MIMO FIR LOWPASS FILTER DESIGN FOR SPATIALLY INTERCONNECTED SYSTEMS USING A BOUNDED REAL LEMMA APPROACH

by Bogdan C. Şicleru and Bogdan Dumitrescu

Prezentăm o metodă de proiectare a filtrelor FIR MIMO pentru sisteme interconectate spaţial. Mărginirea filtrelor pe diferite benzi de frecvenţă este impusă folosind sume de pătrate aplicate unui rezultat cunoscut ca Bounded Real Lemma. Făcând legătura între polinoame sume de pătrate şi matrici pozitiv semidefinite, obţinem o problemă de programare semidefinită. Comparând rezultatele noastre cu o metodă anterioară din literatura de specialitate arătăm că algoritmul nostru este superior.

We present a method for designing MIMO FIR filters for spatially interconnected systems. The boundedness of the filters on different frequency bands is imposed using a sum-of-squares approach for a general result known as the Bounded Real Lemma. By linking the sum-of-squares polynomials to positive semidefinite matrices, we obtain a problem that belongs to semidefinite programming. Comparing our results with a previous example from the literature we show that our algorithm is superior.

Keywords: FIR filter design, MIMO, Bounder Real Lemma, sum-of-squares, semidefinite programming

MSC2000: 93E 11.

1. Introduction

This paper deals with the design of discrete spatially interconnected systems (SISs). Such a system is formed by similar systems which directly interact only with their nearest neighbors [1, 10]. In filter design, these systems must have a frequency response similar to that of a desired ideal filter, for some frequency domains. Applications can be found in e.g. edge detection, image recovery [2].

We treat the boundedness design using some results for a general constraint known as the Bounded Real Lemma (BRL) in conjunction with exploiting the state-space representation of the filters. We characterize a BRL constraint over a (semialgebraic) domain, for multidimensional multi-input

[1]PhD Student, Dept. of Automatic Control and Computers, Politehnica University of Bucharest, Bucharest, Romania, e-mail: bogdan.sicleru@schur.pub.ro

[2]Professor, Dept. of Automatic Control and Computers, Politehnica University of Bucharest, Bucharest, Romania, e-mail: bogdan.dumitrescu@schur.pub.ro

multi-output (MIMO) systems, using sum-of-squares matrix polynomials. Imposing a sum-of-squares constraint amounts to the existence of a positive semidefinite matrix and setting linear constraints between the coefficients of the sum-of-squares polynomial and the positive matrix. Therefore, the design problem becomes a semidefinite programming (SDP) problem which can be solved using dedicated solvers.

We exemplify our algorithm on 2-D MIMO FIR filters and prove that using our approach one can obtain better results than using the method from [10] which is based on a linear matrix inequality condition.

Outline. The remainder of this article is structured as follows. In Section **2** we describe the SISs. Next, in Section **3** we present the characterization for the sum-of-squares polynomials. In Section **4** we introduce the BRL constraint. MIMO FIR filter design, with numerical simulations, is discussed in Section **5**. We conclude in Section **6**.

Notation. $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{T}$ are the sets of integer, real and unit complex numbers, respectively. Bold characters denote multivariate entities (vectors, matrices). The inequality $\boldsymbol{a} \leq \boldsymbol{b}$ is taken elementwise. $\boldsymbol{M} \succeq \boldsymbol{0}$ means that $\boldsymbol{M}$ is a positive semidefinite matrix. $\mathbf{Tr}\,\boldsymbol{M}$ and $\sigma_{\max}(\boldsymbol{M})$ are the trace and maximum singular value of the matrix $\boldsymbol{M}$, respectively. $\operatorname{diag}\left(\boldsymbol{M}_i|_{i=1}^d\right)$ denotes the block diagonal matrix with blocks $\boldsymbol{M}_i$, $i = 1 : d$. The superscript ^{T} denotes transposition. $\otimes$ is the Kronecker product. For a d-dimensional SIS we denote by m_0 the number of subsystem states and with $m(+,i)$ and $m(-,i)$ the number of outputs of a subsystem to the subsystems "immediately after" and "immediately before", respectively. We also denote $m_i = m(+,i) + m(-,i)$ and $m = \sum_{i=0}^d m_i$.

2. System description

We consider a d-dimensional discrete SIS [1]

$$\begin{aligned} \begin{bmatrix} \boldsymbol{x}(k+1,\boldsymbol{s}) \\ \boldsymbol{w}(k,\boldsymbol{s}) \\ \boldsymbol{y}(k,\boldsymbol{s}) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A}_{TT} & \boldsymbol{A}_{TS} & \boldsymbol{B}_T \\ \boldsymbol{A}_{ST} & \boldsymbol{A}_{SS} & \boldsymbol{B}_S \\ \boldsymbol{C}_T & \boldsymbol{C}_S & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(k,\boldsymbol{s}) \\ \boldsymbol{v}(k,\boldsymbol{s}) \\ \boldsymbol{d}(k,\boldsymbol{s}) \end{bmatrix} \\ \boldsymbol{v}(k,\boldsymbol{s}) &= \boldsymbol{Z}_s \boldsymbol{w}(k,\boldsymbol{s}) \end{aligned} \tag{1}$$

where k and $\boldsymbol{s} = [s_1, \ldots, s_d]$ are the temporal and spatial variables, respectively; $\boldsymbol{x}(k,\boldsymbol{s})$ denotes the state vector, $\boldsymbol{v}(k,\boldsymbol{s})$ and $\boldsymbol{w}(k,\boldsymbol{s})$ denote the subsystem input and output vector, respectively and $\boldsymbol{y}(k,\boldsymbol{s})$, $\boldsymbol{d}(k,\boldsymbol{s})$ are the external output and input vectors, respectively. $\boldsymbol{Z}_s$ is a spatial shifting operator and can be expressed as

$$\boldsymbol{Z}_s = \operatorname{diag}\left(\left[\begin{array}{cc} z_i^{-1}\boldsymbol{I}_{m(+,i)} & \boldsymbol{0} \\ \boldsymbol{0} & z_i \boldsymbol{I}_{m(-,i)} \end{array} \right] \Bigg|_{i=1}^{d} \right), \tag{2}$$

where z_i, $i = 1 : d$, is the shift operator for the i-th dimension, namely, $z_i \boldsymbol{w}(k, s_1, \ldots, s_d) = \boldsymbol{w}(k, s_1, \ldots, s_i + 1, \ldots, s_d)$.

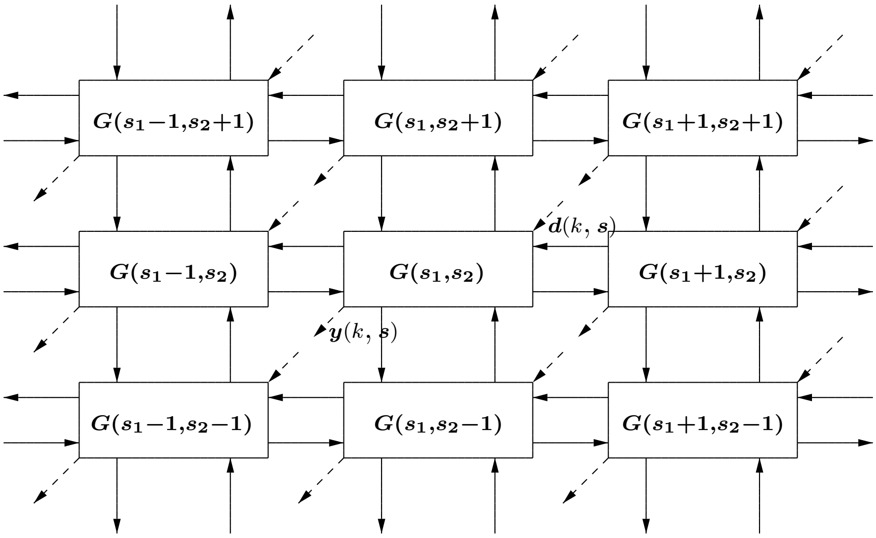

Fig. 1. Spatially interconnected system for $d = 2$.

For brevity, we denote the matrices $\begin{bmatrix} \boldsymbol{A}_{TT} & \boldsymbol{A}_{TS} \\ \boldsymbol{A}_{ST} & \boldsymbol{A}_{SS} \end{bmatrix}$, $\begin{bmatrix} \boldsymbol{B}_T \\ \boldsymbol{B}_S \end{bmatrix}$ and $\begin{bmatrix} \boldsymbol{C}_T & \boldsymbol{C}_S \end{bmatrix}$ by $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, respectively. Furthermore, we define the operator

$$\boldsymbol{Z} = \begin{bmatrix} z_0 \boldsymbol{I}_{m_0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Z}_s^{-1} \end{bmatrix}, \tag{3}$$

in which z_0 plays the role of a time domain shifting operator. Using the notations described above, the transfer function matrix for the system (1) is

$$\boldsymbol{G}(\boldsymbol{Z}) = \boldsymbol{C}(\boldsymbol{Z} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}. \tag{4}$$

Figure 1 presents an SIS for $d = 2$; the dashed arrow towards the subsystem $\boldsymbol{G}(s_1, s_2)$ denotes the external subsystem input $\boldsymbol{d}(k, \boldsymbol{s})$ and the dashed arrow leaving the subsystem denotes the external subsystem output $\boldsymbol{y}(k, \boldsymbol{s})$.

3. Sum-of-squares

Let us consider a d-dimensional trigonometric polynomial [4] with matrix coefficients

$$\boldsymbol{R}(\boldsymbol{z}) = \sum_{\boldsymbol{k}=-\boldsymbol{n}}^{\boldsymbol{n}} \boldsymbol{R}_{\boldsymbol{k}} \boldsymbol{z}^{-\boldsymbol{k}}, \quad \boldsymbol{R}_{-\boldsymbol{k}} = \boldsymbol{R}_{\boldsymbol{k}}^T, \tag{5}$$

with $\boldsymbol{R}_{\boldsymbol{k}} \in \mathbb{R}^{\kappa \times \kappa}$, $\boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ and $\boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} \cdots z_d^{k_d}$, $\boldsymbol{z} \in \mathbb{T}^d$.

A trigonometric causal filter with matrix coefficients is defined as

$$\boldsymbol{H}(\boldsymbol{z}) = \sum_{\boldsymbol{k}=\boldsymbol{0}}^{\boldsymbol{n}} \boldsymbol{H}_{\boldsymbol{k}} \boldsymbol{z}^{-\boldsymbol{k}}, \tag{6}$$

where $\boldsymbol{H}_{\boldsymbol{k}} \in \mathbb{R}^{\kappa_1 \times \kappa_2}$.

The polynomial (5) is sum-of-squares if it can be written as

$$\boldsymbol{R}(\boldsymbol{z}) = \sum_{\ell=1}^{p} \boldsymbol{F}_\ell(\boldsymbol{z}) \boldsymbol{F}_\ell(\boldsymbol{z}^{-1})^T, \tag{7}$$

where $\boldsymbol{F}_\ell(\boldsymbol{z})$, $\ell = 1 : p$, are causal polynomials and $p \in \mathbb{Z}_+$. Note that the degrees of the polynomials from the right-hand side of (7) may exceed $\boldsymbol{n}$.

In order to connect the sum-of-squares polynomials to SDP, we must express the causal polynomials with the standard d-dimensional basis

$$\boldsymbol{\Psi}(\boldsymbol{z}) = \boldsymbol{\Psi}(z_d) \otimes \cdots \otimes \boldsymbol{\Psi}(z_1) \otimes \boldsymbol{I}_\kappa, \tag{8}$$

where

$$\boldsymbol{\Psi}(z_i) = \left[\begin{array}{cccc} 1 & z_i & \ldots & z_i^{n_i} \end{array} \right]^T \tag{9}$$

is a univariate basis. If we stack the matrix coefficients of the polynomial $\boldsymbol{H}(\boldsymbol{z})$ in the order of the basis from (8), we obtain a matrix $\boldsymbol{H}$ of size $N\kappa_1 \times \kappa_2$, with $N = \prod_{i=1}^{d}(n_i + 1)$ being the total number of matrix coefficients of $\boldsymbol{H}(\boldsymbol{z})$. Now, the polynomial $\boldsymbol{H}(\boldsymbol{z})$ can be written as

$$\boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{\Psi}(\boldsymbol{z}^{-1})\boldsymbol{H}. \tag{10}$$

Example 3.1. For a degree $\boldsymbol{n} = (2, 1)$, the matrix $\boldsymbol{H}$ has the form

$$\boldsymbol{H} = \left[\begin{array}{cccccc} \boldsymbol{H}_{0,0}^T & \boldsymbol{H}_{1,0}^T & \boldsymbol{H}_{2,0}^T & \boldsymbol{H}_{0,1}^T & \boldsymbol{H}_{1,1}^T & \boldsymbol{H}_{2,1}^T \end{array} \right]^T. \tag{11}$$

The basis for such a polynomial $\boldsymbol{H}(\boldsymbol{z})$ is

$$\boldsymbol{\Psi}(\boldsymbol{z}) = \left[\begin{array}{cccccc} \boldsymbol{I} & z_1\boldsymbol{I} & z_1^2\boldsymbol{I} & z_2\boldsymbol{I} & z_1z_2\boldsymbol{I} & z_1^2z_2\boldsymbol{I} \end{array} \right]^T. \tag{12}$$

■

The next theorem characterizes a sum-of-squares polynomial in an SDP fashion.

Theorem 3.1. A matrix trigonometric polynomial (5) is sum-of-squares if and only if there exists a matrix $\boldsymbol{Q} \succeq \boldsymbol{0}$, such that

$$\boldsymbol{R}(\boldsymbol{z}) = \boldsymbol{\Psi}(\boldsymbol{z}^{-1})^T \cdot \boldsymbol{Q} \cdot \boldsymbol{\Psi}(\boldsymbol{z}), \tag{13}$$

where $\boldsymbol{Q}$ is of size $N\kappa \times N\kappa$. (The matrix $\boldsymbol{Q}$ is called a Gram matrix.)

Proof. See for instance [7]. ■

Remark 3.1. Explicitly, the equation (13) is implemented using the following linear constraints between the elements of the matrix coefficients of $\boldsymbol{R}(\boldsymbol{z})$ and the matrix $\boldsymbol{Q}$:

$$\boldsymbol{R}_{\boldsymbol{k}}(i,j) = \mathbf{Tr}[\boldsymbol{\Theta}_{k_d} \otimes \cdots \otimes \boldsymbol{\Theta}_{k_1} \otimes \boldsymbol{E}_{j,i} \cdot \boldsymbol{Q}], \quad \boldsymbol{k} \in \mathcal{H}_d,\ i,j = 1 : \kappa, \tag{14}$$

where $\boldsymbol{\Theta}_k$ is the elementary Toeplitz matrix with ones on the k-th diagonal and zeros elsewhere and $\boldsymbol{E}_{i,j}$ is the matrix with one on the position (i,j) and zeros elsewhere. $\mathcal{H}_d$ is the halfspace for the degree $\boldsymbol{n}$, consisting of all the d-tuples $(k_1, \ldots, k_d)$ with $(k_d > 0)$ or $(k_d = 0$ and $(k_1, \ldots, k_{d-1}) \in \mathcal{H}_{d-1})$. ■

We take now a frequency domain

$$\mathcal{D} = \{\boldsymbol{z} \in \mathbb{T}^d \mid D_\ell(\boldsymbol{z}) \geq 0, \ell = 1 : L\}, \tag{15}$$

described by L positive polynomials. The result which characterizes the positivity [4] of the polynomial $\boldsymbol{R}(\boldsymbol{z})$ on the domain $\mathcal{D}$ is given in the following theorem.

Theorem 3.2. A matrix polynomial (5) is positive definite on the domain $\mathcal{D}$, i.e. $\boldsymbol{R}(\boldsymbol{z}) \succ \boldsymbol{0}$, $\forall \boldsymbol{z} \in \mathcal{D}$, if and only if there exist sum-of-squares polynomials $\boldsymbol{S}_\ell(\boldsymbol{z})$, $\ell = 0 : L$, such that

$$\boldsymbol{R}(\boldsymbol{z}) = \boldsymbol{S}_0(\boldsymbol{z}) + \sum_{\ell=1}^{L} D_\ell(\boldsymbol{z})\boldsymbol{S}_\ell(\boldsymbol{z}). \tag{16}$$

Proof. Due to the fact that both scalar and matrix sum-of-squares polynomials are characterized through the trace parameterization in the same manner (recall (14) for the matrix case) with positive semidefinite matrices, the proof of the theorem is similar to the one for scalar polynomials [3]. See also [8] for a real polynomial variant for the Theorem **3**.2. ∎

4. Bounded Real Lemma

A (matrix) BRL constraint, on the domain $\mathcal{D}$, is a characterization of the form

$$\|\boldsymbol{H}(\boldsymbol{z})\| \leq \gamma, \quad \forall \boldsymbol{z} \in \mathcal{D}, \tag{17}$$

where $\gamma \in \mathbb{R}$ and $\|\cdot\|$ is a system norm. Considering the H_∞ norm, (17) is equivalent to

$$\sigma_{\max}(\boldsymbol{H}(\boldsymbol{z})) \leq \gamma, \quad \forall \boldsymbol{z} \in \mathcal{D}. \tag{18}$$

Next, we give a characterization of the BRL from (18) (using sum-of-squares) [5].

Theorem 4.1. For the polynomial $\boldsymbol{H}(\boldsymbol{z})$, the inequality from (18) is true, if and only if there exist sum-of-squares $\boldsymbol{S}_\ell(\boldsymbol{z})$, $\ell = 0 : L$, such that

$$\gamma^2 \boldsymbol{I}_{\kappa_1} = \boldsymbol{S}_0(\boldsymbol{z}) + \sum_{\ell=1}^{L} D_\ell(\boldsymbol{z})\boldsymbol{S}_\ell(\boldsymbol{z}) \tag{19}$$

and

$$\begin{bmatrix} \boldsymbol{Q}_0 & \boldsymbol{H} \\ \boldsymbol{H}^T & \boldsymbol{I}_{\kappa_2} \end{bmatrix} \succeq \boldsymbol{0}, \tag{20}$$

where $\boldsymbol{Q}_0$ is the Gram matrix for the polynomial $\boldsymbol{S}_0(\boldsymbol{z})$.

Proof. The proof is similar to the scalar case presented in [3]; it uses Theorem **3**.2, a majorization and the Schur complement. ∎

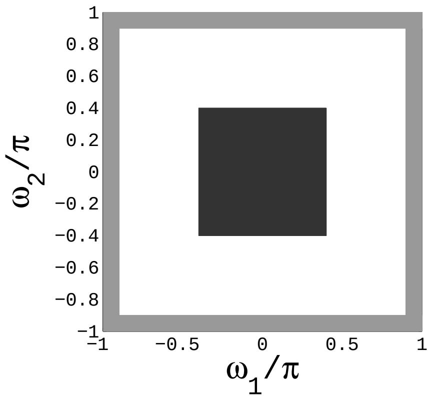

Fig. 2. Passband of lowpass 2-D filter; $\omega_p = 0.4\pi$, $\omega_s = 0.9\pi$.

Remark 4.1. The relation (18) is equivalent to

$$\gamma^2 \boldsymbol{I}_{\kappa_1} > \boldsymbol{H}(\boldsymbol{z})\boldsymbol{H}(\boldsymbol{z}^{-1})^T, \quad \forall \boldsymbol{z} \in \mathcal{D}, \tag{21}$$

which explains the size of the matrix coefficients from (19). ■

Remark 4.2. In implementing (19) we always consider the minimum degree for the sum-of-squares polynomials. In particular we take $\boldsymbol{S}_0(\boldsymbol{z})$ to be of degree $\boldsymbol{n}$. So, we use only a sufficient boundedness condition. Thus, the optimal γ is an upper approximation of the desired H_∞ norm. ■

5. MIMO filter design

We consider designing MIMO FIR lowpass filters. Figure 2 shows the ideal passband for a lowpass filter with square shape; the passband is in black and stopband in grey. We take ω_p and ω_s to be the passband and stopband frequencies, respectively. We also denote γ_p and γ_s to be the passband and stopband error bounds, respectively. Considering a desired response $\boldsymbol{\Delta}(\boldsymbol{z})$ in the passband, finding the optimal filter in the sense of minimal stopband error amounts to solving the optimization problem

$$\begin{array}{ll} \min & \gamma_s \\ \text{s.t.} & \sigma_{\max}(\boldsymbol{H}(\boldsymbol{e}^{j\boldsymbol{\omega}}) - \boldsymbol{\Delta}(\boldsymbol{e}^{j\boldsymbol{\omega}})) \le \gamma_p, \quad \forall |\omega_i| \le \omega_p, i = 1:d \\ & \sigma_{\max}(\boldsymbol{H}(\boldsymbol{e}^{j\boldsymbol{\omega}})) \le \gamma_s, \quad \exists i \in 1:d, |\omega_i| \ge \omega_s \end{array} \tag{22}$$

where $\boldsymbol{H}(\boldsymbol{e}^{j\boldsymbol{\omega}})$ is the frequency response for the filter $\boldsymbol{H}(\boldsymbol{z})$, for $\boldsymbol{z} = \boldsymbol{e}^{j\boldsymbol{\omega}}$. The passband $\mathcal{D}_p$ is a domain like the one in (15), where

$$D_\ell(\boldsymbol{z}) = z_\ell + z_\ell^{-1} - 2\cos\omega_p, \quad \ell = 1:d. \tag{23}$$

The stopband is a union of domains

$$\mathcal{D}_s = \bigcup_{i=1}^{d} \mathcal{D}_{s,i}, \tag{24}$$

where

$$\mathcal{D}_{s,i} = \{\boldsymbol{z} \in \mathbb{T}^d \mid D_s(z_i) \geq 0\}, \quad i = 1 : d, \tag{25}$$

with

$$D_s(z_i) = 2\cos\omega_s - z_i - z_i^{-1}. \tag{26}$$

Hence, the problem (22) can be written as

$$\begin{array}{ll} \min\limits_{\gamma_s, \boldsymbol{H}} & \gamma_s \\ \text{s.t.} & \sigma_{\max}(\boldsymbol{H}(\boldsymbol{z}) - \boldsymbol{\Delta}(\boldsymbol{z})) \leq \gamma_p, \quad \forall \boldsymbol{z} \in \mathcal{D}_p \\ & \sigma_{\max}(\boldsymbol{H}(\boldsymbol{z})) \leq \gamma_s, \quad \forall \boldsymbol{z} \in \mathcal{D}_{s,1} \\ & \vdots \\ & \sigma_{\max}(\boldsymbol{H}(\boldsymbol{z})) \leq \gamma_s, \quad \forall \boldsymbol{z} \in \mathcal{D}_{s,d} \end{array} \tag{27}$$

In order to design a subsystem for an SIS, the filter from (27) must meet the condition (4). Thus, the SIS design problem can be cast as

$$\begin{array}{ll} \min\limits_{\gamma_s, \boldsymbol{H}, \boldsymbol{C}, \boldsymbol{D}} & \gamma_s \\ \text{s.t.} & \boldsymbol{H}_{\boldsymbol{k}} = \boldsymbol{G}_{\boldsymbol{k}}, \quad \forall \boldsymbol{k} \\ & \sigma_{\max}(\boldsymbol{H}(\boldsymbol{z}) - \boldsymbol{\Delta}(\boldsymbol{z})) \leq \gamma_p, \quad \forall \boldsymbol{z} \in \mathcal{D}_p \\ & \sigma_{\max}(\boldsymbol{H}(\boldsymbol{z})) \leq \gamma_s, \quad \forall \boldsymbol{z} \in \mathcal{D}_{s,1} \\ & \vdots \\ & \sigma_{\max}(\boldsymbol{H}(\boldsymbol{z})) \leq \gamma_s, \quad \forall \boldsymbol{z} \in \mathcal{D}_{s,d} \end{array} \tag{28}$$

Note that the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ from (4) are usually given, so we consider only $\boldsymbol{C}$ and $\boldsymbol{D}$ to be variables in the optimization problem.

Let us consider designing 2-D MIMO filters with two inputs and two outputs. Moreover, in filter design, it is assumed that $m_0 = 0$ and $m(-,1) = m(+,1) = m(-,2) = m(+,2) = m'$. (Note that in this case $m = 4m'$ and $\boldsymbol{Z} = \boldsymbol{Z}_s^{-1}$.)

In designing FIR filters the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are designated to be [6]

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0}_{1\times(m-1)} & 0 \\ \boldsymbol{I}_{m-1} & \boldsymbol{0}_{(m-1)\times 1} \end{bmatrix} \tag{29}$$

and

$$\boldsymbol{B} = \Big[\underbrace{\boldsymbol{I}_2 \;\; \ldots \;\; \boldsymbol{I}_2}_{m/2 \text{ blocks}}\Big]^T. \tag{30}$$

The matrix $\boldsymbol{A}$ being subdiagonal, the equation (4) becomes

$$\boldsymbol{G}(\boldsymbol{Z}) = \boldsymbol{C}\left(\boldsymbol{I} + \boldsymbol{Z}^{-1}\boldsymbol{A} + \ldots + (\boldsymbol{Z}^{-1}\boldsymbol{A})^{m-1}\right)\boldsymbol{Z}^{-1}\boldsymbol{B} + \boldsymbol{D}, \tag{31}$$

where $\boldsymbol{C} \in \mathbb{R}^{2\times m}$ and $\boldsymbol{D} \in \mathbb{R}^{2\times 2}$.

In (31) the variables z_1 and z_2 appear with degrees between $-m'$ and m'. As such, we consider the filter $\boldsymbol{H}(\boldsymbol{z})$ to be a causal filter of degree $2m'$. Taking a delay $\boldsymbol{\Delta}(\boldsymbol{z}) = z_1^{m'} z_2^{m'} \boldsymbol{I}$, the problem (28) becomes

$$\begin{array}{ll} \min\limits_{\gamma_s, \boldsymbol{H}, \boldsymbol{C}, \boldsymbol{D}} & \gamma_s \\ \text{s.t.} & \boldsymbol{H}_{k_1,k_2} = \boldsymbol{G}_{k_1-m',k_2-m'}, \quad (0,0) \le (k_1,k_2) \le (2m', 2m') \\ & \sigma_{\max}(\boldsymbol{H}(z_1,z_2) - z_1^{m'} z_2^{m'} \boldsymbol{I}_2) \le \gamma_p, \quad \forall (z_1,z_2) \in \mathcal{D}_p \\ & \sigma_{\max}(\boldsymbol{H}(z_1,z_2)) \le \gamma_s, \quad \forall (z_1,z_2) \in \mathcal{D}_{s,1} \\ & \sigma_{\max}(\boldsymbol{H}(z_1,z_2)) \le \gamma_s, \quad \forall (z_1,z_2) \in \mathcal{D}_{s,2} \end{array} \tag{32}$$

To better understand the equality constraints $\boldsymbol{H}_{\boldsymbol{k}} = \boldsymbol{G}_{\boldsymbol{k}}$ from (32), we detail them for the case $m' = 1$. In this scenario, the relation (31) becomes

$$\boldsymbol{G}(z_1,z_2) = \boldsymbol{C} \begin{bmatrix} 1 & 0 & 0 & 0 \\ z_1 & 1 & 0 & 0 \\ z_1 z_2^{-1} & z_2^{-1} & 1 & 0 \\ z_1 & 1 & z_2 & 1 \end{bmatrix} \begin{bmatrix} z_1^{-1} & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2^{-1} & 0 \\ 0 & 0 & 0 & z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + \boldsymbol{D}. \tag{33}$$

Finally,

$$\boldsymbol{G}(z_1,z_2) = \boldsymbol{C} \begin{bmatrix} z_1^{-1} & 0 \\ 1 & z_1 \\ 2z_2^{-1} & z_1 z_2^{-1} \\ 2 & z_1 + z_2 \end{bmatrix} + \boldsymbol{D}. \tag{34}$$

Separating the degrees of $\boldsymbol{G}(z_1,z_2)$, we obtain

$$\boldsymbol{G}(z_1,z_2) = \boldsymbol{C}(z_1^{-1}\boldsymbol{N}_{-1,0} + z_2^{-1}\boldsymbol{N}_{0,-1} + \boldsymbol{N}_{0,0} + z_1\boldsymbol{N}_{1,0} + z_1 z_2^{-1}\boldsymbol{N}_{1,-1} + z_2\boldsymbol{N}_{0,2}) + \boldsymbol{D}, \tag{35}$$

where

$$\boldsymbol{N}_{-1,0} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{N}_{0,-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{N}_{0,0} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \tag{36}$$

and

$$\boldsymbol{N}_{1,0} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{N}_{1,-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{N}_{0,1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{37}$$

We also denote $\boldsymbol{N}_{o_1,o_2} = \boldsymbol{0}_{4\times 2}$, $\forall$ $(o_1, o_2) \in \{(-1,1), (-1,-1), (1,1)\}$. Therefore, the equality constraints from the problem (32) (for $m' = 1$) can be seen as

$$\begin{array}{ll} \boldsymbol{H}_{k_1,k_2} & = \boldsymbol{C}\boldsymbol{N}_{o_1,o_2}, \quad (0,0) < (k_1,k_2) \le (2,2), \\ \boldsymbol{H}_{0,0} & = \boldsymbol{C}\boldsymbol{N}_{0,0} + \boldsymbol{D}, \end{array} \tag{38}$$

with $o_i = k_i - 1$, $i = 1:2$. The generalization of (38) for $m' > 1$ is straightforward.

Example 5.1. We have solved the problem (32) using the SeDuMi [9] solver and compared our results with the ones from [10]. The stopband error bounds γ_s, considering $m' = 3 : 6$, are presented in Table 1. The values obtained

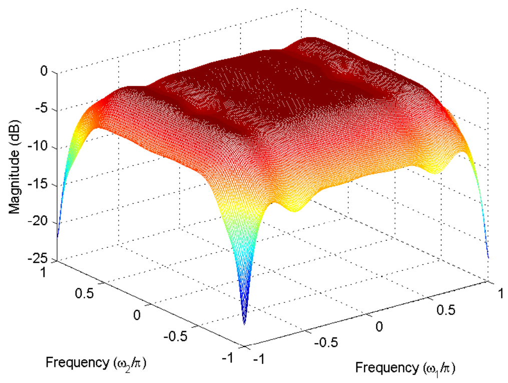

Fig. 3. Frequency response of 2-D FIR filter, $m' = 6$.

for γ_s using the problem (32) are better (smaller) than the ones in [10]. One theoretical explanation for this is that while our algorithm relaxes a necessary condition, the method from [10] implements directly a sufficient condition; sum-of-squares relaxations have been proved to be almost necessary in many practical applications.

Table 1

Stopband error bounds.

Algorithm	γ_s			
	m'			
	3	4	5	6
[10]	0.87128	0.84608	0.82556	0.82086
(32)	0.86213	0.83389	0.81330	0.81056

We present in Figure 3 the frequency response for the 2-D FIR filter, from input 1 to output 1, for the case $m' = 6$. ■

6. Conclusions

We have presented a design method for MIMO filters for SISs. The frequency response constraints were imposed in a BRL manner. To characterize the BRL constraints on semialgebraic domains we have used a sum-of-squares approach, which amounted to setting linear constrains between the coefficients of the sum-of-squares polynomials and positive semidefinite matrices. The design example, performed on 2-D FIR filters, showed that our method can perform better than a previous algorithm, in terms of stopband error bound.

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