* $\Xi = [0,1]$ *and* $\zeta_\varrho(\iota,\eta,t) = \exp\Big\{-\frac{|\iota-\eta|}{t\varrho}\Big\}$. *Define* $\Gamma : [0,1] \to [0,1]$ *via* $\Gamma(\iota) = \frac{\iota}{5}$. *Also, define* $\Upsilon : (0,1] \to [0,\infty)$ *via* $\Upsilon(\iota) = -\ln\iota$. *Note that* Υ *is a strictly decreasing, continuous function and* $\Upsilon(1) = 0$. *Now, we have:*

$$\zeta_\varrho(\Gamma\iota,\Gamma\eta,t) = \exp\Big\{-\frac{|\iota-\eta|}{5t\varrho}\Big\}, \quad \zeta_\varrho(\iota,\eta,t) = \exp\Big\{-\frac{|\iota-\eta|}{t\varrho}\Big\}.$$

So

$$\Upsilon(\zeta_\varrho(\Gamma\iota,\Gamma\eta,t)) = \tfrac{|\iota-\eta|}{5\varrho t}; \quad \Upsilon(\zeta_\varrho(\iota,\eta,t)) = \tfrac{|\iota-\eta|}{\varrho t}.$$

Hence, for $H = \frac{1}{5}$, *we get* $\Upsilon(\zeta_\varrho(\Gamma\iota,\Gamma\eta,t)) = H\Upsilon(\zeta_\varrho(\iota,\eta,t))$. *Thus Theorem (***3***.1) implies that* Γ *has a unique fixed point* $0 \in \Xi$.

The following example satisfies Theorem (**3**.2).

Example 3.5. *Let* $\Xi = [0,1]$ *and* $\zeta_\varrho(\iota,\eta,t) = \frac{t}{t+\frac{|\iota-\eta|}{\varrho}}$. *Define* $\Gamma : [0,1] \to [0,1]$ *via* $\Gamma(\iota) = c$, $c \in [0,1]$. *Also, define* $\Upsilon : (0,1] \to [0,\infty)$ *via* $\Upsilon(\iota) = \frac{1}{\iota} - 1$. *note that* Υ *is a strictly decreasing, continuous function and* $\Upsilon(1) = 0$. *Now, we have:*

$$\zeta_\varrho(\Gamma\iota,\Gamma\eta,t) = \frac{t}{t+|c-c|} = 1; \quad \frac{\zeta_\varrho(\iota,\eta,t)}{4} = \frac{1}{4} \times \frac{\varrho t}{\varrho t + |\iota-\eta|};$$

$$\frac{\zeta_\varrho(\Gamma\iota,\iota,t)}{4} = \frac{1}{4} \times \frac{\varrho t}{\varrho t + |c-\iota|}, \quad \text{and} \quad \frac{\zeta_\varrho(\eta,\Gamma\eta,t)}{2} = \frac{1}{2} \times \frac{\varrho t}{\varrho t + |c-\eta|};.$$

So

$$\begin{aligned}\frac{\zeta_\varrho(\iota,\eta,t) + \zeta_\varrho(\Gamma\iota,\iota,t)}{4} + \frac{\zeta_\varrho(\eta,\Gamma\eta,t)}{2} &= \frac{1}{4} \times \Big(\frac{\varrho t}{\varrho t + |\iota-\eta|} + \frac{\varrho t}{\varrho t + |c-\iota|} + 2 \times \frac{\varrho t}{\varrho t + |c-\eta|}\Big) \\ &\le \frac{1}{4}\Big(1+1+2\Big) = 1 = \zeta_\varrho(\Gamma\iota,\Gamma\eta,t).\end{aligned}$$

So, for H *such that* $0 < H < 1$, *we have*

$$H\Upsilon\Big(\frac{\zeta_\varrho(\iota,\eta,t) + \zeta_\varrho(\Gamma\iota,\iota,t)}{4} + \frac{\zeta_\varrho(\eta,\Gamma\eta,t)}{2}\Big) \ge 0 = \Upsilon(\zeta_\varrho(\Gamma\iota,\Gamma\eta,t)).$$

*Thus Theorem (***3***.2) implies that* Γ *has a unique fixed point* $c \in \Xi$.

4. Application

In this section, we use our obtained results to show that the following integral equation has a solution:

$$\iota(k) = h(k) + \int_0^1 \Omega(k,s)\varsigma(s,\iota(s))ds, \quad k \in [0,1]. \tag{4.1}$$

Let $\Xi = C([0,1])$ be the space of all continuous functions defined on $[0,1]$. Define a modular fuzzy metric:

$$\zeta_\varrho(\iota,\eta,t) : (0,\infty) \times C([0,1]) \times C([0,1]) \times (0,\infty) \to [0,1],$$

by

$$\zeta_\varrho(\iota,\eta,t)=\frac{t}{t+\sup\limits_{k\in[0,1]}\frac{|\iota(k)-\eta(k)|}{\varrho}}.$$

Then $(\Xi,\zeta_\varrho,*)$ is a complete modular fuzzy metric space.

Theorem 4.1. *Suppose we have the following hypotheses:*

(1) $\exists$ *a continuous function* $g:[0,1]\to[0,1]$ *such that*

$$|\varsigma(s,\iota)-\varsigma(s,\eta)|\le g(s)|\iota-\eta|. \tag{4.2}$$

And

$$\sup_{k\in[0,1]}\int_0^1 g(k)dk\le\frac{1}{3}. \tag{4.3}$$

(2)

$$\Omega(k,s)\le 1,\ \forall k,s\in[0,1]. \tag{4.4}$$

*Then the integral equation (**4**.1) has a solution* $\iota^*\in C^2([0,1])$.

Proof. Take the operator:

$$\Gamma\iota(k)=h(k)+\int_0^1\Omega(k,s)\varsigma(s,\iota(s))ds,\ k\in[0,1].$$

For all $\iota,\eta\in C([0,1])$, we have

$$\begin{aligned}\zeta_\varrho(\Gamma\iota,\Gamma\eta,t)&=\frac{t}{t+\sup\limits_{k\in[0,1]}\frac{|\Gamma\iota(k)-\Gamma\eta(k)|}{\varrho}}\\&=\frac{t}{t+\sup\limits_{k\in[0,1]}\frac{1}{\varrho}|h(k)+\int_0^1\Omega(k,s)\varsigma(s,\iota(s))ds-h(k)-\int_0^1\Omega(k,s)\varsigma(s,\eta(s))ds|}\\&=\frac{t}{t+\sup\limits_{k\in[0,1]}\frac{1}{\varrho}|\int_0^1\Omega(k,s)(\varsigma(s,\iota(s))-\varsigma(s,\eta(s)))ds|}.\end{aligned}$$

By (**4**.2) and (**4**.4), we have

$$\begin{aligned}\Big|\int_0^1\Omega(k,s)(\varsigma(s,\iota(s))-\varsigma(s,\eta(s)))ds\Big|&\le\int_0^1\Omega(k,s)\Big|\varsigma(s,\iota(s))-\varsigma(s,\eta(s))\Big|ds\\&\le\int_0^1 g(s)|\iota(s)-\eta(s)|ds.\end{aligned}$$

Hence by (**4**.3), we get

$$\begin{aligned}\frac{t}{t+\sup\limits_{k\in[0,1]}\frac{1}{\varrho}|\int_0^1\Omega(k,s)(\varsigma(s,\iota(s))-\varsigma(s,\eta(s)))ds|}&\ge\frac{t}{t+\sup\limits_{k\in[0,1]}\frac{1}{\varrho}\int_0^1 g(s)|\iota(s)-\eta(s)|ds}\\&\ge\frac{t}{t+\sup\limits_{k\in[0,1]}\frac{1}{3\varrho}|\iota(k)-\eta(k)|}.\end{aligned}$$

Thus

$$\frac{t+\sup\limits_{k\in[0,1]}\frac{1}{3\varrho}|\iota(k)-\eta(k)|}{t}\ge\frac{t+\sup\limits_{k\in[0,1]}\frac{1}{\varrho}|\int_0^1\Omega(k,s)(\varsigma(s,\iota(s))-\varsigma(s,\eta(s)))ds|}{t}$$

So

$$\frac{1}{3}\frac{\sup_{k\in[0,1]} \frac{1}{\varrho}|\iota(k)-\eta(k)|}{t} \geq \frac{\sup_{k\in[0,1]} \frac{1}{\varrho}|\int_0^1 \Omega(k,s)(\varsigma(s,\iota(s))-\varsigma(s,\eta(s)))ds|}{t}$$

Define $\Upsilon : (0,1] \to [0,+\infty)$ by $\Upsilon(t) = \frac{1}{\iota} - 1$. Then Υ is a strictly decreasing, continuous function and $\Upsilon(1) = 0$. For $H = \frac{1}{3}$, we obtain $\Upsilon(\zeta_\varrho(\Gamma\iota, \Gamma\eta, t)) \leq H\Upsilon(\zeta_\varrho(\iota, \eta, t))$. Therefore theorem (**3**.1) implies Γ has a unique fixed point and hence the integral equation (**4**.1) has a solution $\iota^* \in C^2([0,1])$. □

Conclusion: In this paper, we defined a new space called modular fuzzy metric space and stated some examples of this space. Also, we formulate and prove some new fixed point results under this space. In addition, we provided some examples and an application for showing the validity of our results.

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