U.P.B. Sci. Bull. Series D, Vol. 71, Iss. 2, 2009 ISSN 1454-2358

ON KÁRMÁN MODEL FOR COANDA EJECTOR WITH INCOMPRESSIBLE FLOW

Corneliu BERBENTE[1], Sorin DINEA[2]

Modelul Kármán pentru ejectorul Coandă cu regim de curgere incompresibil este utilizat şi completat în vederea precizării parametrilor care influenţează factorul de amplificare a forţei prin ejecţie şi limitele între care pot să varieze aceşti parametri. În acest scop, se stabilesc formule analitice de calcul. Se dau valori numerice pentru mai multe cazuri de interes teoretic şi practic.

Este pus în evidenţă un nou parametru, Λ, ***de care depinde existenţa soluţiilor modelului. Acest parametru, pe lângă gradul de neuniformitate a vitezelor curgerii secundare în secţiunea iniţială, considerat de Kármán****, conţine şi raportul ariilor ocupate de jet şi curgerea secundară iniţială, precum şi raportul dintre viteza maximă, la contact cu jetul, şi viteza medie indusă. Acest ultim raport este evidenţiat ca parametru important pentru stabilirea domeniilor de valori de interes practic. Se extinde profilul de viteze în secţiunea de plecare, şi se dă o metodă unitară de tratare simultană a curgerilor plană şi axial-simetrică.*

Kármán model for Coanda ejector with incompressible regime is used and completed in order to point out the main parameters that influence the augmentation factor of thrust by ejection and the limits allowed for the variation of these parameters. To this aim one obtains analytical formulas of calculation. Numerical values for several cases of theoretical and practical interest are presented.

A new parameter Λ, ***giving the existence conditions of the model is put in evidence. This parameter, besides the degree of flow nonuniformity in the section where the mixing starts considered by Kármán****, also contains the ratio of jet and ejector surface areas, as well as the ratio between the maximal and the average velocities of the secondary flow, induced by mixing. This last ratio is proved to be an important parameter for the domain of practical values. The velocities profile in the starting section is extended and a unitary method to solve together both 2D and axi-symmetrical flows is given.*

1. Introduction

The Coanda effect regarding the jet flow evolution near solid curved walls was subject of many experimental and theoretical studies [1-6], some of them

[1] Prof., Chair of Aerospace Sciences "E. Carafoli",University POLITEHNICA of Buchrest, Romania, e-mail: berbente@yahoo.com

[2] Research assist, Chair of Aerospace Sciences "E.Carafoli", University POLITEHNICA of Buchrest, Romania

done by Coanda himself [5]. A number of these studies were related to the wings of low aspect ratio [4]; others considered the ejector force augmentation [1;2;5].

In order to explain the mechanism of force augmentation obtained by Coanda ejector (an open tube with curved walls at entrance, where a jet is injected -fig.1), Kármán [1] considers an initial section 1, where one supposes that the mixing of jet with the air inside the tube starts, and section 2 at the ejector exit, where the the velocity is totally uniform and the atmospheric pressure is achieved. The chanel is considered of constant area, and the wall friction is neglected. The flow is incompressible.

Defining the force augmentation coefficient ϕ as the ratio:

$$\phi = \frac{\rho u_2^2 \left(a + A\right)}{\rho U^2 a} = \left(1 + \frac{A}{a}\right)\left(\frac{u_2}{U}\right)^2, \tag{1.1}$$

where U is the jet velocity, constant on area a, and u_2 the exit velocity from ejector, through area $(a + A)$. One denoted by ρ the gas density, and by A the area of the secondary flow.

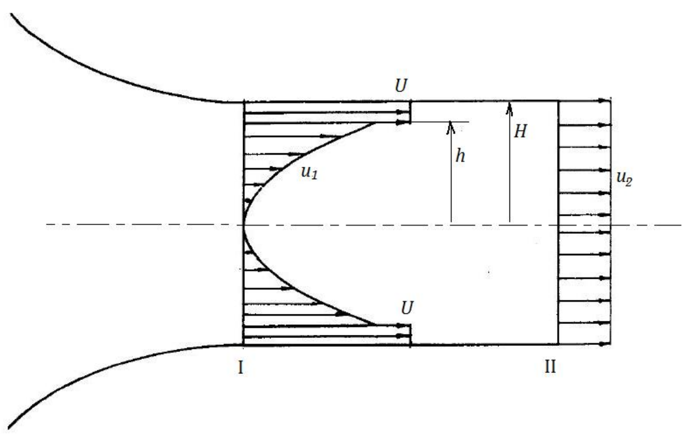

Fig.1 The ejector

Kármán pointed-out that the coefficient ϕ can be larger than unity. It tends to value 2 when the surface area ratio A/a indefinitely increases, if the secondary flow velocity is constant in the initial section 1. For larger nonuniformity degrees of velocity in section 1, the coefficient ϕ can approach values larger than 2. In the following we detail and complete the formulas and the conclusions possible to obtain by using the Kármán model.

2. The velocity profile

First we note the the injection is not necessarily done parallel to the wall, the jet joining however very quickly the wall, due to the pressure decreasing along the wall (so called ***Coanda effect***). Therefore, one takes the section 1 immediately after the wall curvature vanishes.

The mixing starts from the jet to axis; in the axis neighbourhood still remains a zone with potential flow, called ***potential core.***[7;8].

Kármán considers the initial section 1 (mixing start) with a null velocity, u_1=0, on the axis; then the velocity increases along a distance *h*, being maximal at the jet contact, but smaller than U .

In section 1, Kármán introduces a parameter of nonuniformity for the induced flow, denoted by λ, and defined by the relation:

$$\lambda = \frac{\overline{u_1^2}}{\left(\overline{u_1}\right)^2} = A\frac{\int_A u_1^2 dA}{\left(\int_A u_1 dA\right)^2}\ , \tag{2.1}$$

bars indicating the average values.

Retaining this idea, one takes however a more general velocity distribution in the secondary flow, section 1, with a velocity not necessarily null at axis. Moreover, to improve the mixing one can introduce a counter current on axis, for example, by suction. The proposed velocity distribution is:

$$\begin{cases} \dfrac{u_1}{U} = C(\beta + \eta^n);\ \text{for}\ \eta = r/h\ \le 1;\ |\beta| < 1;\ n \ge 0; \\ \dfrac{u_1}{U} = 1;\ \text{for}\ r \le H, \end{cases} \tag{2.2}$$

C being constant, and β ,*n* parameters at hand. For β =0, one obtains the Kármán velocity distribution.

The notations *h* și *H* are given inFig.1.

By using the velocity distribution (2.2) and the definition (2.1), one obtains the average velocities and the expression of the parameter of nonuniformity, λ, in section 1, under the form:

$$\begin{cases} \dfrac{\overline{u_1}}{U} = C(\beta + \dfrac{1}{1+\nu});\ \nu = n/\varepsilon;\ \varepsilon = \overline{1;2}\ ; \\ \dfrac{\overline{u_1^2}}{U^2} = C^2(\beta^2 + \dfrac{2\beta}{1+\nu} + \dfrac{1}{1+2\nu}); \end{cases} \tag{2.3}$$

$$\lambda = 1 + \frac{\nu^2}{(1+2\nu)}\frac{1}{[1+\beta(1+\nu)]^2}, \quad \nu = \frac{n}{\varepsilon}, \varepsilon = \overline{1;2}, \qquad (2.4)$$

where ε=1 for 2D flow, and ε=2 for axi-symmetrical flow. In this way, one can treat the two cases together.

One can see that the degree of nonuniformity increases with ν, having a positive derivative, while for $\nu = 0$ one has $\lambda = 1$. The derivative with respect to β is:

$$\frac{\partial\lambda}{\partial\beta} = -\frac{2\nu^2(1+\nu)}{(1+2\nu)[1+\beta(1+\nu)]^3} < 0, \ (\nu \geq 0), \qquad (2.5)$$

i.e. the degree of nonuniformity decreases with β, at ν=const.

The maximal secondary speed results at $r = h$:

$$\frac{u_{1h}}{U} = C(1+\beta), \qquad (2.6)$$

such that the ratio $\frac{u_{1h}}{\overline{u_1}}$ and its derivative have the expressions:

$$\frac{u_{1h}}{\overline{u_1}} = \frac{(1+\beta)(1+\nu)}{1+\beta(1+\nu)}; \ \frac{\partial}{\partial\beta}\left(\frac{u_{1h}}{\overline{u_1}}\right) = -\frac{\nu(1+\nu)}{[1+\beta(1+\nu)]^2}. \qquad (2.7)$$

Therefore the ratio $\frac{u_{1h}}{\overline{u_1}}$ decreases with β, at ν=const.

3.The conservation laws.

a) From the tube entrance up to the section 1 , the motion is potential (the potential core); therefore one can apply the Bernoulli law, as a particular case of the law of momentum conservation.

At the jet – secondary flow contact (section1), the pressures are equal, such that the pressure is denoted p_{1h} .

One yields the pressure - velocity relations in section 1:

$$\begin{cases} p_1 = p_\infty - \dfrac{\rho u_1^{\ 2}}{2}; \ r \in [0;h]; \\ p_1 = p_{1h} = p_\infty - \dfrac{\rho u_{1h}^{\ 2}}{2}; \ r \in [h; H-h], \end{cases} \qquad (3.1)$$

whilst the pressure p_2 at exit is constant and equal to the atmospheric prssure: $p_2 = p_\infty$.

The equation of continuity [8;9] between the sections 1 and 2 ($A_1 = A_2 = a + A$), gives:

$$\int_{A_1} \rho u dA = \int_{A_2} \rho u dA \tag{3.2}$$

The equation of momentum [8;9] between the sections 1 and 2 yields:

$$\int_{A_1} \left(\rho u^2 + p\right) dA = \int_{A_2} \left(\rho u^2 + p\right) dA$$

From (3.2) one gets:

$$\frac{\overline{u}_1}{U} = \left(1+\alpha\right)\frac{u_2}{U} - \alpha,\ \alpha = \frac{a}{A},$$

whereas from (3.3) results:

$$\left(\frac{\overline{u}_1}{U}\right)^2 \left(\lambda - \alpha\left(\frac{u_{1h}}{U}\right)^2\right) = 2\left(\alpha+1\right)\left(\frac{u_2}{U}\right)^2 - 2\alpha \tag{3.5}$$

Denoting by Λ the combined parameter:

$$\Lambda = \lambda - \alpha\left(\frac{u_{1h}}{\overline{\overline{u}}_1}\right)^2, \tag{3.6}$$

containing the degree of nonuniformity, λ , as well as the velocity ratio, , one obtains the following algebraic equation for the exit velocity $\frac{u_2}{U}$:

$$\left(\frac{u_2}{U}\right)^2 \left(2-\left(1+\alpha\right)\Lambda\right) + 2\alpha\Lambda\left(\frac{u_2}{U}\right) - \frac{\alpha\left(2+\alpha\Lambda\right)}{1+\alpha} = 0.$$

The second degree algebraic equation (3.7) has the reduced determinant:

$$\Delta' = \frac{2\alpha(2-\Lambda)}{1+\alpha};\ \Lambda \le 2. \tag{3.8}$$

Therefore, real solutions exist if and only if $\Lambda \le 2$.

We write the two real solutions under the form:

$$\begin{cases} \left(\frac{u_2}{U}\right)_I = \frac{\alpha\left(2+\alpha\Lambda\right)}{(1+\alpha)\left(\sqrt{\Delta'}+\alpha\Lambda\right)} = \frac{\sqrt{\Delta'}-\alpha\Lambda}{2-\left(1+\alpha\right)\Lambda}; \\ \left(\frac{u_2}{U}\right)_{II} = \frac{-\alpha\left(2+\alpha\Lambda\right)}{(1+\alpha)\left(\sqrt{\Delta'}-\alpha\Lambda\right)} = -\frac{\sqrt{\Delta'}+\alpha\Lambda}{2-\left(1+\alpha\right)\Lambda}. \end{cases} \tag{3.9}$$

From the expressions (3.9) one can see that the solution, $\left(\frac{u_2}{U}\right)_I$, is positive for any $\Lambda \le 2$, while the other solution, $\left(\frac{u_2}{U}\right)_{II}$, can be negative, or can change the sign, so that cannot represent a solution for our problem.

Case n = 0 (ν = 0).

In this case, the secondary speed u_1 is uniform. The constant β plays no role, being included in the general speed constant *C*. All quantities are now completely determinated as functions of surface area ratio *α*. Below one gives explicit expressions for the basic quantities in this case:

$$\nu = 0\ ;\ \lambda = 1\ ;\ \Lambda = 1 - \alpha\ ;\ \Delta' = 2\alpha\ ; \tag{3.10}$$

$$\left(\frac{u_2}{U}\right)_I = \frac{\alpha + \sqrt{2\alpha}}{1 + \alpha + \sqrt{2\alpha}};\ \frac{\overline{u}_1}{U} = \frac{\sqrt{2\alpha}}{1 + \alpha + \sqrt{2\alpha}} = \frac{u_{1h}}{U} < 1. \tag{3.11}$$

$$\phi = \left(1 + \frac{1}{\alpha}\right)\left(\frac{\alpha + \sqrt{2\alpha}}{1 + \alpha + \sqrt{2\alpha}}\right)^2. \tag{3.12}$$

When $\alpha \to 0$, one gets the maximum force amplification ϕ_{max} = 2. One can see that u_{1h}/U<1, for any *α*.

Study of several limit cases.

For $\Lambda = 2$, $\Delta' = 0$, one gets:

$$\left(\frac{u_2}{U}\right)_I = \left(\frac{u_2}{U}\right)_{II} = 1;\ \frac{u_1}{U} = 1;\ (\Lambda = 2)\ . \tag{3.13}$$

It is clear that the situation (3.13) is not a practical one, because the jet cannot suddenly entail the entire air quantity from the ejector entrance. However the conditions (3.13) do not necessarily represent the trivial case of a uniform current in a tube of constant section.

In Table 3.1 values for the exponent ν and for the speed ratio u_{1h}/U are given, when $\Lambda = 2$, β= 0. One remarks that solutions exist only in a certain interval of surface aria ratio (α ≤ 1/28). On the other hand, values larger than unity for the ratio u_{1h}/U are not acceptable for simple mixing.

Table 3.1 ; Values for $\Lambda = 2$, *β= 0* .

α	0.0	1/50	1/40	1/28
υ	2.4142	3.1406	3.4072	5.1276
$\frac{u_{1h}}{U}$	3.4142	4.1406	4.4072	6.1276

Therefore one should take $\Lambda < 2$.

A limit situation of interest for the study of the ejector performances is to have, as a maximal value, equal contact velocities at the initial section, that is :

$$\frac{u_{1h}}{U} = 1. \tag{3.14}$$

As a consequence, between the parameters α, β, ν of the model a relation should exist. Considering α, β independent, the exponent ν will result.

In the Table 3.2, values for the parameters ν, Λ, ϕ , r_d as functions of the surface area ratio α and of β, the parameter of the initial velocity on axis, are given.

One denotes by r_d the flow rates ratio, defined by the relation:

$$r_d = \frac{\rho(A+a)u_2}{\rho a U} = \phi\left(\frac{U}{u_2}\right) = \sqrt{\left(1+\frac{1}{\alpha}\right)\phi} \ , \tag{3.15}$$

φ being the force amplification.

Table 3.2.

***Values* for** $\frac{u_{1h}}{U} = 1.$

α	1/100	1/50	1/40	1/20	1/10	1/2
			β = – 0.05			
ν	1.8716	1.8820	1.8880	1.9220	2.010	3.537
Λ	1.9054	1.8102	1.7625	1.5175	0.9822	-11.947
φ	10.389	5.4203	4.4279	2.4360	1.4334	0.6155
r_d	32.393	16.626	13.474	7.1523	3.9708	1.3589

			$\beta = 0$			
ν	2.4170	2.4230	2.428	2.4600	2.5670	5.050
Λ	1.8846	1.7716	1.7129	1.4237	0.8019	-15.004
φ	9.o84	4.801	3.919	2.1892	1.3151	0.5901
r_d	30.310	15.648	12.676	6.7804	3.8034	1.3305
			$\beta = 0.05$			
ν	3.378	3.354	3.348	3.353	3.460	8.280
Λ	1.8480	1.7023	1.6312	1.2795	0.5444	- 19.328
φ	7.4999	4.018	3.3201	1.9096	1.1856	0.5637
r_d	27.523	14.315	11.667	6.333	3.6113	1.3004

From Table 3.2, one can see that the mixing (Λ) is diminished with β and, accordingly, the force augmentation coefficient φ also decreases. The exponent ν increases with β, at u_{1h}/U imposed. For negative values of the combined parameter, Λ, one yields force augmentation coefficients less than unity. The flow rate amplification, r_d, is also large.

The surface area ratio α, remains the main parameter to influence mixing and the force augmentation coefficient, leading to amplifications even by an order of magnitude. Certainly, by reducing the ratio u_{1h}/U, one decreases the force amplification. For example, for $u_{1h}/U = 0.500$, $\beta = 0$, one gets the values: $\nu = 1.743$; $\Lambda = 1.602$; $\varphi = 3.664$ (as compared to $\varphi = 9.084$, for $u_{1h}/U = 1$).

4. Using the velocity ratio u_{1h}/U as parameter.

Because the velocity ratio u_{1h}/U proved itself to play an important role in our analysis. Using it as parameter in place of the exponent ν is therefore of interest, as it can be easily controled.

By noting that the ratio between the maximum and average velocities in the secondary flow, $\overline{u_1}/u_{1h}$, has the expression:

$$\frac{\overline{u_1}}{u_{1h}} = \frac{1+\beta(1+\nu)}{(1+\beta)(1+\nu)}, \tag{4.1}$$

we will express successively the entrance and the exit velocities, as follows:

$$\frac{\overline{u}_1}{U}=\left(\frac{\overline{u}_1}{u_{1h}}\right)\frac{u_{1h}}{U}\ ;\ \frac{u_2}{U}=\frac{1}{1+\alpha}\left(\alpha+\left(\frac{\overline{u}_1}{u_{1h}}\right)\frac{u_{1h}}{U}\right). \qquad (4.2)$$

The force augmentation coefficient, ϕ, is expressed too as function of the section 1 parameters only:

$$\phi=\frac{1}{\alpha(1+\alpha)}\left(\alpha+\left(\frac{\overline{u}_1}{u_{1h}}\right)\frac{u_{1h}}{U}\right)^2. \qquad (4.3)$$

The momentum theorem is then used to obtain an equation for the exponent ν, β being known. This equation is (see also (3.5)):

$$\left(\frac{u_{1h}}{U}\right)^2\left(\lambda-\alpha\left(\frac{\overline{u}_1}{u_{1h}}\right)^2\right)=\frac{2}{(1+\alpha)}\left(\alpha+\left(\frac{\overline{u}_1}{u_{1h}}\right)\frac{u_{1h}}{U}\right)^2-2\alpha\ ;$$
$$\left(\lambda=1+\frac{\nu^2}{(1+2\nu)}\frac{1}{[1+\beta(1+\nu)]^2}\right) \qquad (4.4)$$

We remark that this strategy was already used to calculate the values given in Table 3.2.

On the other hand, because u_{1h}/U and ϕ are closely related parameters, by the expression:

$$\frac{u_{1h}}{U}=\left(\frac{u_{1h}}{\overline{u}_1}\right)\left(\sqrt{\phi\alpha(1+\alpha)}-\alpha\right)\ , \qquad (4.5)$$

one can use as well $\varphi > 1$ as parameter, then calculating the velocity ratio $u_{1h}/U \le 1$.

5. Conclusions

The Kármán model for Coanda ejector with incompressible flow and its extension are able to explain the force augmentation by mixing and aditional flow rate. The initial nonuniformity of the secondary flow is essential for the ejector performance. On the other hand, the mixing is complete when the exit velocity distribution is uniform, what requires a sufficiently long tube and small wall effects. The frictions effects were neglected; a diminution of the obtained thrust due to viscosity is also expected. If the turbulent boundary layer is considered in connection with a divergent nozzle, a boundary layer separation can occur [7;8;9], affecting the uniformity of the exit velocity distribution. The surface area ratio α, remains the main parameter to influence mixing and the force augmentation coefficient. Besides α, an important parameter was proved to be the ratio of the contact velocities between the primary and the secondary flows in the starting section where the potential core ends (u_{1h}/U).

Further theoretical and experimental studies are necessary in order to include the specified aspects, as well as a more accurate description of the jet characteristics around slit .

B I B L I O G R A F Y

[1]. *Th.von Kármán*, "Theoretical Remarks on Thrust Augmentation", Reissner Aniversary Volume, Contributions to Applied Mechanics, J.W. Eduards, Ann Arbor, Michigan, 1949.

[2]. *H. Fasel, St. Wernz,* "Numerical Investigation of the Control Separation from Curved and Blunt Trailing Edges Using DNS and LES", (Report), Tucson, Aerospace and Mechanical University, Arizona, USA, 2002

[3]. *T. Mehus*, "An Experimental Investigation into the Shape of Thrust Augmenenting Surface in Conjunction with Coanda Effect Deflector Jet Sheets" (Report),UTIAS TECHNICAL NOTE, AD614617, 1965

[4] *E.O.Rogers, M.J.Donnelly*, "Characteristics of Dual-Slotted Circulation Control Wing of Low Aspect Ratio", AIAA J.,Vols.5-8, jan.2004

[5] *Sferi-Coanda*, "Analysis ofThrust Due to Coanda Effect Phenomenon", (Report), Centre National D'Etudes et de Recherches Aeronautique Brussels, Contract AF61(052)382 USAF, AD0255741U, 1960 (unclassified)

[6] *C. Teodorescu-Ţintea*, "Asupra determinării câmpurilor de viteze şi temperaturi în secţiunile transversale ale jetului liber", Revista Transporturilor, nr.4,vol. 7, 1960

[7]. *H. Schlichting,* "Grenzschicht Theorie", Braun Verlag, 8. Auflage, Karlsruhe, 1982.

[8]. *C. Berbente, N.V. Constantinescu,* "Dinamica Gazelor şi Aerotermochimia", Vol.I; II, UPB, Bucureşti, 1980.

[9]. *S. Dănăilă, C. Berbente*, "Metode Numerice în Dinamica Fluidelor", Ed.Academiei, Bucureşti, 2003.